MATHEMATICA, Tome 50 (73), N° 1, 2008, pp. 13–30

# THE WILSON VERSION OF D'ALEMBERT'S FUNCTIONAL EQUATION ON A CLASS OF 2-DIVISIBLE NILPOTENT GROUPS

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**Abstract.** Consider the functional equation

$$f, g, h, k \colon G \to K, \quad f(xy) + g(xy^{-1}) = h(x)k(y), \tag{$*$}$$

where $G$ is a group and $K$ a field with char $K \neq 2$.

Wilson [13] and Aczél [1] have solved the equation $(*)$ where $G$ is the additive group of real numbers $\mathbb{R}$ and $K = \mathbb{R}$.

In the present paper we obtain the general solution of the equation $(*)$ when $G$ belongs to a special class of nilpotent or generalized nilpotent groups.

**MSC 2000.** 39B52, 20B99.

**Key words.** Functional equation, nilpotent group, Lie group.

## INTRODUCTION

Consider the functional equation

$$f, g, h, k : G \to K, \quad f(xy) + g(xy^{-1}) = h(x)k(y), \tag{1.1}$$

where $G$ is a group and $K$ a field. This equation is called sometimes Wilson's second generalization of d'Alembert's functional equation (see [1, §3.2.2]).

Several papers deal with the equation (1.1). In Wilson [13], and cf. also Aczél [1, §3.2.2], equation (1.1) is solved when $G$ is the additive group of real numbers $\mathbb{R}$ and $K = \mathbb{R}$. Vincze [12] has solved equation (1.1) when $G$ is a subgroup of the additive group of complex numbers $\mathbb{C}$ and $K = \mathbb{C}$. In [2] Aczél and Vincze study an equation of the type (1.1) where $G$ is a subgroup of the additive group of $\mathbb{C}$ and $K$ is a field of characteristic equal to zero.

The equation (1.1) was solved by the author in [7] when $G$ is a generalized nilpotent group provided that all its element have the odd order and $K$ is a field with char $K = 0$.

Friis [9] solved Wilson's functional equation when $G$ is a connected nilpotent Lie group, except the case when it is the Jensen equation. He also pointed out the role of the class $\mathcal{N}$ that we introduce below.

Investigations of the particular cases of the equation (1.1) on non-abelian groups revealed that other solutions than the classical ones sometimes occur.

Definition 1. A group $G$ is said to be a *generalized nilpotent group* (see [3]) if

$$G = \bigcup_{\alpha<\gamma} Z_\alpha,$$

where

$$\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_\alpha \subset \dots, \quad \alpha < \gamma,$$

is the ascending central chain of the group $G$ ($\alpha$ and $\gamma$ are ordinal numbers). The groups $Z_\alpha$ are defined as follows: suppose $Z_\beta$ are defined for $\beta < \alpha$; if $\alpha - 1$ exists we have

$$Z_\alpha / Z_{\alpha-1} = Z(G/Z_{\alpha-1}),$$

if $\alpha$ is a limit-ordinal, then

$$Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta.$$

The group $G$ is said to be nilpotent if it is swept out by its ascending central chain, i.e.,

$$\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_m = G,$$

where $m$ is a non-negative integer and $Z_1 = Z(G)$ is the center of the group $G$.

Definition 2. The group $G$ is said to be *divisible by* 2 if $G = \{x^2 |\ x \in G\}$.

Definition 3. We denote by $\mathcal{N}$ the class of nilpotent or generalized nilpotent groups for which the condition $G \in \mathcal{N}$ implies that $G$ and $Z_\alpha$, are 2-divisible and $G/Z_\alpha \in \mathcal{N}$.

Note. All generalized nilpotent groups all of whose elements are of odd order, and connected nilpotent Lie groups belong to the class $\mathcal{N}$. Theorem 7 below is a generalization of Theorem 17 of [7] and Theorem 2 is a generalization of Theorem 3.4 of [9].

## 1. FORMULAS AND RELATIONS

If $h$ or $k$ are zero functions and $G$ is 2-divisible group then the functions $f$ and $g$ are constant functions.

In this case the equation (1.1) has the following solutions:

$$f(x) = A,\ g(x) = -A,$$

$$h(x) = 0 \quad (\text{resp. } k(x) = 0), \quad x \in G$$

and $k$ (resp. $h$) is any $K$-valued function.

Because these two cases occur many times during proofs we skip them.

Definition 4. The system of functions $(f, g, h, k)$ is called the solution of the equation (1.1) if it verifies the equation (1.1) and the mappings $h$ and $k$ are not zero function.

First we will derive results about equation (1.1) that are valid on any group, then we solve equation (1.1) on the class $\mathcal{N}$.

Taking $y = e$ in (1.1) and then $x = e$, we have

$$f(x) + g(x) = Lh(x), \quad L = k(e) \tag{1.2}$$

and

$$f(x) + g(x^{-1}) = Ek(x), \quad E = h(e). \tag{1.3}$$

We split a function $f\colon G \to K$ into its even and odd components

$$f(x) = f_1(x) + f_2(x), \quad x \in G,$$

where

$$f_1(x) = f_1(x^{-1})$$

and

$$f_2(x) = -f_2(x^{-1}).$$

Setting $y^{-1}$ for $y$ in (1.1), adding the equality such obtained with (1.1) and considering (1.2), we have

$$L[h(xy) + h(xy^{-1})] = 2h(x)k_1(y). \tag{1.4}$$

If $L \neq 0$ then (1.4) becomes the Wilson equation

$$h(xy) + h(xy^{-1}) = 2h(x)l(y), \tag{1.5}$$

where $l(y) = k_1(y)/L$. If $l(y) = 1$ for all $y \in G$ we get the Jensen equation

$$h(xy) + h(xy^{-1}) = 2h(x). \tag{1.6}$$

REMARK 1. If $L = 0$ then $k_1(x) = 0$ for all $x \in G$.

Replacing $y$ first by $x$ and then by $x^{-1}$ in (1.1), we find

$$f(x^2) = h(x)k(x) - C, \quad C = g(e) \tag{1.7}$$

and

$$g(x^2) = h(x)k(x^{-1}) - A, \quad A = f(e). \tag{1.8}$$

Let $H$ be a subgroup of $G$ and let $\pi\colon G \to G/H$ be the canonical projection. We say that *a function $f : G \to K$ is a function on $G/H$* if it can be written in the form $f = F \circ \pi$ for some function $F\colon G/H \to K$, i.e., $f(x) = f(xu)$ for all $x \in G$ and $u \in H$, and $F(\overline{x}) = f(x)$, where $\overline{x} = xH$ (this means that the function $f$ takes the same value on the residue class $\overline{x} = xH$).

