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MONOTONICITY, COMPARISON AND MINKOWSKI'S INEQUALITY FOR GENERALIZED MUIRHEAD MEANS IN TWO VARIABLES

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Abstract. Given the real numbers a and b with $a+b\neq 0$, the generalized Muirhead (or symmetric) mean with parameters a, b is the function $\Sigma_{a,b}(\cdot,\cdot)$, defined by

$$\Sigma_{a,b}(x,y)=\left(\frac{x^a y^b+x^b y^a}{2}\right)^{\frac{1}{a+b}}.$$

The aim of the paper is to investigate the monotonicity of $\Sigma_{a,b}$ with respect to a or b. Likewise, a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead means $\Sigma_{a,b}$ are established.

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Key words. Generalized Muirhead means, comparison of means, Minkowski's inequality.

1. INTRODUCTION

A *mean* in two variables is any function $M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ (where $\mathbb{R}_+:=(0,\infty)$ is the set of positive real numbers), satisfying for all $x,y\in\mathbb{R}_+$ the following conditions:

(i) $M(x,y)=M(y,x)$ (symmetry property);
(ii) $\min(x,y)\leq M(x,y)\leq\max(x,y)$ (mean value property).

Sometimes, condition (ii) is replaced by the weaker requirement (e.g. [7]):

(iii) $M(x,x)=x$ (reflexivity property).

Clearly, (ii) implies (iii), but the converse is not always true.

The means in two variables are special and they have already found a number of applications. Due to these facts, there is a rich literature concerning these means. Especially the following two-parameter families of bivariate means have evoked the interest of many mathematicians in the last three decades.

The first family is that of Stolarsky means (sometimes called difference means). Given $a,b\in\mathbb{R}$ and $x,y\in\mathbb{R}_+$, satisfying $ab(a-b)(x-y)\neq 0$, the Stolarsky mean of x and y with parameters a,b is the value

$$\xi=D_{a,b}(x,y)=\left(\frac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{\frac{1}{a-b}},$$

obtained by applying the Cauchy mean value theorem to the functions $t\mapsto t^a$ and $t\mapsto t^b$ on the interval $[y,x]$ or $[x,y]$. This mean was first defined by

K.B. Stolarsky [17], who showed that it can be extended continuously to the domain

$$\{ (a,b,x,y) \mid a,b \in \mathbb{R},\ x,y \in \mathbb{R}_+ \}. \tag{1.1}$$

The extended Stolarsky mean with parameters $a, b \in \mathbb{R}$ is the function (see, for instance, [11]) $D_{a,b} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by

$$D_{a,b}(x,y) = \begin{cases} \left(\frac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{\frac{1}{a-b}} & \text{if} \quad ab(a-b)(x-y) \neq 0 \\ \left(\frac{x^a-y^a}{a(\ln x - \ln y)}\right)^{\frac{1}{a}} & \text{if} \quad a(x-y) \neq 0, \quad b = 0 \\ \left(\frac{b(\ln x - \ln y)}{x^b-y^b}\right)^{-\frac{1}{b}} & \text{if} \quad b(x-y) \neq 0, \quad a = 0 \\ \exp\left(-\frac{1}{a} + \frac{x^a \ln x - y^a \ln y}{x^a - y^a}\right) & \text{if} \quad a(x-y) \neq 0, \quad a = b \\ \sqrt{xy} & \text{if} \quad x - y \neq 0, \quad a = b = 0 \\ x & \text{if} \quad x - y = 0. \end{cases}$$

The mean $D_{a,b}$ satisfies both (i) and (ii). Moreover, the function

$$(a,b,x,y) \longmapsto D_{a,b}(x,y)$$

is infinitely many times differentiable on the domain (1.1).

The second family is that of Gini means (sometimes called sum means). Given $a, b \in \mathbb{R}$, $a \neq b$ and $x, y \in \mathbb{R}_+$, the Gini mean of x and y with parameters a, b is

$$S_{a,b}(x,y) = \left(\frac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}}.$$

This mean was first defined by C. Gini [5] and it can be also extended continuously to the domain (1.1). The extended Gini mean with parameters $a, b \in \mathbb{R}$ is the function $S_{a,b} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by

$$S_{a,b}(x,y) = \begin{cases} \left(\frac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}} & \text{if} \quad a - b \neq 0 \\ \exp\left(\frac{x^a \ln x + y^a \ln y}{x^a + y^a}\right) & \text{if} \quad a - b = 0. \end{cases}$$

The mean $S_{a,b}$ satisfies also (i) and (ii).

