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EXPONENTIAL STABILITY AND EXPONENTIAL DICHOTOMY OF SEMIGROUPS OF LINEAR OPERATORS

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Abstract. The aim of this paper is to establish necessary and sufficient conditions for exponential stability of semigroups of linear operators and to show how this conditions can be applied in order to characterize the exponential dichotomy. First, we prove that an exponentially bounded semigroup is exponentially stable if and only if it is $(l^p(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$-stable, where $p \in (1, \infty)$. After that this result is applied at the study of the exponential dichotomy of exponentially bounded semigroups. We deduce that an exponentially bounded semigroup is exponentially dichotomic if and only if the pair $(l^\infty(\mathbb{N}, X), l^p(\mathbb{N}, X))$ is admissible for it and an associated subspace is closed and complemented.

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1. INTRODUCTION

The theory of the asymptotic behavior of semigroups of linear operators in infinite dimensional spaces is a well-developed area in the field of differential equations and their applications. In the last decades significant monographs were devoted to the analysis of the asymptotic properties of semigroups (see [1], [3], [4], [5], [9], [15]). The main studies of the last years concerning the asymptotic behavior of evolution equations focuses on the input-output conditions (see [1], [7], [8], [11], [13], [14], [15]). Moreover, recent studies have shown that in certain situations the non-autonomous problems can be solved by passing to the autonomous case: this can be done by associating to an evolution family, or to a linear skew-product flow, diverse evolution semigroups on function spaces (see [1], [14]).

An interesting question concerning the autonomous case is whether one of the basic concepts can be used in order to deduce information concerning the others. The main purpose of this paper is to answer this question. We will show that in certain situations it is sufficient to have a characterization for exponential stability and this one can be exploited in order to obtain important properties concerning the exponential expansiveness and also the exponential dichotomy. First, we establish the connections between the $(l^p(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$-stability, with $p \in (1, \infty)$, and the exponential stability of an exponentially bounded semigroup. We motivate the choice of the above discrete pair by examples, showing that the $(l^1(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$-stability and the $(l^1(\mathbb{N}, X), c_0(\mathbb{N}, X))$-stability are not sufficient conditions for exponential stability. We obtain that an exponentially bounded semigroup is exponentially stable if and only if there is $p \in (1, \infty)$ such that the semigroup is $(l^p(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$-stable. Finally, we apply our result at the study of the exponential dichotomy of semigroups. Under the assumption that an associated subspace is closed and complemented, we present a detailed and constructive study for the equivalence between the admissibility of the pair $(l^\infty(\mathbb{N}, X), l^p(\mathbb{N}, X))$, with $p \in (1, \infty)$, and the exponential dichotomy of an exponentially bounded semigroup.

2. EXPONENTIAL STABILITY OF SEMIGROUPS

Let X be a real or a complex Banach space. Throughout this paper, the norm on X and on $\mathcal{B}(X)$ – the Banach algebra of all bounded linear operators on X, will be denoted by $||\cdot||$.

Definition 1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be a *semigroup* on X, if $T(0) = I$ and $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.

Definition 2. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be:
(i) *exponentially bounded* if there are $M \geq 1$ and $\omega > 0$ such that $||T(t)|| \leq M\mathrm{e}^{\omega t}$, for all $t \geq 0$;
(ii) C_0*-semigroup* if $\lim\limits_{t\searrow 0} T(t)x = x$, for all $x \in X$.

Remark 1. If $\mathbf{T}$ is a C_0-semigroup, then it is exponentially bounded (see [9, Theorem 2.2, p. 4]).

Definition 3. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *exponentially stable* if there are two constants $N, \nu > 0$ such that

$$||T(t)x|| \leq N\,\mathrm{e}^{-\nu t}||x||, \quad \forall (t, x) \in \mathbb{R}_+ \times X.$$

Remark 2. It is easy to see that an exponentially bounded semigroup is exponentially stable if and only if there are $\delta > 0$ and $c \in (0,1)$ such that $||T(\delta)|| \leq c$.

