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MEROMORPHIC FUNCTIONS WITH MISSING AND ALTERNATING COEFFICIENTS

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Abstract. In this paper, we introduce a new class of meromorphic functions with missing and alternating coefficients. The usual properties such as coefficient inequalities, distortion theorems, the radii of starlikeness and convexity, closure theorem of the functions belonging to this class are obtained.

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1. INTRODUCTION

Let $\sum$ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \tag{1}$$

which are analytic in the punctured unit disc $D=\{z:0<z<1\}$ with a simple pole at $z=0$ and residue 1 there. Also let $\sum_s$ denote the class of functions in $\sum$ which are univalent in D. A function $f\in\sum_s$ is said to be meromorphically starlike of order α if it satisfies the following:

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in D;\ 0\leq\alpha<1). \tag{2}$$

Similarly, a function $f\in\sum_s$ is said to be meromorphically convex of order α if it satisfies the following:

$$\operatorname{Re}\left(-1-\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in D;\ 0\leq\alpha<1). \tag{3}$$

Further, let $\sum(p)$ be the class of functions f defined by (1) with

$$a_j=0 \ \ (j=1,...,p-1;\ p\in N=\{1,2,...\}),$$

i.e. by

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n} \tag{4}$$

which are analytic in D.

Definition 1. A function $f\in\sum(p)$ is said to be in the class $\sum_s(p)$ if it is also univalent in D.

Definition 2. A function f, analytic and univalent in D, is said to be in the class $\sum_s^+(p)$ if it is given by (1) with

$$a_j = 0 \ \ (j = 1, ..., p-1; p \in N) \text{ and } a_{p+j} \geq 0 \ \ (j \in N_0 = N \cup \{0\}),$$

that is, by

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, \ \ p \in N). \tag{5}$$

We have the relationship

$$\textstyle\sum_s^+(p) \subseteq \sum_s(p) \subseteq \sum_s \subseteq \sum \text{ and } \sum(p) \subseteq \sum.$$

Let Ω be the subclass of $\sum_s^+(p)$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{p+n-1} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, \ \ p \in N). \tag{6}$$

Definition 3. A function $f \in \sum_s^+(p)$ is said to be in the class $\sum_s^+(p, \alpha, \beta, k)$ if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} + k}{\frac{zf'(z)}{f(z)} + (2\alpha - k)} \right| < \beta. \tag{7}$$

for α $(0 \leq \alpha < 1)$ and $\alpha \leq k \leq 1$.

Now let $\Omega_s^+(p, \alpha, \beta, k) = \Omega \cap \sum_s^+(p, \alpha, \beta, k)$. The class of meromorphically starlike functions of order α $(0 \leq \alpha < 1)$ and various other subclasses of $\sum_s$ have been studied rather extensively by Nehari and Netanyahu [19], Clunie [6], Pommerenke ([4], [5]), Miller [8], Royster [18], and others(cf., e.g., Bajpai [16], Goel and Sohi[14], Mogra et. al. [10], Uralegaddi and Ganigi[1], Cho et. al. [12], Aouf[11], and Uralegaddi and Somanatha ([2],[3]); see also Duren([13], pages 29 and 137), and Srivastava and Owa([7], pages 86 and 429). Note that $\sum^*(\alpha, k)$ is the class of meromorphic starlike functions obtained by Owa and Pascu [17]. We also note that $\sum^*(\alpha, \beta, k)$ is the class of meromorphic starlike functions studied by Darus [9]. The class $\sum^*(\alpha, \beta, 1)$ and the class $\Omega_s^+(p, \alpha, \beta, 1)$ were studied by Joshi [15].

2. COEFFICIENT INEQUALITIES

Theorem 1. *Let the function f be defined by* (4). *If*

$$\sum_{n=0}^{\infty} \Phi_n(p, \alpha, \beta, k) \left| a_{p+n} \right| \leq \beta(k + 1 - 2\alpha) + k - 1, \tag{8}$$

where $\Phi_n(p, \alpha, \beta, k) = (p + n + k) + \beta \left[\left| p + n + 2\alpha - k \right| \right]$ *and for some* k $(\alpha \leq k \leq 1)$, α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ *and* $p \in N$, *then* $f \in \sum_s^+(p, \alpha, \beta, k)$.

Proof. Suppose that (8) holds and let $|z| = 1$. Then we have

$$\left|\frac{\frac{zf'(z)}{f(z)}+k}{\frac{zf'(z)}{f(z)}+(2\alpha-k)}\right| = \left|\frac{(k-1)+\sum_{n=0}^{\infty}(p+n+k)a_{p+n}z^{p+n+1}}{(2\alpha-1-k)+\sum_{n=0}^{\infty}(p+n+2\alpha-k)a_{p+n}z^{p+n+1}}\right|$$

$$\leq \frac{(1-k)+\sum_{n=0}^{\infty}(p+n+k)|a_{p+n}||z|^{p+n+1}}{(k+1-2\alpha)-\sum_{n=0}^{\infty}|p+n+2\alpha-k||a_{p+n}||z|^{p+n+1}}$$

$$\leq \frac{(1-k)+\sum_{n=0}^{\infty}(p+n+k)|a_{p+n}|}{(k+1-2\alpha)-\sum_{n=0}^{\infty}|p+n+2\alpha-k||a_{p+n}|}.$$

The last expression is bounded above by β if $(1-k)+\sum_{n=0}^{\infty}(p+n+k)\,|a_{p+n}| \leq \beta\left\{(k+1-2\alpha)-\sum_{n=0}^{\infty}|p+n+2\alpha-k|\,|a_{p+n}|\right\}$ which is equivalent to our condition (8) of the theorem.