But in the literature one can find other means, not belonging to the above mentioned two families. One such important mean is the generalized Muirhead (or symmetric) mean. Given $a, b \in \mathbb{R}$ with $a + b \neq 0$, the generalized Muirhead mean with parameters a and b is the function $\Sigma_{a,b} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by (see, for instance, [2, p. 333] or [1])

$$\Sigma_{a,b}(x,y) = \left(\frac{x^a y^b + x^b y^a}{2}\right)^{\frac{1}{a+b}}.$$

In the special case when $a+b=1$, i.e., $a=\alpha$ and $b=1-\alpha$, the Muirhead (or symmetric) mean is obtained:

$$\Sigma_{\alpha,1-\alpha}(x,y)=\widetilde{\Sigma}_{\alpha}(x,y)=\frac{x^{\alpha}y^{1-\alpha}+x^{1-\alpha}y^{\alpha}}{2}.$$

On the other hand, for $p\in\mathbb{R}\setminus\{0\}$, $\Sigma_{p,0}(x,y)$ reduces to the power mean of order p of x and y:

$$\Sigma_{p,0}(x,y)=M_p(x,y)=\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}.$$

The generalized Muirhead mean $\Sigma_{a,b}$ satisfies (i) and (iii). It is not difficult to see that $\Sigma_{a,b}$ satisfies the mean value property (ii) if and only if $ab\geq 0$. However, in this paper we will consider the mean $\Sigma_{a,b}$ also for parameters a and b not satisfying this condition. (The symmetric mean $\widetilde{\Sigma}_{\alpha}$ for values of α not lying in $[0,1]$ has been considered in [15] and [18].) Finally, we point out that, unlike the Stolarsky or Gini means, the generalized Muirhead mean $\Sigma_{a,b}$ cannot be extended continuously to the domain (1.1).

2. MAIN RESULTS

We present first a list of immediate basic properties of the generalized Muirhead mean $\Sigma_{a,b}$. They are similar to the corresponding properties of the Stolarsky or Gini means (see [12]).

(P1) The generalized Muirhead mean Σ is symmetric in its parameters, i.e., $\Sigma_{a,b}(\cdot,\cdot)=\Sigma_{b,a}(\cdot,\cdot)$ for all $a,b\in\mathbb{R}$ with $a+b\neq 0$.

(P2) Given $a,b\in\mathbb{R}$ with $a+b\neq 0$, the function $\Sigma_{a,b}(\cdot,\cdot)$ is symmetric in its variables, i.e., $\Sigma_{a,b}(x,y)=\Sigma_{a,b}(y,x)$ for all $x,y\in\mathbb{R}_+$.

(P3) Given $a,b\in\mathbb{R}$ with $a+b\neq 0$, the function $\Sigma_{a,b}(\cdot,\cdot)$ is homogeneous of order one in its variables, i.e.,

$$\Sigma_{a,b}(\lambda x,\lambda y)=\lambda\Sigma_{a,b}(x,y)\qquad\text{for all }\ \lambda,x,y\in\mathbb{R}_+.$$

(P4) For all $a,b,c\in\mathbb{R}$ with $(a+b)c\neq 0$ and all $x,y\in\mathbb{R}_+$ it holds that

$$\Sigma_{a,b}(x^c,y^c)=[\Sigma_{ac,bc}(x,y)]^c.$$

(P5) For all $a,b\in\mathbb{R}$ with $a+b\neq 0$ and all $x,y\in\mathbb{R}_+$ it holds that

$$\Sigma_{a,b}(x,y)\Sigma_{-a,-b}(x,y)=xy.$$

(P6) For all $a,b,c\in\mathbb{R}$ with $a+b\neq 0$ and all $x,y\in\mathbb{R}_+$ it holds that

$$\Sigma_{a,b}(x^c,y^c)=(xy)^c\Sigma_{a,b}(x^{-c},y^{-c}).$$

It is well known that both $D_{a,b}$ and $S_{a,b}$ increase with increase in either a or b. For fixed $x,y\in\mathbb{R}_+$, the monotonicity of the functions $a\mapsto D_{a,b}(x,y)$ and $b\mapsto D_{a,b}(x,y)$ is established in [17] and [8]. For the monotonicity of the functions $a\mapsto S_{a,b}(x,y)$ and $b\mapsto S_{a,b}(x,y)$ the reader is referred to [16] or to [12]. The corresponding monotonicity property for the generalized Muirhead mean $\Sigma_{a,b}$ is more complicated and it is stated in the following theorem.