Let $\ell^\infty(\mathbb{N}, X) = \{s : \mathbb{N} \to X | \sup\limits_{n\in\mathbb{N}} ||s(n)|| < \infty\}$, which is a Banach space with respect to the norm $||s||_\infty := \sup\limits_{n\in\mathbb{N}} ||s(n)||$. If $c_0(\mathbb{N}, X) = \{s : \mathbb{N} \to X | \lim\limits_{n\to\infty} s(n) = 0\}$, then $c_0(\mathbb{N}, X)$ is a closed linear subspace of $\ell^\infty(\mathbb{N}, X)$.

For $p \in [1, \infty)$, we consider $\ell^p(\mathbb{N}, X) = \{s : \mathbb{N} \to X \mid \sum\limits_{k=0}^{\infty} ||s(k)||^p < \infty\}$. With respect to the norm $||s||_p := (\sum\limits_{k=0}^{\infty} ||s(k)||^p)^{1/p}$, the space $\ell^p(\mathbb{N}, X)$ is a Banach space.

Remark 3. For every $p, q \in [1, \infty)$ with $p \leq q$ we have that

$$\ell^1(\mathbb{N}, X) \subset \ell^p(\mathbb{N}, X) \subset \ell^q(\mathbb{N}, X) \subset c_0(\mathbb{N}, X) \subset \ell^\infty(\mathbb{N}, X).$$

Definition 4. The semigroup $\mathbf{T}$ is said to be $(I(\mathbb{N},X),O(\mathbb{N},X))$-*stable* if for every $s\in I(\mathbb{N},X)$ the sequence

$$\gamma_s:\mathbb{N}\to X,\quad \gamma_s(n)=\sum_{k=0}^{n}T(n-k)s(k)$$

belongs to $O(\mathbb{N},X)$.

Remark 4. $I(\mathbb{N},X)$ is called *the input space* and the space $O(\mathbb{N},X)$ is called *the output space.*

In what follows we establish connections between the $(I(\mathbb{N},X),O(\mathbb{N},X))$-stability of $\mathbf{T}$ and the exponential stability of $\mathbf{T}$.

Naturally, from Remark 3 the problem arises whether the $(\ell^1(\mathbb{N},X),\ell^\infty(\mathbb{N},X))$-stability or the $(\ell^1(\mathbb{N},X),c_0(\mathbb{N},X))$-stability implies the exponential stability. The answer is negative as the following example shows:

Example 1. Let $X=C_0(\mathbb{R}_+,\mathbb{R})$ be the linear space of all continuous functions $x:\mathbb{R}_+\to\mathbb{R}$ with $\lim\limits_{t\to\infty}x(t)=0$. With respect to the norm $||x||:=\sup\limits_{t\geq 0}|x(t)|$, we have that X is a Banach space.

For every $t\geq 0$ consider the operator $T(t):X\to X$, $(T(t)x)(s)=x(s+t)$. Then $\mathbf{T}=\{T(t)\}_{t\geq 0}$ is a C_0-semigroup called *the translation semigroup* on X. It is easy to see that $||T(t)||=1$, for all $t\geq 0$.

We prove that $\mathbf{T}$ is $(l^1(\mathbb{N},X),c_0(\mathbb{N},X))$-stable. Let $s\in l^1(\mathbb{N},X)$ and let $\varepsilon>0$. From $s\in l^1(\mathbb{N},X)$ it follows that there is $n_0\in\mathbb{N}^*$ such that

$$\sum_{k=n_0}^{\infty}||s(k)||<\frac{\varepsilon}{2}.$$

Let $y=\sum_{k=0}^{n_0-1}T(n_0-k)s(k)$. Since $y\in X$, there is $p\in\mathbb{N}^*$ such that $|y(t)|<(\varepsilon/2)$, for all $t\geq p$. Then, for $n\geq n_0+p$ we have that