Theorem 2. *If the functions f defined by* (6) *belongs to Ω, then $f \in \Omega_s^+(p,\alpha,\beta,k)$ if and only if* (8) *is satisfied.*

Proof. In view of Theorem 1, it is sufficient to show the "only if" part,

$$\left|\frac{\frac{zf'(z)}{f(z)}+k}{\frac{zf'(z)}{f(z)}+(2\alpha-k)}\right|$$

$$= \left|\frac{(k-1)+\sum_{n=0}^{\infty}(-1)^{n+p-1}(p+n+k)a_{p+n}z^{p+n+1}}{(k+1-2\alpha)-\sum_{n=0}^{\infty}(-1)^{n+p-1}(p+n+2\alpha-k)a_{p+n}z^{p+n+1}}\right| \leq \beta.$$

Since $Re(z) \leq |z|$ for all z, we have

$$\text{Re}\left\{\frac{(k-1)+\sum_{n=0}^{\infty}(-1)^{n+p-1}(p+n+k)a_{p+n}z^{p+n+1}}{(k+1-2\alpha)-\sum_{n=0}^{\infty}(-1)^{n+p-1}(p+n+2\alpha-k)a_{p+n}z^{p+n+1}}\right\} \leq \beta. \tag{9}$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing denominator in (9) and letting $z \to 1-$, through real values we get

$$\sum_{n=0}^{\infty}\Phi_n(p,\alpha,\beta,k)a_{p+n} \leq \beta(k+1-2\alpha)+k-1.$$

Corollary 1. *Let the function f be defined by* (6) *and let $f \in \Omega$. If $f \in \Omega_s^+(p,\alpha,\beta,k)$, then*

$$a_{p+n} \leq \frac{\beta(k+1-2\alpha)+k-1}{\Phi_n(p,\alpha,\beta,k)}. \tag{10}$$

Equality holds for the functions of the form

$$f_n(z) = \frac{1}{z} + (-1)^{p+n-1}\frac{\beta(k+1-2\alpha)+k-1}{\Phi_n(p,\alpha,\beta,k)} z^{p+n} \qquad (n = 0,1,2,...)\,. \tag{11}$$

3. DISTORTION THEOREM

A distortion property for function $f \in \Omega_s^+(p,\alpha,\beta,k)$ is given as follows:

Theorem 3. *If the function f defined by* (6) *is in the class $\Omega_s^+(p,\alpha,\beta,k)$, then for $0 < |z| = r < 1$,*

$$\frac{1}{r} - \frac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)} r^p \leq |f(z)| \leq \frac{1}{r} + \frac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)} r^p \tag{12}$$

and

$$\frac{1}{r^2} - \frac{p\left[\beta(k+1-2\alpha)+k-1\right]}{\Phi_0(p,\alpha,\beta,k)} r^{p-1} \leq \left|f'(z)\right| \leq \frac{1}{r^2} + \frac{p\left[\beta(k+1-2\alpha)+k-1\right]}{\Phi_0(p,\alpha,\beta,k)}. \tag{13}$$

The bounds in (12) *and* (13) *are attained for the functions f given by*

$$f(z) = \frac{1}{z} + \frac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)} z^p. \tag{14}$$

Proof. Suppose f is in $\Omega_s^+(p,\alpha,\beta,k)$. In view of Theorem 1, we have

$$\sum_{n=o}^{\infty} a_{p+n} \leq \frac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)}. \tag{15}$$

Then, for $0 < |z| < 1$,

$$|f(z)| = \left| \tfrac{1}{z} + \sum_{n=0}^{\infty} (-1)^{p+n-1} a_{p+n} z^{p+n} \right|$$

$$\leq \tfrac{1}{|z|} + \sum_{n=0}^{\infty} a_{p+n} \, |z|^{p+n}$$

$$\leq \tfrac{1}{r} + r^p \sum_{n=0}^{\infty} a_{p+n}$$

$$\leq \tfrac{1}{r} + \tfrac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)} r^p$$

and

$$|f(z)| \geq \tfrac{1}{r} - \sum_{n=0}^{\infty} a_{p+n} r^{p+n}$$
$$\geq \tfrac{1}{r} - r^p \sum_{n=0}^{\infty} a_{p+n}$$
$$\geq \tfrac{1}{r} - \tfrac{\beta(k+1-2\alpha)+k-1}{\Phi_0(p,\alpha,\beta,k)} r^p$$