THEOREM 1. *Let* $b \in \mathbb{R}$, *let* $x, y \in \mathbb{R}_+$ *with* $x \neq y$, *let* $b^* = \dfrac{\ln 2}{|\ln x - \ln y|}$, *and let* $f : \mathbb{R} \setminus \{-b\} \to \mathbb{R}$ *be the function defined by* $f(a) = \Sigma_{a,b}(x, y)$. *Then the following assertions, concerning the monotonicity of* f, *are true:*

1° *If* $b > 0$, *then*

$$f \quad is \quad \begin{cases} \text{decreasing on } \ (-b, b] \\ \text{increasing on } \ [b, \infty). \end{cases}$$

In addition, if $b \geq b^*$, *then* f *is decreasing on* $(-\infty, -b)$, *whilst if* $0 < b < b^*$, *then there is a unique* $a_0 \in (-\infty, -b)$ *such that*

$$f \quad is \quad \begin{cases} \text{increasing on } \ (-\infty, a_0] \\ \text{decreasing on } \ [a_0, -b). \end{cases}$$

2° *If* $b < 0$, *then*

$$f \quad is \quad \begin{cases} \text{decreasing on } \ [b, -b) \\ \text{increasing on } \ (-\infty, b]. \end{cases}$$

In addition, if $b \leq -b^*$, *then* f *is decreasing on* $(-b, \infty)$, *whilst if* $-b^* < b < 0$, *then there is a unique* $a_0 \in (-b, \infty)$ *such that*

$$f \quad is \quad \begin{cases} \text{decreasing on } \ (-b, a_0] \\ \text{increasing on } \ [a_0, \infty). \end{cases}$$

3° *If* $b = 0$, *then* f *is increasing on* $(-\infty, 0) \cup (0, \infty)$.

Our next main result is a comparison theorem for the generalized Muirhead means $\Sigma_{a,b}$. Recall that the comparison between the Stolarsky means has been settled by E. B. Leach and M. C. Sholander [9] and Zs. Páles [14]. A similar comparison theorem for the Gini means was established by Zs. Páles [13] (see also the paper by P. Czinder and Zs. Páles [4]).

THEOREM 2. *Let* $a, b, c, d \in \mathbb{R}$ *with* $(a + b)(c + d) \neq 0$. *The inequality*

$$\Sigma_{a,b}(x, y) \leq \Sigma_{c,d}(x, y) \tag{2.1}$$

holds true for all $x, y \in \mathbb{R}_+$ *if and only if*

$$\frac{|a - b|}{a + b} \leq \frac{|c - d|}{c + d} \qquad \text{and} \qquad \frac{(a - b)^2}{a + b} \leq \frac{(c - d)^2}{c + d}. \tag{2.2}$$

REMARK 1. In the special case when $b = 1 - a$ and $d = 1 - c$, from Theorem 2 we deduce that

$$\widetilde{\Sigma}_a(x, y) \leq \widetilde{\Sigma}_c(x, y) \qquad \text{for all } \ x, y \in \mathbb{R}_+$$

if and only if $\left| a - \dfrac{1}{2} \right| \leq \left| c - \dfrac{1}{2} \right|$.

REMARK 2. Suppose that the real numbers a, b, c, d satisfy $a \geq b$, $c \geq d$, and $(a, b) \prec (c, d)$, i.e. (see [2, p. 18]), $a \leq c$ and $a + b = c + d$. If, in addition $a + b \neq 0$, then it is easily seen that (2.2) is satisfied. By Theorem 1 we deduce that (2.1) holds true for all $x, y \in \mathbb{R}_+$. This is a special case of Corollary 19 from [2, p. 335].

Finally, our last main result is a Minkowski-type inequality involving the generalized Muirhead means $\Sigma_{a,b}$. It relates to similar inequalities obtained by L. Losonczi and Zs. Páles [11] for the Stolarsky means and by L. Losonczi and Zs. Páles [10], P. Czinder and Zs. Páles [3] for the Gini means.