$$||T(n-n_0)y||=\sup_{t\geq 0}|y(t+n-n_0)|\leq\frac{\varepsilon}{2}.$$

We obtain that

$$||\gamma_s(n)||\leq||T(n-n_0)y||+\sum_{k=n_0}^{n}||T(n-k)||\;||s(k)||<\varepsilon,\quad\forall n\geq n_0+p,$$

so $\gamma_s\in c_0(\mathbb{N},X)$. It follows that $\mathbf{T}$ is $(l^1(\mathbb{N},X),c_0(\mathbb{N},X))$-stable and $(l^1(\mathbb{N},X),l^\infty(\mathbb{N},X))$-stable, respectively. But, for all that $\mathbf{T}$ is not exponentially stable.

The main result of this section is:

Theorem 1. *Let* $\mathbf{T}=\{T(t)\}_{t\geq 0}$ *be an exponentially bounded semigroup on the Banach space* X *and let* $p\in(1,\infty)$. *Then* $\mathbf{T}$ *is exponentially stable if and only if* $\mathbf{T}$ *is* $(l^p(\mathbb{N},X),l^\infty(\mathbb{N},X))$-*stable.*

Proof. The necessity is immediate.

To prove the sufficiency, let $D : l^p(\mathbb{N}, X) \to l^\infty(\mathbb{N}, X)$, $Ds = \gamma_s$. We have that D is a closed linear operator, so it is bounded. We set $K = ||D||$.

Let $x \in X$ and

$$s : \mathbb{N} \to X, \quad s(n) = \begin{cases} x, & n = 0 \\ 0, & n \in \mathbb{N}^*. \end{cases}$$

Then $s \in l^p(\mathbb{N}, X)$ and $||s||_p = ||x||$. Observing that $\gamma_s(n) = T(n)x$, for all $n \in \mathbb{N}$, we deduce that

$$||T(n)x|| = ||\gamma_s(n)|| \le ||\gamma_s||_\infty \le K||s||_p = K||x||.$$

It follows that $||T(n)|| \le K$, for all $n \in \mathbb{N}$. We consider the sequence

$$\alpha : \mathbb{N} \to \mathbb{R}_+, \quad \alpha(n) = \frac{1}{n+1}.$$

Let $x \in X$ and let $s : \mathbb{N} \to X$, $s(n) = \alpha(n)\, T(n)x$. Then $s \in l^p(\mathbb{N}, X)$ and $||s||_p \le K||\alpha||_p\, ||x||$. Observing that

$$\gamma_s(n) = \left(\sum_{k=0}^{n} \frac{1}{k+1}\right) T(n)x, \quad \forall n \in \mathbb{N},$$

we obtain that

$$||T(n)x|| \left(\sum_{k=0}^{n} \frac{1}{k+1}\right) = ||\gamma_s(n)|| \le ||\gamma_s||_\infty \le K||s||_p \le K^2||\alpha||_p\, ||x||,\ \forall n \in \mathbb{N}.$$

Let $n_0 \in \mathbb{N}^*$ be such that $\sum_{k=0}^{n_0} [1/(k+1)] \ge 2K^2\, ||\alpha||_p$. Then, from the above relation, we deduce that $||T(n_0)x|| \le (1/2)||x||$.

Taking into account that n_0 does not depend on x it follows that $||T(n_0)|| \le 1/2$. Using Remark 2 we conclude that $\mathbf{T}$ is exponentially stable. □

3. EXPONENTIAL DICHOTOMY OF SEMIGROUPS

In what follows we apply the results obtained in the previous section in order to obtain input-output characterizations for exponential dichotomy of exponentially bounded semigroups.

Let X be a real or a complex Banach space and let $\mathbf{T} = \{T(t)\}_{t \ge 0}$ be an exponentially bounded semigroup on X.