Assume furthermore that $H$ is a normal subgroup of $G$. It is easy to see that if the system of the functions $(f, g, h, k)$ is a solution of the equation (1.1) and $f, g, h, k$ are functions on $G/H$ then $(f, g, h, k)$ is a solution of (1.1) on $G/H$, too.

We denote by $Z_1 = Z(G)$ the *center* of the group $G$.

LEMMA 1. *Let $G$ be a group with the 2-divisible center $Z_1$ and $(f, g, h, k)$ a solution of the equation* (1.1). *If $h$ is a function on $G/Z_1$, $k_1(y) = L$ and $k_2(y) = 0$, $y \in Z_1$, then $f, g, h, k$ are functions on $G/Z_1$ and $(f, g, h, k)$ is a solution of* (1.1) *on $G/Z_1$.*

*Proof.* From (1.1) and (1.2) we get

$$f(xy) - f(xy^{-1}) = h(x)k(y) - Lh(xy^{-1}). \tag{1.9}$$

Set $y \in Z_1$; because $k_2(y) = 0$ and $k_1(y) = L$ we have

$$f(xy) = f(xy^{-1}) \text{ for all } x \in G \text{ and } y \in Z_1.$$

Taking $xy$ for $x$ in the above equality we obtain

$$f(xy^2) = f(x) \text{ for all } x \in G,\ y \in Z_1.$$

Since $Z_1$ is 2-divisible, $f$ is a function on $G/Z_1$.

Setting in (1.9) the element $yu$ for $y$, $u \in Z_1$, since $f$ and $h$ are functions on $G/Z_1$, we get

$$h(x)k(y) = h(x)k(yu).$$

Therefore

$$k(yu) = k(y) \text{ for all } y \in G,\ u \in Z_1.$$

Consequently $k$ is a function on $G/Z_1$.

Now, it is easy to see from (1.2) that $g$ is a function on $G/Z_1$ and that $(f, g, h, k)$ is a solution of (1.1) on $G/Z_1$. This completes the proof. □

Putting $(y^{-1}, x)$, $(y, x^{-1})$ and $(x^{-1}, y^{-1})$ in (1.1) instead of $(x, y)$, adding the resulting identities with (1.1), we get, using (1.3), that

$$E[k(xy) + k(y^{-1}x) + k(yx^{-1}) + k(x^{-1}y^{-1})] =$$
$$= h(x)k(y) + h(y^{-1})k(x) + h(y)k(x^{-1}) + h(x^{-1})k(y^{-1}).$$

Interchanging $x$ and $y$ in this equality and subtracting the equality such obtained from this, we have

$$E[k(xy) - k(y^{-1}x^{-1}) + k(y^{-1}x) - k(x^{-1}y) + k(yx^{-1}) -$$
$$- k(xy^{-1}) + k(x^{-1}y^{-1}) - k(yx)] =$$
$$= [h(x) - h(x^{-1})][k(y) - k(y^{-1})] - [h(y) - h(y^{-1})][k(x) - k(x^{-1})].$$

Using in this relation the even and odd component of $h$ and $k$, we find

$$E[k_2(xy) + k_2(y^{-1}x) + k_2(yx^{-1}) + k_2(x^{-1}y^{-1})] = \quad (1.10)$$
$$= 2[h_2(x)k_2(y) - h_2(y)k_2(x)].$$

For $y \in Z_1$ we then obtain

$$h_2(x)k_2(y) = h_2(y)k_2(x), \quad x \in G,\ y \in Z_1. \quad (1.11)$$

It will be convenient to record the following fact, because it will be used a couple of times during proofs.

Remark 2. If there exists $y_0 \in Z_1$, such that $h_2(y_0) \neq 0$ or $E = 0$ and there exists $y_0 \in G$ such that $h_2(y_0) \neq 0$, then

$$k_2(x) = Nh_2(x),\ \forall x \in G, \quad (1.12)$$

where $N = k_2(y_0)/h_2(y_0)$.

Remark 3. If $h_2(u) = 0$ for all $u \in Z_1$ and there exists $x \in G$ such that $h_2(x) \neq 0$ it follows that

$$k_2(u) = 0 \text{ for all } u \in Z_1. \quad (1.13)$$

LEMMA 2. *Let $G$ be a 2-divisible group and let $(h, l)$ be a solution of the equation* (1.5). *If*

a) $k_1(u) = L$, $u \in Z_1$, $L \neq 0$ *and there exists* $x \in G$ *such that* $k_1(x) \neq L$ *or*

b) $k_1(x) = L$, $x \in G$, $L \neq 0$ *and* $h_2(u) = 0$ *for all* $u \in Z_1$,

*then $h$ and $l$ are functions on $G/Z_1$ and $(h, l)$ is a solution of* (1.5) *on* $G/Z_1$.

*Proof.* a) This is [5, Lemma 2] or [9, Lemma 3.3].

b) The function $h$ verifies the Jensen's equation (1.6), hence $h(x) = h_2(x) + E$ and

$$h_2(xy) + h_2(xy^{-1}) = 2h_2(x). \tag{1.14}$$

Interchanging $x$ and $y$ in this equality and adding the resulting identity with (1.14), we get

$$h_2(xy) + h_2(yx) = 2h_2(x) + 2h_2(y). \tag{1.15}$$

If $y = u \in Z_1$ we obtain $h_2(xu) = h_2(x)$ for all $x \in G$ and $u \in Z_1$. Hence $h$ is a function on $G/Z_1$. □

LEMMA 3. *Let $G$ and $Z_1$ be 2-divisible and $(f, g, h, k)$ a solution of the equation* (1.1). *If*

a) $k_1(u) = L$, $u \in Z_1$ *and there exists* $x \in G$ *such that* $k_1(x) \neq L$ *and* $L \neq 0$, *or*

b) $k_1(x) = L$, $x \in G$, $L \neq 0$, $h_2(u) = 0$ *for all* $u \in Z_1$ *and there exists* $x \in G$ *such that* $h_2(x) \neq 0$,

*then $f, g, h, k$ are functions on $G/Z_1$ and $(f, g, h, k)$ is a solution of* (1.1) *on* $G/Z_1$.