THEOREM 3. *Given $a, b \in \mathbb{R}$ with $a + b \neq 0$, the following assertions are true:*

1° *The inequality*

$$\Sigma_{a,b}(x_1 + x_2, y_1 + y_2) \leq \Sigma_{a,b}(x_1, y_1) + \Sigma_{a,b}(x_2, y_2) \tag{2.3}$$

holds true for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$ if and only if

$$\begin{cases} ab \leq 0 \\ \text{and} \\ (a-b)^2(a+b-1) - 4ab \geq 0. \end{cases} \tag{2.4}$$

2° *The inequality*

$$\Sigma_{a,b}(x_1 + x_2, y_1 + y_2) \geq \Sigma_{a,b}(x_1, y_1) + \Sigma_{a,b}(x_2, y_2) \tag{2.5}$$

holds true for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$ if and only if

$$\begin{cases} ab \geq 0 \\ \text{and} \\ (a-b)^2(a+b-1) - 4ab \leq 0. \end{cases} \tag{2.6}$$

Moreover, if $(a, b) = (0, 1)$ or $(a, b) = (1, 0)$, then equality holds in (2.3) and (2.5) for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$. If $(a, b) \neq (0, 1)$ and $(a, b) \neq (1, 0)$, then equality occurs in (2.3) or in (2.5) if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

Figure 2.1 shows the domain of points (a, b) satisfying (2.4), while figure 2.2 shows the domain of points (a, b) satisfying (2.6). Note that

$$(a-b)^2(a+b-1) - 4ab = (a+b)[(a-b)^2 - (a+b)].$$

The following examples illustrate the typical inequalities that can be obtained from Theorem 3. Since the point $(2, -1)$ satisfies (2.4), it follows that

$$\frac{(x_1+x_2)^3 + (y_1+y_2)^3}{(x_1+x_2)(y_1+y_2)} \leq \frac{x_1^3 + y_1^3}{x_1 y_1} + \frac{x_2^3 + y_2^3}{x_2 y_2}.$$

On the other hand, because the point $(2, 1)$ satisfies (2.6), it follows that

$$\sqrt[3]{(x_1+x_2)^2(y_1+y_2) + (x_1+x_2)(y_1+y_2)^2} \leq \sqrt[3]{x_1^2 y_1 + x_1 y_1^2} + \sqrt[3]{x_2^2 y_2 + x_2 y_2^2}.$$

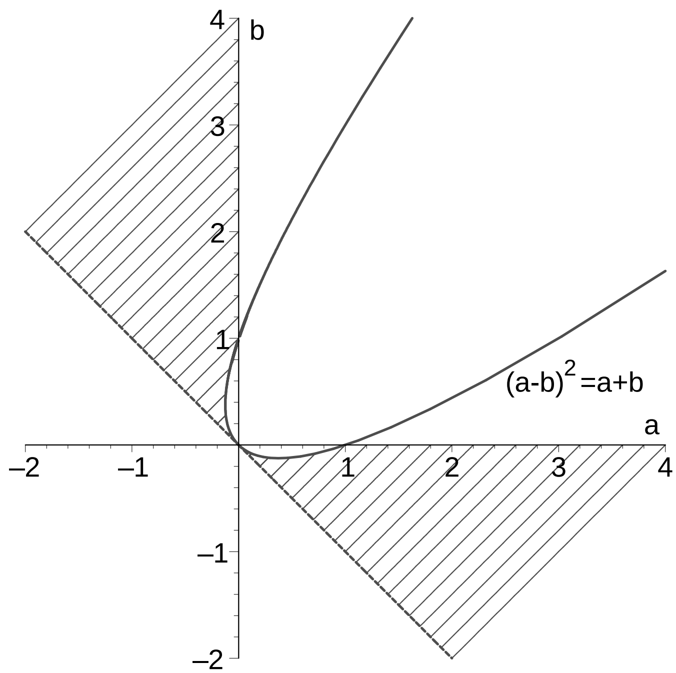

Fig. 2.1 – The domain of points (a,b) satisfying (2.4).

REMARK 3. In the special case when $a=\alpha$ and $b=1-\alpha$, from Theorem 3 we deduce that

$$\widetilde{\Sigma}_\alpha(x_1+x_2,y_1+y_2)\le\widetilde{\Sigma}_\alpha(x_1,y_1)+\widetilde{\Sigma}_\alpha(x_2,y_2) \tag{2.7}$$

holds true for all $x_1,x_2,y_1,y_2\in\mathbb{R}_+$ if and only if $\alpha\in(-\infty,0]\cup[1,\infty)$. Likewise, the converse of (2.7) holds for all $x_1,x_2,y_1,y_2\in\mathbb{R}_+$ if and only if $\alpha\in[0,1]$.