which proves the assertion (12).
Next, we also observe that

$$\frac{\Phi_0(p,\alpha,\beta,k)}{p} \sum_{n=0}^{\infty} (p+n) a_{p+n} \leq \beta(k+1-2\alpha) + k - 1 \tag{16}$$

which readily yields the following distortion inequalities:

$$|f'(z)| \leq \tfrac{1}{|z|^2} + \sum_{n=0}^{\infty} (p+n) a_{p+n} |z|^{p+n-1}$$

$$\leq \tfrac{1}{r^2} + r^{p-1} \sum_{n=0}^{\infty} (p+n) a_{p+n}$$

$$\leq \tfrac{1}{r^2} + \tfrac{p[\beta(k+1-2\alpha)+k-1]}{\Phi_0(p,\alpha,\beta,k)} r^{p-1}$$

and

$$|f'(z)| \geq \tfrac{1}{|z|^2} - \sum_{n=0}^{\infty} (p+n) a_{p+n} |z|^{p+n-1}$$

$$\geq \tfrac{1}{r^2} - r^{p-1} \sum_{n=0}^{\infty} (p+n) a_{p+n}$$

$$\geq \tfrac{1}{r^2} - \tfrac{p[\beta(k+1-2\alpha)+k-1]}{\Phi_0(p,\alpha,\beta,k)} r^{p-1}$$

which proves the assertion (13) of Theorem 3.

4. RADII OF STARLIKENESS AND CONVEXITY.

The radii of starlikeness and convexity for the class $\Omega_s^+(p,\alpha,\beta,k)$ are given as the following theorem:

THEOREM 4. *If the function f defined by (6) is in the class $\Omega_s^+(p,\alpha,\beta,k)$, then f is meromorphically starlike of order $\delta(0 \le \delta < 1)$ in the disk $|z| < r_1(p,\alpha,\beta,\delta)$, where $r_1(p,\alpha,\beta,\delta)$ is the largest value for which*

(17)

$$r_1 = r_1(p,\alpha,\beta,\delta) = \inf_{n\ge 0}\left[\frac{(1-p)\Phi_n(p,\alpha,\beta,k)}{(p+n+2-\delta)\left[\beta(k+1-2\alpha)+k-1\right]}\right]^{\frac{1}{p+n+1}}.$$

Furthermore, f is meromorphically convex of order $\delta(0 \le \delta < 1)$ in the disk $|z| < r_2(p,\alpha,\beta,\delta)$, where $r_1(p,\alpha,\beta,\delta)$ is the largest value for which

$$(18)\qquad r_2 = r_2(p,\alpha,\beta,\delta) = \inf_{n\ge 0}\left[\frac{(1-p)\Phi_n(p,\alpha,\beta,k)}{(p+n)(p+n+2-\delta)\left[\beta(k+1-2\alpha)+k-1\right]}\right]^{\frac{1}{p+n+1}}.$$

The results (17) *and* (18) *are sharp for the function f given by* (11).

Proof. It suffices to show that

$$(19)\qquad \left|\frac{zf'(z)}{f(z)}+1\right| \le 1-\delta$$

for $|z| \le r_1$. We have

$$(20)\qquad \left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{n=0}^{\infty}(p+n+1)a_{p+n}z^{p+n}}{\frac{1}{z}+\sum_{n=0}^{\infty}a_{p+n}z^{p+n}}\right| \le \frac{\sum_{n=0}^{\infty}(p+n+1)|a_{p+n}||z|^{p+n+1}}{1-\sum_{n=0}^{\infty}|a_{p+n}||z|^{p+n+1}} \le 1-\delta.$$

Hence (20) holds true if

$$(21)\qquad \sum_{n=0}^{\infty}(p+n+1)\left|a_{p+n}\right|\left|z^{p+n+1}\right| \le (1-\delta)\left(1-\sum_{n=0}^{\infty}\left|a_{p+n}\right|\left|z^{p+n+1}\right|\right)$$

or

$$(22)\qquad \sum_{n=0}^{\infty}\frac{(p+n+2-\delta)}{1-\delta}\left|a_{p+n}\right|\left|z^{p+n+1}\right| \le 1$$

which, with the aid of (8), (22), is true if

$$\frac{(p+n+2-\delta)}{1-\delta}\,|z|^{p+n+1} \le \frac{\Phi_n(p,\alpha,\beta,k)}{\beta(k+1-2\alpha)+k-1}. \tag{23}$$

Solving (23) for $|z|$, we obtain

$$|z| \le \left[\frac{(1-p)\Phi_n(p,\alpha,\beta,k)}{(p+n+2-\delta)\left[\beta(k+1-2\alpha)+k-1\right]}\right]^{\frac{1}{p+n+1}},\ n \ge 0.$$

In precisely the same manner, we can find the radius of convexity asserted by (18) by requiring that

$$\left|\frac{zf''(z)}{f'(z)}+p+1\right| \le p-\delta \tag{24}$$

in view of (8). This completes the proof of Theorem 4.

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