Definition 5. The semigroup $\mathbf{T}$ is said to be *exponentially dichotomic* if there are a projection $P \in \mathcal{B}(X)$ and two constants $K, \nu > 0$ such that:

(i) $T(t)P = PT(t)$, for all $t \ge 0$;

(ii) for every $t \ge 0$, the restriction $T(t)_| : \mathrm{Ker}\, P \to \mathrm{Ker}\, P$ is an isomorphism;

(iii) $||T(t)x|| \le K\mathrm{e}^{-\nu t}||x||$, for all $t \ge 0$ and all $x \in \mathrm{Im}\, P$;

(iv) $||T(t)x|| \ge \frac{1}{K}\, \mathrm{e}^{\nu t}||x||$, for all $t \ge 0$ and all $x \in \mathrm{Ker}\, P$.

Let $p \in (1, \infty)$. We consider the discrete-time equation:

$$(E_d) \qquad \gamma(n+1) = T(1)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{N}$$

with $\gamma \in \ell^\infty(\mathbb{N}, X)$ and $s \in \ell^p(\mathbb{N}, X)$.

Definition 6. The pair $(\ell^\infty(\mathbb{N}, X), \ell^p(\mathbb{N}, X))$ is said to be *admissible* for $\mathbf{T}$ if for every $s \in \ell^p(\mathbb{N}, X)$ there is $\gamma \in \ell^\infty(\mathbb{N}, X)$ such that the pair (γ, s) verifies the equation (E_d).

Definition 7. A subspace $Y \subset X$ is said to be $\mathbf{T}$-*invariant* if $T(t)Y \subset Y$, for all $t \geq 0$.

In all what follows we suppose that the subspace

$$X_1 = \{x \in X : \sup_{t \geq 0} ||T(t)x|| < \infty\}$$

is closed and it has a $\mathbf{T}$-invariant complement, i.e. there is a closed $\mathbf{T}$-invariant subspace X_2 such that $X = X_1 \oplus X_2$. We denote by P the projection corresponding to this decomposition, i.e. $\operatorname{Im} P = X_1$ and $\operatorname{Ker} P = X_2$.

Remark 5. We have that $T(t)P = PT(t)$, for all $t \geq 0$.

Denoting $T_1(t) = T(t)_{|\operatorname{Im} P}$ and $T_2(t) = T(t)_{|\operatorname{Ker} P}$, $\forall t \geq 0$ we have that $\mathbf{T}_1 = \{T_1(t)\}_{t \geq 0}$ is an exponentially bounded semigroup on $\operatorname{Im} P$ and $\mathbf{T}_2 = \{T_2(t)\}_{t \geq 0}$ is an exponentially bounded semigroup on $\operatorname{Ker} P$.

Theorem 2. *If the pair $(\ell^\infty(\mathbb{N}, X), \ell^p(\mathbb{N}, X))$ is admissible for $\mathbf{T}$, then $\mathbf{T_1}$ is exponentially stable.*

Proof. From hypothesis it follows that the pair $(l^\infty(\mathbb{N}, \operatorname{Im} P), l^p(\mathbb{N}, \operatorname{Im} P))$ is admissible for the semigroup $\mathbf{T_1}$.

Let $s \in l^p(\mathbb{N}, \operatorname{Im} P)$. From the admissibility of the pair $(l^\infty(\mathbb{N}, \operatorname{Im} P), l^p(\mathbb{N}, \operatorname{Im} P))$ we obtain that there is $\alpha \in l^\infty(\mathbb{N}, \operatorname{Im} P)$ such that

$$(1) \qquad \alpha(n+1) = T_1(1)\alpha(n) + s(n+1), \quad \forall n \in \mathbb{N}.$$

From relation (1) it follows that

$$(2) \qquad \alpha(n) = T_1(n)\alpha(0) + \sum_{k=1}^{n} T_1(n-k)s(k), \quad \forall n \in \mathbb{N}.$$

From $\alpha(0), s(0) \in \operatorname{Im} P$ we deduce that the sequences $\beta : \mathbb{N} \to \operatorname{Im} P$, $\beta(n) = T_1(n)\alpha(0)$ and $\delta : \mathbb{N} \to \operatorname{Im} P$, $\delta(n) = T_1(n)s(0)$ belong to $l^\infty(\mathbb{N}, \operatorname{Im} P)$. Denoting

$$\gamma_s : \mathbb{N} \to \operatorname{Im} P, \quad \gamma_s(n) = \sum_{k=0}^{n} T_1(n-k)s(k)$$

from relation (2) we obtain that $\gamma_s(n) = \alpha(n) - \beta(n) + \delta(n)$, $\forall n \in \mathbb{N}$, so $\gamma_s \in l^\infty(\mathbb{N}, \operatorname{Im} P)$.