3. PROOF OF THEOREM 1

On account of (P2) we may assume that $x>y$. On the other hand, by virtue of (P3) we have

$$f(a)=\sqrt{xy}\;\Sigma_{a,b}\left(\sqrt{\frac{x}{y}},\,\sqrt{\frac{y}{x}}\right)=\sqrt{xy}\;\Sigma_{a,b}\left(\mathrm{e}^t,\,\mathrm{e}^{-t}\right),$$

where $t=\ln\sqrt{\dfrac{x}{y}}>0$. Therefore, the monotonicity of f is the same as that of the function

$$g(a):=\ln\Sigma_{a,b}\left(\mathrm{e}^t,\,\mathrm{e}^{-t}\right)=\frac{\ln(\cosh((a-b)t))}{a+b}.$$

Note that

$$g'(a)=-\frac{\ln(\cosh((a-b)t))}{(a+b)^2}+\frac{t}{a+b}\,\tanh((a-b)t)$$

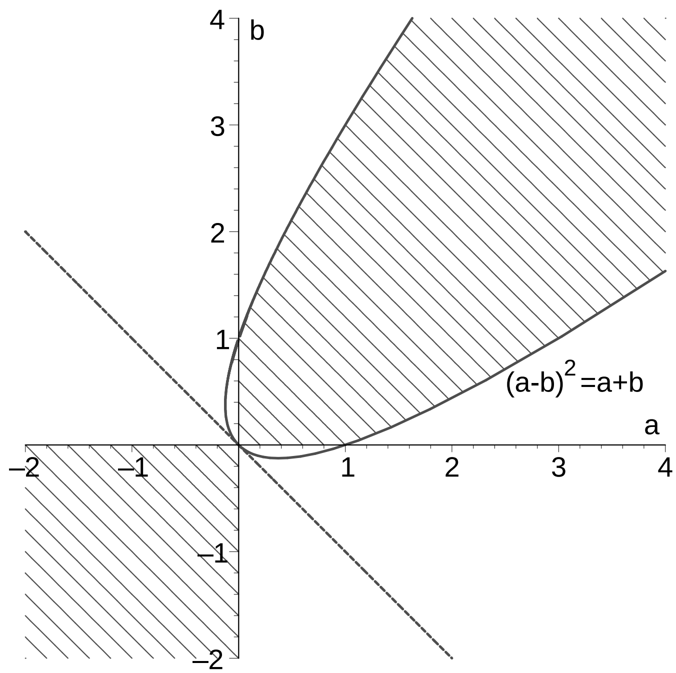

Fig. 2.2 – The domain of points (a, b) satisfying (2.6).

for all $a \in \mathbb{R} \setminus \{-b\}$. We have to examine the sign of $g'(a)$ which is the same as that of $h(a) := (a+b)^2 g'(a)$. Since

$$h(a) = -\ln(\cosh((a-b)t)) + (a+b)t\tanh((a-b)t),$$

we have

$$h'(a) = \frac{(a+b)t^2}{\cosh^2((a-b)t)}.$$

1° Suppose that $b > 0$. Since $h'(a) > 0$ for all $a \in (-b, \infty)$, it follows that h is increasing on $(-b, \infty)$. Taking into account that $h(b) = 0$, we deduce that

$$\begin{cases} h(a) < 0 & \text{for all } \ a \in (-b, b), \\ h(a) > 0 & \text{for all } \ a \in (b, \infty). \end{cases}$$

This means that

$$\begin{cases} g'(a) < 0 & \text{for all } \ a \in (-b, b), \\ g'(a) > 0 & \text{for all } \ a \in (b, \infty). \end{cases}$$

Consequently, g (i.e. f) is decreasing on $(-b, b]$ and increasing on $[b, \infty)$.

On the other hand, since $h'(a) < 0$ for all $a \in (-\infty, -b)$, it follows that h is decreasing on $(-\infty, -b)$. It is easily seen that

$$\lim_{a \to -\infty} h(a) = \ln 2 - 2bt = \ln 2 - b|\ln x - \ln y|,$$
$$h(-b) = -\ln(\cosh(2bt)) < 0.$$

Therefore, if $b \geq b^* = \dfrac{\ln 2}{|\ln x - \ln y|}$, then $\lim\limits_{a\to-\infty} h(a) \leq 0$, hence $h(a) < 0$ for all $a \in (-\infty, -b)$. As above, this means that g (i.e. f) is decreasing on $(-\infty, -b)$.

If $0 < b < b^*$, then we have $\lim\limits_{a\to-\infty} h(a) > 0$, hence there is a unique $a_0 \in (-\infty, -b)$ such that $h(a_0) = 0$. Because h is decreasing on $(-\infty, -b)$, we deduce that

$$\begin{cases} h(a) > 0 & \text{for all } \ a \in (-\infty, a_0), \\ h(a) < 0 & \text{for all } \ a \in (a_0, -b). \end{cases}$$

Reasoning as above, we conclude that g (i.e. f) is increasing on $(-\infty, a_0]$ and decreasing on $[a_0, -b)$.