*Proof.* Since $(h, l)$ is a solution of the equation (1.5) it follows from Lemma 2 that $h$ and $k_1$ are functions on $G/Z_1$.

If there exists $y_0 \in Z_1$ such that $h_2(y_0) \neq 0$ then from (1.12) follows that $k_2$ is a function on $G/Z_1$ consequently and $k$ is a function on $G/Z_1$. From (1.7) we obtain $f(x^2u^2) = f(x^2)$ for all $x \in G$ and $u \in Z_1$, but $G$ and $Z_1$ are divisible by 2, hence $f(xu) = f(x)$ and $f$ is a function on $G/Z_1$. Similarly we deduce from (1.8) that $g$ is a function on $G/Z_1$.

If $h_2(u) = 0$, $u \in Z_1$ and there exists $x \in G$ such that $h(x) \neq 0$ then we get using Remark 3 that $k_2(u) = 0$ for all $u \in Z_1$ and from Lemma 1 follows that Lemma 3 is true. This completes the proof. □

It is left to discuss the case when $h_2(x) = 0$ for all $x \in G$.

LEMMA 4. *Let $G$ be a 2-divisible group. If $k_1(x) = L$, $x \in G$, $L \neq 0$ and $h_2(x) = 0$, $x \in G$, then the solution $(f, g, h, k)$ of* (1.1) *has the form*

$$\begin{cases} f(x) = \dfrac{E}{2}\beta(x) + A, & g(x) = -\dfrac{E}{2}\beta(x) + C \\ h(x) = E, & k(x) = L + \beta(x), \end{cases} \tag{1.16}$$

*where $\beta$ is a homomorphism from $G$ into the additive group of $K$, and $A, C, E, L$ are arbitrary elements of $K$ and $A + C = EL$.*

*Proof.* From (1.5) we have

$$h_1(xy) + h_1(xy^{-1}) = 2h_1(x).$$

Putting $x = e$, we get $h_1(x) = E$. Because $h(x) = E$, (1.2) can be written as

$$f(x) + g(x) = EL.$$

This equality yields

$$f_1(x) + g_1(x) = EL \tag{1.17}$$

and

$$f_2(x) + g_2(x) = 0.$$

From (1.3) we get

$$f(x) + g(x^{-1}) = EL + Ek_2(x),$$

consequently

$$f_2(x) - g_2(x) = Ek_2(x).$$

Hence

$$f_2(x) = \frac{E}{2}k_2(x) = -g_2(x).$$

Because $h_2(x) = 0$, from (1.1) we find

$$f(xy) + g(xy^{-1}) = h_1(x)k(y).$$

Replacing $x$ by $x^{-1}$ in this equality, because the right hand side remains unchanged, we have

$$f(xy) + g(xy^{-1}) = f(x^{-1}y) + g(x^{-1}y^{-1}).$$

Putting $y = x$, yields

$$f(x^2) + C = g(x^{-2}) + A.$$

Hence

$$f(x) - g(x^{-1}) = A - C.$$

It is easy to see that $f_1(x) - g_1(x) = A - C$. Using (1.17), we get $f_1(x) = A$ and $g_1(x) = C$. Taking these relations in (1.1), we obtain

$$A + \frac{E}{2}k_2(xy) + C - \frac{E}{2}k_2(xy^{-1}) = E[L + k_2(y)].$$

Hence

$$k_2(xy) - k_2(xy^{-1}) = 2k_2(y). \tag{1.18}$$

This is a variant of Jensen's equation. Now we use the following result.

LEMMA 5. (see [10], eq. 2.14) *The solutions $f\colon G \to K$ of* (1.18) *are the functions of the form*

$$k_2(x) = \beta(x),$$

*where $\beta$ is a homomorphism from $G$ into the additive group of $K$.*

Consequently the solution $(f, g, h, k)$ of (1.1) has the form (1.16). □

Lemma 6. *Suppose $G \in \mathcal{N}$ and let $(f, g, h, k)$ be a solution of the equation* (1.1). *If:*

a) *for certain $\alpha < \gamma$, $k_1(x) = L$, $x \in Z_\alpha$ and there exists $x \in G$ such that $k_1(x) \neq L$, $L \neq 0$,*

*or*

b) *$k_1(x) = L$, $L \neq 0$, $x \in G$, and for certain $\alpha < \gamma$, $h_2(x) = 0$, $x \in Z_\alpha$ and there exists $x \in G$ such that $h_2(x) \neq 0$,*

*then $f, g, h, k$ are functions on $G/Z_\alpha$ and $(f, g, h, k)$ is a solution of* (1.1) *on $G/Z_\alpha$.*

*Proof.* We will prove Lemma 6 by transfinite induction on $\alpha$.

The case $\alpha = 1$ reduces to Lemma 3. The proof for $\alpha$ is the same as the proof of Lemma 6 from [7] so we will omit it. □

If $(h, l)$ is a solution of the equation (1.5) then $l$ verifies d'Alembert's long functional equation (Lemma 1 from [5])

$$l(xy) + l(yx) + l(xy^{-1}) + l(y^{-1}x) = 4l(x)l(y), \quad x, y \in G. \tag{1.19}$$

Theorem 1. *Let $G \in \mathcal{N}$ and let $K$ be a quadratically closed field of* char $K \neq 2$. *If $l\colon G \to K$ is a non-zero solution of the equation* (1.19) *then it has the form*

$$l(x) = A\frac{Q(x) + Q^*(x)}{2},$$

*where $Q$ is a homomorphism from $G$ into the multiplicative group of $K$.*

The proof is analogous as that of Theorem 3 from [4] if we use Lemma 7 below instead of Lemma 3 from [5].

Lemma 7. *Let $G$ and $Z_1$ be 2-divisible, $K$ a field with* char $K \neq 2$ *and $l$ a solution of the equation* (1.19). *If $l^2(x) = 1$, $\forall x \in Z_1$, then $l(x) = 1$ for all $x \in Z_1$.*

*Proof.* It is easy to see that $l$ verifies the equality $l(x^2) + 1 = 2l^2(x)$, $x \in G$, hence $l(x^2) = 1$, $\forall x \in Z_1$. Since $Z_1$ is 2-divisible, we have $l(x) = 1$, $x \in Z_1$. □

## 2. SOLUTIONS WHEN $L \neq 0$

In this section we first obtain the solution of the Wilson's equation (1.5) and after that of the equation (1.1) in the case where $L \neq 0$.