It follows that for every $s \in l^p(\mathbb{N}, \operatorname{Im} P)$ the sequence $\gamma_s \in l^\infty(\mathbb{N}, \operatorname{Im} P)$, so the semigroup $\mathbf{T_1}$ is $(l^p(\mathbb{N}, \operatorname{Im} P), l^\infty(\mathbb{N}, \operatorname{Im} P))$-stable. Applying Theorem 1 we deduce that $\mathbf{T_1}$ is exponentially stable. □

THEOREM 3. *If the pair* $(\ell^\infty(\mathbb{N},X), \ell^p(\mathbb{N},X))$ *is admissible for* $\mathbf{T}$*, then* $T_2(t)$ *is invertible, for all* $t \geq 0$.

Proof. It is sufficient to prove that $T_2(1)$ is invertible.

If $x \in \operatorname{Ker} T_2(1)$ then $\sup\limits_{t\geq 0} ||T(t)x|| < \infty$. This shows that $x \in \operatorname{Im} P$. Since $\operatorname{Ker} T_2(1) \subset \operatorname{Ker} P$, it follows that $x = 0$, so $T_2(1)$ is injective.

Let $y \in X$. We consider the sequence

$$s : \mathbb{N} \to X, \quad s(n) = \begin{cases} -y, & n = 1 \\ 0, & n \neq 1. \end{cases}$$

From hypothesis, there is $\gamma \in \ell^\infty(\mathbb{N},X)$ such that the pair (γ, s) verifies the equation (E_d). Then we have that

$$\gamma(n) = T(n-1)\gamma(1), \quad \forall n \in \mathbb{N}^*. \tag{3}$$

Since $\gamma \in \ell^\infty(\mathbb{N},X)$, from relation (3) it follows that $\gamma(1) \in \operatorname{Im} P$. In particular, this implies that $(I-P)\gamma(1) = 0$. Because $\gamma(1) = T(1)\gamma(0) - y$, we obtain that $T(1)(I-P)\gamma(0) = y$. Setting $x = (I-P)\gamma(0)$ we deduce that $T_2(1)x = y$, so $T_2(1)$ is surjective. □

THEOREM 4. *If the pair* $(\ell^\infty(\mathbb{N},X), \ell^p(\mathbb{N},X))$ *is admissible for* $\mathbf{T}$*, then there are* $K, \nu > 0$ *such that*

$$||T(t)x|| \geq \frac{1}{K}\,\mathrm{e}^{\nu t}\,||x||, \quad \forall t \geq 0, \quad \forall x \in \operatorname{Ker} P.$$

Proof. Step 1. We prove that there is $M > 0$ such that

$$||\gamma||_\infty \leq M\,||s||_p, \tag{4}$$

for every pair $(\gamma, s) \in \ell^\infty(\mathbb{N},X) \times \ell^p(\mathbb{N},X)$ with the property that (γ, s) verifies the equation (E_d) and $\gamma(0) \in \operatorname{Ker} P$.

Let $s \in \ell^p(\mathbb{N},X)$. From hypothesis, there is $\lambda \in \ell^\infty(\mathbb{N},X)$ such that the pair (λ, s) verifies the equation (E_d). Then $\gamma : \mathbb{N} \to X$, $\gamma(n) = \lambda(n) - T(n)P\lambda(0)$ belongs to $\ell^\infty(\mathbb{N},X)$, $\gamma(0) \in \operatorname{Ker} P$ and the pair (γ, s) verifies the equation (E_d).