2° This assertion follows immediately from assertion 1°, by virtue of (P5).

3° Although the case $b = 0$ reduces to the monotonicity of the power means, which is well-known, we include a proof. In this case we have

$$h'(a) = \frac{at^2}{\cosh^2(at)}.$$

Consequently, $h'(a) < 0$, hence h is decreasing on $(-\infty, 0)$ and $h'(a) > 0$, hence h is increasing on $(0, \infty)$. Since $h(a) = 0$, we deduce that $h(a) > 0$, hence $g'(a) > 0$ for all $a \in (-\infty, 0) \cup (0, \infty)$. Therefore g (i.e. f) is incresing on $(-\infty, 0) \cup (0, \infty)$. □

4. PROOF OF THEOREM 2

Let

$$\alpha := a - b, \qquad \gamma := c - d,$$
$$\beta := a + b, \qquad \delta := c + d.$$

Due to (P1), we may assume that $\alpha \geq 0$ and $\gamma \geq 0$. On the other hand, as it has already been remarked in the proof of Theorem 1, we have

$$\Sigma_{a,b}(x, y) = \sqrt{xy}\ \Sigma_{a,b}\left(e^t, e^{-t}\right), \qquad \Sigma_{c,d}(x, y) = \sqrt{xy}\ \Sigma_{c,d}\left(e^t, e^{-t}\right),$$

where $t = \ln\sqrt{\dfrac{x}{y}}$. Taking into account (P2), we can restrict ourselves to the case $x > y$ ($t > 0$). In other words, (2.1) holds true for all $x, y \in \mathbb{R}_+$ if and only if

$$\Sigma_{a,b}\left(e^t, e^{-t}\right) \leq \Sigma_{c,d}\left(e^t, e^{-t}\right) \qquad \text{for all } \ t > 0.$$

But this inequality is equivalent to

$$(4.1) \qquad (\cosh(\alpha t))^{\frac{1}{\beta}} \leq (\cosh(\gamma t))^{\frac{1}{\delta}} \qquad \text{for all } \ t > 0.$$

Further, let $f : (0, \infty) \to \mathbb{R}$ be the function defined by

$$f(t) = \frac{\ln(\cosh(\alpha t))}{\beta} - \frac{\ln(\cosh(\gamma t))}{\delta}.$$

Then we have

$$f'(t) = \frac{\alpha \sinh(\alpha t)}{\beta \cosh(\alpha t)} - \frac{\gamma \sinh(\gamma t)}{\delta \cosh(\gamma t)}.$$

Also, letting $g(t) := \cosh(\alpha t)\cosh(\gamma t) f'(t)$, we have

$$\begin{aligned} (4.2)\quad g(t) &= \frac{\alpha}{\beta}\sinh(\alpha t)\cosh(\gamma t) - \frac{\gamma}{\delta}\sinh(\gamma t)\cosh(\alpha t) \\ &= \frac{1}{2}\left(\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right)\sinh((\alpha+\gamma)t) + \frac{1}{2}\left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right)\sinh((\alpha-\gamma)t) \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left[\left(\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right)(\alpha+\gamma)^{2k+1} \right. \\ &\qquad \left. + \left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right)(\alpha-\gamma)^{2k+1}\right] t^{2k+1}. \end{aligned}$$

The problem now is to prove that (4.1) holds true (or, equivalently, that $f(t) \le 0$ for all $t \in (0,\infty)$) if and only if

$$(4.3)\qquad \frac{\alpha}{\beta} \le \frac{\gamma}{\delta} \quad \text{and} \quad \frac{\alpha^2}{\beta} \le \frac{\gamma^2}{\delta}.$$

Necessity. Suppose that (4.1) holds true, i.e., $f(t) \le 0$ for all $t \in (0,\infty)$. Letting $t \to \infty$ in (4.1) we get $\frac{\alpha}{\beta} \le \frac{\gamma}{\delta}$. On the other hand, we must have

$$(4.4)\qquad \left(\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right)(\alpha+\gamma) + \left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right)(\alpha-\gamma) \le 0,$$

which is obviously equivalent to the second inequality in (4.3). Indeed, if (4.4) does not hold, by (4.2) it follows that there is an $\varepsilon > 0$ such that $g(t) > 0$ on $(0,\varepsilon)$, hence $f'(t) > 0$ on $(0,\varepsilon)$. Since $f(0) = 0$ and f is increasing on $(0,\varepsilon)$, we deduce that $f(t) > 0$ for all $t \in (0,\varepsilon)$, in contradiction with our assumption. This contradiction shows that (4.4) holds true.