Theorem 2. *Let $G \in \mathcal{N}$, let $K$ be a quadratically closed field of* char $K \neq 2$ *and $(h, k_1)$, $h, k_1\colon G \to K$ a solution of the equation* (1.5). *If there exists $x \in G$, such that $k_1(x) \neq L$, $L \neq 0$, then $h$ and $k_1$ have the form*

$$h(x) = A\frac{Q(x) + Q^*(x)}{2} + B\frac{Q(x) - Q^*(x)}{2} \tag{2.1}$$

*and*

$$k_1(x) = L\frac{Q(x) + Q^*(x)}{2}, \tag{2.2}$$

*where $Q$ is a homomorphism of $G$ into the multiplicative group of $K$, $A$, $B$ and $L$ are arbitrary elements of $K$ and $Q^*(x) = [Q(x)]^{-1}$.*

The proof of this theorem is analogous as that of Theorem 3 from [5] case i) if we use Theorem 1 and Lemma 7 instead of Lemma 3 from [5], so we will omit it.

The *commutator subgroup* of $G$, i.e. the subgroup generated by the commutators $[x, y] = xyx^{-1}y^{-1}$, $x, y \in G$, will be denoted by $[G, G]$. The group $G$ is said to be *step* 2 *nilpotent* if $[G, G] \subseteq Z(G)$.

The investigations of Jensen's functional equation revealed that other solutions than classical ones sometimes occur. Stetkaer [11] showed that any solution of Jensen's functional equation on any group $G$ is a function on the quotient group $G/[G, [G, G]]$. This quotient group is always step 2-nilpotent, so the study of Jensen's functional equation reduces to a study of it on step 2-nilpotent groups.

We will determine solutions of the equation (1.1) using homomorphisms and odd solution of Jensen's equation.

Lemma 8. (see [6], Proposition 1.5). *Let $G$ be an arbitrary group and $H$ an abelian group divisible by 2. The functional equation*

$$\varphi(xy) + \varphi(yx) = 2\varphi(x) + 2\varphi(y), \quad \varphi\colon G \to H, \tag{2.3}$$

*is equivalent with Jensen's equation*

$$\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x). \tag{2.4}$$

*Proof.* Due to the fact that $\varphi$ is odd, interchanging $x$ and $y$ in (2.4) and adding the resulting identity with (2.4) we get (2.3).

Conversely, it is easy to see that from (2.3) we have

$$\varphi(e) = 0, \quad \varphi(x) = -\varphi(x^{-1}) \quad \text{and} \quad \varphi(x^2) = 2\varphi(x)$$

for all $x \in G$. Using (2.3) in the expression $4[\varphi(x) + \varphi(y) + \varphi(u) + \varphi(v)]$, we have

$$\varphi(xyuv) + \varphi(uvxy) + \varphi(yxvu) + \varphi(vuyx) =$$
$$= \varphi(xuvy) + \varphi(vyxu) + \varphi(uxyv) + \varphi(yvux).$$

Replacing $x$ by $y$ and $v$ by $e$ in this identity we find

$$\varphi(x^2u) + \varphi(ux^2) = 2\varphi(xux),$$

which, together with (2.3), yield

$$\varphi(xux) = 2\varphi(x) + \varphi(u).$$

Setting $ux^{-1}$ for $u$ in this identity, we obtain

$$\varphi(xu) = 2\varphi(x) + \varphi(ux^{-1}) = 2\varphi(x) - \varphi(xu^{-1}).$$

Consequently $\varphi$ is a solution of (2.4). □

THEOREM 3. *Let $G \in \mathcal{N}$ and let $K$ be a quadratically closed field of* char $K \neq 2$. *If $f, g, h, k\colon G \to K$ and there exists $x \in G$ such that $k_1(x) \neq L$ and $L \neq 0$, then the solution $(f, g, h, k)$ of equation* (1.1) *is of the following form:*

$$\text{(2.5)} \qquad \begin{cases} f(x) = A\dfrac{Q(x)+Q^*(x)}{2} + B\dfrac{Q(x)-Q^*(x)}{2} + \gamma \\[2ex] g(x) = C\dfrac{Q(x)+Q^*(x)}{2} + D\dfrac{Q(x)-Q^*(x)}{2} - \gamma \\[2ex] h(x) = E\dfrac{Q(x)+Q^*(x)}{2} + F\dfrac{Q(x)-Q^*(x)}{2} \\[2ex] k(x) = L\dfrac{Q(x)+Q^*(x)}{2} + M\dfrac{Q(x)-Q^*(x)}{2}, \end{cases}$$

*where $Q$ is a homomorphism from $G$ in the multiplicative group of $K$ and $A, B, C, D, E, F, L, M, \gamma$ are arbitrary elements in $K$, which verify the following relations*

$$2A = EL + FM,\ 2B = LF + EM,\ 2C = EL - FM,\ 2D = FL - EM.$$

*Proof.* This is [7, Theorem 11]. □

THEOREM 4. *Suppose $G$ belongs to $\mathcal{N}$ and $K$ is a field of* char $K \neq 2$. *If $k_1(x) = L$, $L \neq 0$, for all $x$ in $G$, then the solution $(f, g, h, k)$ of equation* (1.1) *has the form* (1.16) *or the following form*

$$\text{(2.6)} \qquad \begin{cases} f(x) = \dfrac{N}{4}\beta^2(x) + \dfrac{L+EN}{2}\beta(x) + A \\[2ex] g(x) = -\dfrac{N}{4}\beta^2(x) + \dfrac{L-EN}{2}\beta(x) + C \\[2ex] h(x) = \beta(x) + E, \quad k(x) = N\beta(x) + L, \end{cases}$$

*where $\beta$ is a homomorphism from $G$ into the additive group of $K$ and $A, C, E, L, N$ are elements of $K$ which satisfy the relation*

$$A + C = EL,$$

*or*

$$\text{(2.7)} \qquad \begin{cases} f(x) = \dfrac{L}{2}\varphi(x) + A, \quad g(x) = \dfrac{L}{2}\varphi(x) + C \\[2ex] h(x) = \varphi(x) + E, \qquad k(x) = L, \end{cases}$$

*where $\varphi$ is an odd solution of Jensen's equation* (1.6) *and $A, C, E, L$ are arbitrary constants which verify the relation $A + C = EL$.*

*Proof.* The function $h$ verifies Jensen's equation (1.6), hence it has the form

$$h(x) = \varphi(x) + E, \tag{2.8}$$

where $\varphi$ is an odd solution of Jensen's equation.