Let $\tilde{\gamma} \in \ell^\infty(\mathbb{N},X)$ with the property that $(\tilde{\gamma}, s)$ verifies the equation (E_d) and $\tilde{\gamma}(0) \in \operatorname{Ker} P$. Setting $\alpha = \tilde{\gamma} - \gamma$ we have that

$$\alpha(n) = T(n)\alpha(0), \quad \forall n \in \mathbb{N}. \tag{5}$$

From relation (5) we obtain that $\alpha(0) \in \operatorname{Im} P$. But $\alpha(0) = \tilde{\gamma}(0) - \gamma(0) \in \operatorname{Ker} P$. It follows that $\alpha(0) = 0$, so $\alpha = 0$.

This shows that for every $s \in \ell^p(\mathbb{N},X)$ there is a unique $\gamma \in \ell^\infty(\mathbb{N},X)$ with the property that the pair (γ, s) verifies the equation (E_d) and $\gamma(0) \in \operatorname{Ker} P$. Then it makes sense to consider the operator $Q : \ell^p(\mathbb{N},X) \to \ell^\infty(\mathbb{N},X)$, $Q(s) = \gamma$, where $\gamma \in \ell^\infty(\mathbb{N},X)$ with the property that the pair (γ, s) verifies the equation (E_d) and $\gamma(0) \in \operatorname{Ker} P$. It is easy to see that Q is a closed linear operator, so it is bounded. Taking $M = ||Q||$ we obtain the relation (4).

Step 2. From Theorem 3, it makes sense to consider $S(t) = T_2(t)^{-1}$, for all $t \geq 0$. We have that $\mathbf{S} = \{S(t)\}_{t\geq 0}$ is an exponentially bounded semigroup on $\operatorname{Ker} P$. Let $u \in \ell^p(\mathbb{N}, \operatorname{Ker} P)$ and let

$$\gamma_u : \mathbb{N} \to \text{Ker } P, \quad \gamma_u(n) = \sum_{k=0}^{n} S(n-k)u(k).$$

Let $n \in \mathbb{N}$. We consider the sequences

$$\alpha : \mathbb{N} \to X, \quad \alpha(k) = \begin{cases} \quad 0 \qquad , & k \geq n+2 \text{ or } k = 0 \\ -T(1)u(n+1-k), & k \in \{1, ..., n+1\}. \end{cases}$$

$$\delta : \mathbb{N} \to X, \quad \delta(k) = \begin{cases} \quad 0 \qquad , & k \geq n+1 \\ \gamma_u(n-k), & k \in \{0, ..., n\}. \end{cases}$$

An easy computation shows that the pair (δ, α) verifies the equation (E_d). Since $\delta(0) = \gamma_u(n) \in \operatorname{Ker} P$ from Step 1 it follows that $||\delta||_\infty \leq M\,||\alpha||_p$. Taking into account that

$$||\alpha||_p = (\sum_{k=1}^{n+1} ||T(1)u(n+1-k)||^p)^{1/p} \leq ||T(1)||\,||u||_p,$$

we obtain that $||\gamma_u(n)|| = ||\alpha(0)|| \leq M\,||T(1)||\,||u||_p$. Since $n \in \mathbb{N}$ was arbitrary, it follows that $\gamma_u \in \ell^\infty(\mathbb{N}, \operatorname{Ker} P)$.

Thus, it results that the semigroup $\mathbf{S} = \{S(t)\}_{t\geq 0}$ is $(\ell^p(\mathbb{N}, \operatorname{Ker} P), \ell^\infty(\mathbb{N}, \operatorname{Ker} P))$-stable. Applying Theorem 1, we deduce that $\mathbf{S}$ is uniformly exponentially stable, which completes the proof. □

The main result of this section is:

Theorem 5. *An exponentially bounded semigroup* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *is exponentially dichotomic if and only if the pair* $(\ell^\infty(\mathbb{N}, X), \ell^p(\mathbb{N}, X))$ *is admissible for* $\mathbf{T}$ *and the subspace*

$$X_1 = \{x \in X : \sup_{t\geq 0} ||T(t)x|| < \infty\}$$

is closed and complemented in X.