Sufficiency. Suppose now that (4.3) holds true. Depending on β and δ, we have the following possible cases:

Case 1. $\beta > 0$ and $\delta < 0$.

In this case by (4.3) it follows that $0 \le \frac{\alpha^2}{\beta} \le \frac{\gamma^2}{\delta} \le 0$, that is $\alpha = \gamma = 0$, hence in (4.1) we have equality.

Case 2. $\beta < 0$ and $\delta > 0$.

In this case it is obvious that $f(t) \le 0$ for all $t \in (0,\infty)$.

Case 3. $\beta > 0$ and $\delta > 0$.

If $\alpha \le \gamma$, then by (4.3) we deduce that

$$(4.5)\qquad \left(\frac{\alpha}{\beta} - \frac{\gamma}{\delta}\right)(\alpha+\gamma)^{2k+1} + \left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right)(\alpha-\gamma)^{2k+1} \le 0$$

for all $k = 0, 1, 2, \ldots$, hence $g(t) \le 0$ for all $t > 0$. This means that $f'(t) \le 0$ for all $t > 0$, hence f is nonincreasing on $(0,\infty)$. Since $f(0) = 0$, we conclude that $f(t) \le 0$ for all $t \in (0,\infty)$.

Assume now that $\alpha > \gamma$. Because the second inequality in (4.3) is equivalent to (4.4), it follows that (4.4) holds true. Therefore we have

$$-\frac{\frac{\alpha}{\beta} - \frac{\gamma}{\delta}}{\frac{\alpha}{\beta} + \frac{\gamma}{\delta}} \geq \frac{\alpha - \gamma}{\alpha + \gamma} \geq \left(\frac{\alpha - \gamma}{\alpha + \gamma}\right)^{2k+1}$$

for all $k = 0, 1, 2, \ldots$. Consequently, (4.5) holds true for all $k = 0, 1, 2, \ldots$, hence $g(t) \leq 0$ for all $t > 0$. Reasoning as above, we conclude that $f(t) \leq 0$ for all $t \in (0, \infty)$.

Case 4. $\beta < 0$ and $\delta < 0$.
From (4.3) it follows that

$$\frac{\gamma}{-\delta} \leq \frac{\alpha}{-\beta} \qquad \text{and} \qquad \frac{\gamma^2}{-\delta} \leq \frac{\alpha^2}{-\beta}.$$

By virtue of Case 3, we have

$$\Sigma_{-c,-d}(x, y) \leq \Sigma_{-a,-b}(x, y) \qquad \text{for all} \;\; x, y \in \mathbb{R}_+.$$

Taking now into account (P5), we conclude that (2.1) holds true. □

5. PROOF OF THEOREM 3

Following L. Losonczi and Zs. Páles [10] in their proof of Theorem 1, remark that $\Sigma_{a,b}(x, y) = y\Sigma_{a,b}\left(\frac{x}{y}, 1\right) = y\varphi_{a,b}\left(\frac{x}{y}\right)$, where

$$\begin{aligned} \varphi_{a,b}(u) &= \left(\frac{u^a + u^b}{2}\right)^{\frac{1}{a+b}} = \sqrt{u}\left(\frac{u^{\frac{a-b}{2}} + u^{-\frac{a-b}{2}}}{2}\right)^{\frac{1}{a+b}} \\ &= \sqrt{u}[\cosh((a-b)\ln\sqrt{u})]^{\frac{1}{a+b}}. \end{aligned}$$

Taking this into account, (2.3) becomes

$$\varphi_{a,b}\left(\frac{y_1}{y_1+y_2}\cdot\frac{x_1}{y_1} + \frac{y_2}{y_1+y_2}\cdot\frac{x_2}{y_2}\right) \leq \frac{y_1}{y_1+y_2}\varphi_{a,b}\left(\frac{x_1}{y_1}\right) + \frac{y_2}{y_1+y_2}\varphi_{a,b}\left(\frac{x_2}{y_2}\right).$$

Using the notations $\frac{x_1}{y_1} = u \in \mathbb{R}_+$, $\frac{x_2}{y_2} = v \in \mathbb{R}_+$, $\frac{y_1}{y_1+y_2} = \lambda \in (0, 1)$ we see that the last inequality is equivalent to

$$\varphi_{a,b}(\lambda u + (1-\lambda)v) \leq \lambda\varphi_{a,b}(u) + (1-\lambda)\varphi_{a,b}(v). \tag{5.1}$$

Consequently, the validity of (2.3) for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$ is equivalent to the convexity of the function $\varphi_{a,b}$ on $\mathbb{R}_+$. Similarly, the validity of (2.5) for all $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$ is equivalent to the concavity of $\varphi_{a,b}$ on $\mathbb{R}_+$.