We distinguish two cases:

a) There exist $x \in G$ such that $h_2(x) = \varphi(x) \neq 0$

and

b) $h_2(x) = 0$, $x \in G$.

a) If there exists $y_0 \in Z_1$ such that $h_2(y_0) \neq 0$, then we have (1.12). Hence $k$ has the form

$$k(x) = N\varphi(x) + L. \tag{2.9}$$

If $h_2(x) = 0$ for certain $\alpha < \gamma$ and there exists $y_0 \in Z_{\alpha+1}$ such that $h_2(y_0) \neq 0$, then from Lemma 6 case b) there exist the functions $F_\alpha, G_\alpha, H_\alpha, K_\alpha \colon G/Z_\alpha \to K$ which verify the equation (1.1) on $G/Z_\alpha$.

Hence between $H_{\alpha_2}$ and $K_{\alpha_2}$ the relation (1.11) holds and there exists $y_0^{(\alpha)} = y_0 Z_\alpha \in Z(G/Z_\alpha)$ such that $H_{\alpha_2}(y_0^{(\alpha)}) = h_2(y_0) \neq 0$. Using Remark 2, we have (1.12) for $H_{\alpha_2}$ and $K_{\alpha_2}$, therefore we have (1.12) and for $h_2$ and $k_2$ too, hence

$$k_2(x) = N\varphi(x)$$

and $k$ has the form (2.9).

Replacing the expression of $h$ and $k$ in (1.7) and (1.8), respectively, using the equality $\varphi(x^2) = 2\varphi(x)$ and due to the fact that $G$ is 2-divisible, we obtain that $(f, g, h, k)$ have the form

$$\begin{cases} f(x) = \dfrac{N}{4}\varphi^2(x) + \dfrac{L + EN}{2}\varphi(x) + A, \\[2ex] g(x) = -\dfrac{N}{4}\varphi^2(x) + \dfrac{L - EN}{2}\varphi(x) + C, \\[2ex] h(x) = \varphi(x) + E, \quad k(x) = N\varphi(x) + L, \end{cases} \tag{2.10}$$

where $\varphi$ is an odd solution of Jensen's equation and $A, C, E, L, N$ are elements of $K$.

Conversely, assume that the system of the functions $f, g, h, k$ of the form (2.10) is a solution of the equation (1.1). Putting these expressions in (1.1), a simple computation gives us

$$N[\varphi(x) + E][\varphi(xy) - \varphi(xy^{-1}) - 2\varphi(y)] = 0.$$

Because $h$ is non-zero function from this equality we have

$$\varphi(xy) - \varphi(xy^{-1}) = 2\varphi(y).$$

This is the equation (1.18). By Lemma 5 we have that its solution is $\varphi(x) = \beta(x)$, where $\beta$ is a homomorphism from $G$ into the additive group of $K$. In this case we obtain the solution of the form (2.6).

If $N = 0$, we have the solution $(f, g, h, k)$ of the form (2.7).

b) The hypotheses of Lemma 4 are satisfied, hence $(f, g, h, k)$ has the form (1.16). □

## 3. SOLUTIONS WHEN $L = 0$

First we deduce a Wilson's equation that may be used to obtain the solution of the equation (1.1) in the case that $k(e) = L = 0$ and $h(e) = E \neq 0$.

LEMMA 9. *Let $G$ be an arbitrary group, $K$ a field of* char $K \neq 2$ *and let $(f, g, h, k)$ be a solution of the equation* (1.1). *If $L = 0$, $E \neq 0$, and there exists $u \in Z_1$ such that $k_2(u) \neq 0$, then we have*

$$h(xy) + h(xy^{-1}) = 2h(x)m(y), \text{ for all } x, y \in G, \tag{3.1}$$

*where $m(y) = h_1(y)/E$.*

*Proof.* This is [7, Lemma 14]. □

It is left to consider the case $L = 0$ and $E = 0$.

LEMMA 10. *Let $G$ be a group with $Z_1 \neq \{e\}$. If $(f, g, h, k)$ is a solution of the equation* (1.1), *there exists $y_0 \in Z_1$ such that $k_2(y_0) \neq 0$, $L = 0$ and $E = 0$, then the function $k_2$ verifies the sine equation*

$$k_2(xy)k_2(xy^{-1}) = k_2^2(x) - k_2^2(y). \tag{4.1}$$

*Proof.* This is [7, Lemma 16]. □

THEOREM 5. *Let $G$ be a 2-divisible group such that the commutator subgroup $[G, G]$ is divisible by 2. Let $K$ be a quadratically closed field of* char $K \neq 2$. *The general solutions $k_2 \colon G \to K$ of the sine functional equation* (4.1) *are given by*

$$k_2(x) = M\frac{Q(x) - Q^*(x)}{2} \tag{4.2}$$

*or*

$$k_2(x) = \beta(x), \tag{4.3}$$

*where $Q$ is a homomorphism from $G$ in the multiplicative group of $K$ and $\beta$ is a homomorphism from $G$ in the additive group of $K$.*

*Proof.* See [8, Theorem 3]. □

LEMMA 11. *Let $G \in \mathcal{N}$ and let $(f, g, h, k)$ be a solution of the equation* (1.1). *If $L = 0$ and there exists $\alpha < \gamma$ such that $k_2(x) = 0$ for all $x \in Z_\alpha$, then $f, g, h, k$ are functions on $G/Z_\alpha$ and $(f, g, h, k)$ is a solution of* (1.1) *on $G/Z_\alpha$.*

*Proof.* If $L=0$, from (1.2) we have $g(x)=-f(x)$ and, using Remark 1, $k_1(x)\equiv 0$, the equation (1.1) can be rewritten as

$$f(xy)-f(xy^{-1})=h(x)k_2(y). \tag{4.4}$$

We will prove Lemma 11 by transfinite induction on $\alpha$. Suppose that $\alpha=1$.