Proof. Necessity. Let P be the projection and let $K, \nu > 0$ be given by Definition 5. Obviously, $\operatorname{Im} P \subset X_1$. Conversely, if $x \in X_1$ then

$$||x - Px|| \leq K\mathrm{e}^{-\nu t}||T(t)(I-P)x|| \leq K\mathrm{e}^{-\nu t}(||T(t)x|| + K\mathrm{e}^{-\nu t}||Px||), \quad \forall t \geq 0.$$

This implies that $x \in \operatorname{Im} P$. It follows that $X_1 = \operatorname{Im} P$, so it is closed and it has a complement – $\operatorname{Ker} P$ – which is $\mathbf{T}$-invariant.

For $s \in \ell^p(\mathbb{N}, X)$, we consider the sequence

$$\gamma : \mathbb{N} \to X, \quad \gamma(n) = \sum_{k=0}^{n} T(n-k)Ps(k) - \sum_{k=n+1}^{\infty} T(k-n)_|^{-1}(I-P)s(k),$$

where $T(j)_{|}^{-1}$ denotes the inverse of the operator $T(j)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$. We have that $\gamma \in \ell^\infty(\mathbb{N}, X)$ and an easy computation shows that the pair (γ, s) verifies the equation (E_d). It results that the pair $(\ell^\infty(\mathbb{N}, X), \ell^p(\mathbb{N}, X))$ is admissible for $\mathbf{T}$.

Sufficiency. It follows from Remark 1 and Theorems 2–4. □

REFERENCES

[1] Chicone, C. and Latushkin, Y., *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys Monogr., **70**, Providence, R. I., Amer. Math. Soc., 1999.

[2] Da Prato, G. and Zabczyk, J., *Second Order Partial Differential Equations in Hilbert Spaces*, London Mathematical Society, Lecture Note Series, **293**, Cambridge University Press, 2002.

[3] Daleckii, J.L. and Krein, M.G., *Stability of Solutions of Differential Equations in Banach Spaces*, Trans. Math. Monographs, **43**, Amer. Math. Soc., Providence R.I., 1974.

[4] Goldstein, J., *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, New York, Oxford University Press, 1985.

[5] Massera, J.L. and Schäffer, J.J., *Linear Differential Equations and Function Spaces*, Academic Press, New York, Londra, 1966.

[6] Megan, M., Sasu, A.L., Sasu, B. and Pogan, A., *Exponential stability and instability of semigroups of linear operators in Banach spaces*, Math. Ineq. Appl., **5** (2002), 557–568.

[7] Megan, M., Sasu, A.L. and Sasu, B., *The Asymptotic Behaviour of Evolution Families*, Editura Mirton, 2003.

[8] Megan, M., Sasu, A.L. and Sasu, B., *Discrete admissibility and exponential dichotomy for evolution families*, Discret. Contin. Dynam. Systems, **9** (2003), 383–397.

[9] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, Heidelberg, New York, 1983.

[10] Rus, I.A., Petruşel, A. and Sîntămărian, A., *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., Theory Methods Appl., **52** (2003), 1947–1959.

[11] Sasu, B., *Perron conditions for exponential expansiveness of one-parameter semigroups*, Matematiche (Catania), **58** (2003), 101–115.

[12] Sasu, B., *Generalizations of a theorem of Rolewicz*, Appl. Anal., **84** (2005), 1165–1172.

[13] Sasu, B. and Sasu, A.L., *Stability and stabilizability for linear systems of difference equations*, J. Differ. Equations Appl., **12** (2004), 1085–1105.

[14] Van Minh, N., Räbiger, F. and Schnaubelt, R., *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory, **32** (1998), 332–353.

[15] van Neerven, J., *The Asymptotic Behaviour of Semigroups of Linear Operators*, Operator Theory Adv. Appl., **88**, Birkhäuser, Bassel, 1996.

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