Next we show that $\varphi_{a,b}$ is convex on $\mathbb{R}_+$ if and only if (2.4) holds true. The fact that $\varphi_{a,b}$ is concave on $\mathbb{R}_+$ if and only if (2.6) holds true can be proved analogously.

It is well-known that $\varphi_{a,b}$ is convex on $\mathbb{R}_+$ if and only if $\varphi''_{a,b}(u) \geq 0$ for all $u \in \mathbb{R}_+$. Since

$$\ln \varphi_{a,b}(u) = \frac{1}{2} \ln u + \frac{\ln(\cosh((a-b)\ln \sqrt{u}))}{a+b},$$

we deduce that

$$\frac{\varphi'_{a,b}(u)}{\varphi_{a,b}(u)} = \frac{1}{2u} + \frac{a-b}{2(a+b)u} \tanh((a-b)\ln \sqrt{u}). \tag{5.2}$$

Therefore we have

$$\begin{aligned} \frac{\varphi''_{a,b}(u)\varphi_{a,b}(u) - [\varphi'_{a,b}(u)]^2}{[\varphi_{a,b}(u)]^2} & \\ = -\frac{1}{2u^2} - \frac{(a-b)\tanh((a-b)\ln \sqrt{u})}{2(a+b)u^2} & \\ + \frac{(a-b)^2}{4(a+b)u^2 \cosh^2((a-b)\ln \sqrt{u})}. & \end{aligned} \tag{5.3}$$

From (5.2) and (5.3) it follows that

$$\frac{\varphi''_{a,b}(u)}{\varphi_{a,b}(u)} = -\frac{1}{4u^2} + \frac{(a-b)^2}{4(a+b)^2u^2} + \frac{(a-b)^2(a+b-1)}{4(a+b)^2u^2\cosh^2((a-b)\ln \sqrt{u})},$$

hence

$$4(a+b)^2u^2\cosh^2((a-b)\ln \sqrt{u})\frac{\varphi''_{a,b}(u)}{\varphi_{a,b}(u)} = \psi_{a,b}(\ln \sqrt{u}), \tag{5.4}$$

where

$$\begin{aligned} \psi_{a,b}(s) &= -(a+b)^2\cosh^2((a-b)s) + (a-b)^2\cosh^2((a-b)s) \\ &\quad +(a-b)^2(a+b-1) \\ &= (a-b)^2(a+b-1) - 4ab\cosh^2((a-b)s). \end{aligned} \tag{5.5}$$

By (5.4) it follows that $\varphi''_{a,b}(u) \geq 0$ for all $u \in \mathbb{R}_+$ if and only if $\psi_{a,b}(s) \geq 0$ for all $s \in \mathbb{R}$. But this is clearly equivalent to (2.4).

It remains to clarify the cases of equality.

If $ab < 0$ or $(a-b)^2(a+b-1) - 4ab > 0$, then by (5.4) and (5.5) we deduce that $\varphi''_{a,b}(u) > 0$ for all $u \in \mathbb{R}_+$. This means that $\varphi_{a,b}$ is strictly convex on $\mathbb{R}_+$. Therefore, equality occurs in (5.1) if and only if $u = v$. Equivalently, equality occurs in (2.3) if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

Analogously, it can be proved that for $ab > 0$ or $(a-b)^2(a+b-1) - 4ab < 0$, equality occurs in (2.5) if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$.

Finally, remark that for $a + b \neq 0$ we have

$$ab = 0 \qquad \text{and} \qquad (a-b)^2(a+b-1) = 0$$

if and only if $(a,b)=(0,1)$ or $(a,b)=(1,0)$. In this case $\varphi_{a,b}$ is affine on $\mathbb{R}_+$, hence equality holds in (5.1) for all $u,v\in\mathbb{R}_+$. Equivalently, equality holds in (2.3) and (2.5) for all $x_1,x_2,y_1,y_2\in\mathbb{R}_+$. □

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