If $k_2(y)=0$, $y\in Z_1$, we get $f(xy)=f(xy^{-1})$. Setting $xy$ for $x$ we have $f(xy^2)=f(x)$, hence $f(xu)=f(x)$ for all $x\in G$, $u\in Z_1$. Setting $xu$ for $x$ in (4.4) we obtain

$$h(xu)k_2(y)=h(x)k_2(y);$$

because $k_2\neq 0$, it follows that $h(xu)=h(x)$. Replacing $y$ by $yu$ in (4.4), we have $k_2(yu)=k_2(y)$ for all $y\in G$, $u\in Z_1$.

Hence $f,g,h,k$ are functions on $G/Z_1$ and they verify the equation (1.1) on $G/Z_1$. Thus Lemma 11 is true for $\alpha=1$. We shall prove it for $\alpha$ and admit therefore that $k_2(x)=0$ for all $x\in Z_\alpha$.

Case i). If $\alpha-1$ exists, we have the functions $F_{\alpha-1},H_{\alpha-1},K_{\alpha-1,2}\colon G/Z_{\alpha-1}\to K$ which verify equation (4.4) on $G/Z_{\alpha-1}$.

Since $Z_\alpha/Z_{\alpha-1}=Z(G/Z_{\alpha-1})$, for all $xZ_{\alpha-1}\in Z_\alpha/Z_{\alpha-1}$, $x\in Z_\alpha$, we have $K_{\alpha-1,2}(xZ_{\alpha-1})=k_2(x)=0$. We can apply the case $\alpha=1$ to get new functions $F^*,H^*,K_2^*\colon G/Z_{\alpha-1}\Big/Z_\alpha/Z_{\alpha-1}$ with

$$F^*(x^{(\alpha-1)}Z_\alpha/Z_{\alpha-1})=F_{\alpha-1}(x^{(\alpha-1)})=f(x) \text{ for all } x^{(\alpha-1)}\in G/Z_{\alpha-1}$$

and analogously for $H^*$ and $K_2^*$.

By the isomorphism theorem there exists an isomorphism

$$\psi\colon G/Z_\alpha\to (G/Z_{\alpha-1})\Big/(Z_\alpha/Z_{\alpha-1}).$$

For the function $F_\alpha=F^*\circ\psi$ we get

$$F_\alpha(x^{(\alpha)})=F_\alpha(xZ_\alpha)=F^*(x^{(\alpha-1)}Z_\alpha(Z_{\alpha-1})=f(x).$$

In a similar way we obtain $H_\alpha(x^{(\alpha)})=h(x)$ and $K_{2,\alpha}(x^{(\alpha)})=k_2(x)$, $x\in G$.

Case ii). If $\alpha$ is a limit-ordinal consider first two elements $x,x'$ in the same residue class of $Z_\alpha$. We have $x'=xz$ with $z\in Z_\alpha$; one can find, by definition of $Z_\alpha$, $\beta<\alpha$ such that $z\in Z_\beta$. Applying the induction hypothesis to $Z_\beta$ we obtain

$$f(xz)=F_\beta(xzZ_\beta)=F_\beta(xZ_\beta)=f(x),$$

that is $f(x')=f(x)$.

In a similar way we deduce that the function $g,h$ and $k$ are functions on $G/Z_\alpha$. □

Theorem 6. *Suppose that $G\in\mathcal{N}$ and $[G,G]$ is divisible by 2 and $Z(G)\neq\{e\}$. If $E=L=0$ then the solution $(f,g,h,k)$ of (1.1) has the form (2.5) or the following form*

$$\begin{cases} f(x)=C\beta(x)+A, & g(x)=-C\beta^2(x)-A\\ h(x)=\frac{C}{4}\beta(x), & k(x)=\beta(x), \end{cases} \tag{4.5}$$

*where $\beta$ is a homomorphism from $G$ into the additive group of $K$.*

*Proof.* From Remark 1 we have $k_1(x) = 0$ for all $x \in G$, hence the function $k_2$ is different from the identically zero function. If there exists $x \in Z_1$ such that $k_2(x) \neq 0$, then, using Lemma 10, the function $k_2$ verifies the sine equation, hence it follows from Theorem 5 that $k_2$ has the form (4.2) or (4.3). If for $\alpha < \gamma$, $k_2(x) = 0$ for all $x \in Z_\alpha$, and there exists $y_0 \in Z_{\alpha+1}$ such that $k_2(y_0) \neq 0$, then, using Lemma 11, we infer that $(f, g, h, k)$ is a solution of the equation (1.1) on $G/Z_\alpha$.

We have $K_{\alpha 2}(y_0^{(\alpha)}) = k_2(y_0) \neq 0$ for $y_0^{(\alpha)} = y_0 Z_\alpha$. By Lemma 10 the function $K_{\alpha_2}$ verifies the sine equation and from Theorem 5 we find that it has one of the forms (4.2) or (4.3), consequently $k_2$ is of the same forms.

From (1.10) for $E = 0$ we find

$$h_2(x)k_2(y) = h_2(y)k_2(x) \text{ for all } x, y \in G.$$

Taking $y = y_0$, we get

$$h_2(x) = Mk_2(x), \tag{4.6}$$

where $M = h_2(y_0)/k_2(y_0)$.

The functions $F_\alpha, H_\alpha, K_{\alpha 2}$ verify the equation (4.4) on $G/Z_\alpha$, that is

$$F_\alpha(x^{(\alpha)}y^{(\alpha)}) - F_\alpha(x^{(\alpha)}y^{-(\alpha)}) = H_\alpha(x^{(\alpha)})K_{\alpha 2}(y^{(\alpha)}).$$

Interchanging $x^{(\alpha)}$ and $y^{(\alpha)}$ and replacing $x^{(\alpha)}$ by $x^{-(\alpha)}$, due to the fact that $F_\alpha$ is even, we conclude

$$F_\alpha(y^{-(\alpha)}x^{(\alpha)}) - F_\alpha(y^{(\alpha)}x^{(\alpha)}) = H_\alpha(x^{-(\alpha)})K_{\alpha 2}(y^{(\alpha)}).$$

Adding these two equations and supposing that $y^{(\alpha)} \in Z_{\alpha+1}/Z_\alpha$, we have

$$H_{\alpha 1}(x^{(\alpha)})K_{\alpha 2}(y^{(\alpha)}) = 0,$$

hence

$$H_{\alpha 1}(x^{(\alpha)}) = h_1(x) = 0, \qquad x \in G.$$

Setting $y = x$ in (4.4), we get

$$f(x^2) = h_2(x)k_2(x) + A. \tag{4.7}$$

If $k_2$ has the form (4.2), then from (4.6) we obtain the function $h_2$ and from (4.7) we find the function $f$. Therefore the solution $(f, g, h, k)$ has the form (2.5).

If $k_2$ has the form (4.3) in a similar way we obtain that the solution $(f, g, h, k)$ of (1.1) has the form (4.5). □

## 4. MAIN RESULT

By help of the previous Lemmata and Theorems we now describe the complete solution of (1.1) under the assumption that $G \in \mathcal{N}$.

THEOREM 7. *Let $G \in \mathcal{N}$ and let $K$ be a quadratically closed field of* char $K \neq 2$. *Furthermore suppose that $[G,G]$ is divisible by* 2. *If $(f,g,h,k)$ is a solution of the equation* (1.1), *then it has one of the following forms* (1.16), (2.5), (2.6), (2.7) *or* (4.5).

*Proof.* Distinguish three cases: i) $L \neq 0$, ii) $L = 0$, $E \neq 0$ and iii) $L = 0$, $E = 0$.

Case i). From Theorem 3 and Theorem 4 infer that the solution has the forms (2.5), (1.16), (2.6) or (2.7).

Case ii). If $L = 0$, then Remark 1 yields that $k_1(x) = 0$ for all $x \in G$.

First we find the functions $h$ and $k = k_2$. For this we consider two cases.

a) There exists $x \in G$ such that $h_2(x) \neq 0$. If there exists $u \in Z_1$ such that $h_2(u) \neq 0$, then, by Remark 3, it follows that $k_2(u) \neq 0$. According to Lemma 9 the functions $h$ and $h_1$ verify Wilson's equation (3.1). Now we can apply Theorem 2. If $h_1(x) \neq E$, $E \neq 0$, then $h$ has the form (2.1).

If $h_1(x) = E$, $E \neq 0$, then the equation (3.1) becomes Jensen's equation, hence $h$ has the form

$$h(x) = \varphi(x) + E, \tag{5.1}$$

where $\varphi$ is an odd solution of Jensen's equation and $E$ is an arbitrary element of $K$. Now, using Remark 2 we find that

$$k(x) = k_2(x) = M\frac{Q(x) - Q^*(x)}{2} \tag{5.2}$$

or

$$k(x) = k_2(x) = N\varphi(x). \tag{5.3}$$

Replacing the form (2.1) of $h$ and (5.2) of $k$ in (1.7) and (1.8), respectively, we obtain the functions $f$ and $g$ claimed in (2.5). In a similar way we get from (5.1) and (5.3) that the system $(f,g,h,k)$ has the form (2.10). From Theorem 4 we find that the solution $(f,g,h,k)$ has the form (2.6) or (2.7).

We prove by induction on $\alpha$ that the solution $(f,g,h,k)$ has the forms (2.5), (2.6) or (2.7). First we consider $\alpha = 1$.

If $h_2(u) = 0$, $u \in Z_1$, then from Remark 3 we deduce that $k_2(u) = 0$ for all $u \in Z_1$. By Lemma 11 there exist functions $F_1, H_1, K_{12} \colon G/Z_1 \to K$ which verify the equation (4.4). If there exists $u^{(1)} \in Z_2/Z_1$ such that $H_{12}(u^{(1)}) \neq 0$ and because between $H_{12}$ and $K_{12}$ there exists relation (1.11), Remark 3 implies that $K_{12}(u^{(1)}) \neq 0$. As above we obtain in this case that $H_1$ and $K_{12}$ have the forms (2.1) and (5.2), or (5.1) and (5.3), respectively. Consequently $h$ and $k$ have those forms. The solution $(f,g,h,k)$ of (1.1) has form (2.5), (2.6) or (2.7) in this case. Therefore for $\alpha = 1$ the statement is true.

According to the inductive hypothesis the solution $(f, g, h, k)$ has the forms (2.5), (2.6) or (2.7) or there exists $\alpha < \gamma$ such that $k_2(x) = 0$ for all $x \in Z_\alpha$ and there exists $u \in Z_{\alpha+1}$ such that $h_2(u) \neq 0$ and $k_2(u) \neq 0$.

Lemma 11 tells us that there exist the functions $F_\alpha, H_\alpha, K_{\alpha 2}\colon G/Z_\alpha \to K$ which verify the equation (4.4).

Because $K_{\alpha 2}(u^{(\alpha)}) = k_2(u) \neq 0$, where $u^{(\alpha)} \in Z_{\alpha+1}/Z_\alpha = Z(G/Z_\alpha)$, the hypotheses of Lemma 9 are satisfied, consequently $(H_\alpha, H_{\alpha 1})$ is a solution of Wilson's equation (3.1). By Theorem 2, if $H_{\alpha 1}(x^{(\alpha)}) \neq E$, $x^{(\alpha)} \in G/Z_\alpha$, then $H_\alpha$ has the form (2.1), if $H_{\alpha 1}(x^{(\alpha)}) = E$ for all $x^{(\alpha)} \in G/Z_\alpha$, then $H_\alpha$ has the form (5.1). The functions $H_{\alpha 2}$ and $K_{\alpha 2}$ verify the relation (1.11) and there exists $u \in Z(G/Z_\alpha)$ such that $K_{\alpha 2}(u^{(\alpha)}) \neq 0$, hence $K_{\alpha 2}$ is given by (1.12). As above we conclude that the solution $(f, g, h, k)$ has the forms (2.5), (2.6) or (2.7).

b) If $h_2(x) = 0$ for all $x \in G$, $h(x) = h_1(x)$, then the equation (4.4) becomes

$$f(xy) - f(xy^{-1}) = h_1(x)k_2(y). \tag{5.4}$$

If there exists $u \in Z_1$ such that $k_2(u) \neq 0$, then the function $h_1$ verifies the Wilson equation

$$h_1(xy) + h_1(xy^{-1}) = h_1(x)\frac{h_1(y)}{E}. \tag{5.5}$$

Using Theorem 2, if there exist $y \in G$ such that $h_1(y) \neq E$, we get

$$h_1(x) = E\frac{Q(x) + Q^*(x)}{2} = h(x), \tag{5.6}$$

where $Q$ is a homomorphism from $G$ into the multiplicative group of $K$.

First we will obtain the function $k_2$.

Taking $x = e$ in (5.4), we get

$$f_2(y) = \frac{E}{2}k_2(y).$$

Since $h_1$ is an even function, putting $x^{-1}$ for $x$, the right hand side of (5.4) remains unchanged and

$$f(xy) - f(xy^{-1}) = f(x^{-1}y) - f(x^{-1}y^{-1}).$$

Setting $y = x$ in this equality, we obtain

$$f(x^2) - A = A - f(x^{-2}).$$

Because $G$ is 2-divisible one gets $f(x) + f(x^{-1}) = 2A$, therefore $f_1(x) = A$, and $f(x) = \frac{E}{2}k_2(x) + A$. From this equality and (5.4) we deduce

$$E[k_2(xy) - k_2(xy^{-1})] = 2h_1(x)k_2(y). \tag{5.7}$$

Permuting $x$ and $y$ in this equality and adding the equality such obtained with (5.7), we have

$$\frac{E}{2}[k_2(xy) + k_2(yx)] = h_1(x)k_2(y) + h_1(y)k_2(x). \tag{5.8}$$

In view of (5.6) we obtain

$$k_2(xy) + k_2(yx) = [Q(x) + Q^*(x)]k_2(y) + [Q(y) + Q^*(y)]k_2(x). \tag{5.9}$$

Because $h_1(x) \neq E$, there exists $\alpha < \gamma$ such that $h_1(x) = E$ for all $x \in Z_\alpha$ and there exists $u \in Z_{\alpha+1}$ such that $h_1(x) \neq E$. Using Lemma 2 and Lemma 6, cases a) for Wilson's equation (5.5) instead of the Wilson equation (1.5), we get that there exist the functions $H_\alpha$ and $H_{\alpha 1}$ which verify the Wilson's equation (5.5) on $G/Z_\alpha$. Hence $K_{\alpha 2}$ satisfies the equation (5.9), which is equation (1.12) from [5]. To conclude that $K_{\alpha 2}$ has the form (4.2) is necessary to show that there exists $u^{(\alpha)} \in Z(G/Z_\alpha)$, such that $Q^2(u^{(\alpha)}) \neq 1$. Indeed, if $u^{(\alpha)} \in Z_{\alpha+1}/Z_\alpha$, $Q^2(u^{(\alpha)}) = Q((u^{(\alpha)})^2) = 1$, then $Q(u^{(\alpha)}) = 1$ implies $Q(u) = 1$, i.e., $h_1(u) = E$, $u \in Z_{\alpha+1}$ contradicting our hypothesis. Therefore $k_2$ has the form (4.2).

It is easy to see that in this case the solution $(f, g, h, k)$ has the form (2.5).

If $h_1(x) = E$ for all $x \in G$, the equation (5.7) can be written as

$$k_2(xy) - k_2(xy^{-1}) = 2k_2(y). \tag{5.10}$$

This is equation (2.9). If $G$ is 2-divisible, then the solution can be obtained very easy.

Set $y = x$ in (5.10), thus

$$k_2(x^2) = 2k_2(x).$$

Taking $xy$ for $x$ in (5.10), yields

$$k_2(xy^2) = 2k_2(y) + k_2(x) = k_2(x) + k_2(y^2).$$

Consequently, $G$ being divisible by 2, $k_2(x) = \beta(x)$, where $\beta$ is a homomorphism from $G$ into additive group of $K$.

We have the solution $(f, g, h, k)$ of (1.1) in this case of the form

$$\begin{cases} f(x) = \dfrac{E}{2}\beta(x) + A, & g(x) = -\dfrac{E}{2}\beta(x) - A \\ h(x) = E, & k(x) = \beta(x), \end{cases}$$

where $\beta$ is a homomorphism from $G$ into the additive group of $K$ and $A, E$ are arbitrary elements of $K$.

iii) We distinguish two cases:

a) If $E = 0$, $L = 0$ and there exists $y_0 \in Z_1$ such that $k_2(y_0) \neq 0$, then according to Lemma 10 the function $k_2$ verifies the sine equation (4.1). Using

Theorem 5, we obtain that $k_2$ has the forms (4.2) or (4.3). Taking $L = 0$ in (1.2) and $E = 0$ in (1.3), it follows

$$g(x) = -f(x) \text{ and } f(x) = f(x^{-1}). \tag{5.11}$$

Permuting in (4.4) $x$ and $y$ and subtracting the equality such obtained from (4.4), due to the fact that $f$ is even, we have

$$f(xy) - f(yx) = h(x)k_2(y) - h(y)k_2(x).$$

Setting $y = y_0 \in Z_1$ in this equality, we get

$$h(x) = ck_2(x), \tag{5.12}$$

where $c = h(y_0)/k_2(y_0)$. From this equality and (4.4) we deduce

$$f(xy) - f(xy^{-1}) = ck_2(x)k_2(y).$$

Taking in this equality $y = x$, we have

$$f(x^2) = A + ck_2^2(x). \tag{5.13}$$

If we consider for $k_2$ the form (4.2), from (5.11), (5.12) and (5.13) we obtain the solution $(f, g, h, k)$ of (1.1) of the form (2.5). If we take $k_2$ of the form (4.3), we obtain that the solution $(f, g, h, k)$ has the form (2.6).

b) If $L = 0$, $E = 0$ and $k_2(x) = 0$ for all $x \in Z_\alpha$ and there exists $u \in Z_{\alpha+1}$ such that $k_2(u) \neq 0$, then it follows from Lemma 11 that $f, g, h, k$ are functions on $G/Z_\alpha$ and $(f, g, h, k)$ is a solution of (1.1) on $G/Z_\alpha$. But for this solution the hypotheses of the case a) are verified, hence the solution $(f, g, h, k)$ has the forms (2.5) or (2.6) on $G/Z_\alpha$, in turn $(f, g, h, k)$ has the same forms on $G$. This finishes the proof. □

*Acknowledgements.* The authors wish to thank the referees for their useful remarks that improved the presentation of this paper.

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Received April 26, 2007

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