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SOME PROPERTIES OF CERTAIN FUNCTIONS CONCERNING HYPERGEOMETRIC FUNCTIONS IN SOME CLASSES OF UNIVALENT FUNCTIONS

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Abstract. In this paper we define the function $E(a,b,c,d,z)$ concerning hypergeometric function. The main aim of the present paper is to obtain conditions for a,b,c,d such that the functions $zE(a,b,c,d,z)$ and $z(2-E(a,b,c,d,z))$ belong to some classes of univalent functions. Also several operators related to the above functions are investigated.

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1. INTRODUCTION

Let A denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the open unit disk U and satisfy the normalized conditions $f(0)=f'(0)-1=0$. The subclass of A consisting of functions of the form $f(z)=z-\sum_{n=2}^{\infty}a_nz^n, a_n\geq 0$, is denoted by T. Recently the authors have investigated in [8] the following class:

$$\mathbb{D}(\alpha,\beta,\lambda)=\left\{f\in T:\mathrm{Re}\frac{z(\mathcal{D}^{\lambda}f(z))'}{\mathcal{D}^{\lambda}f(z)}>\alpha\left|\frac{z(\mathcal{D}^{\lambda}f(z))'}{\mathcal{D}^{\lambda}f(z)}-1\right|+\beta\right\},$$

where $\alpha\geq 0, 0\leq\beta<1, \lambda\geq 1$ and $D^{\lambda}f(z)$ given by

$$\mathcal{D}^{\lambda}f(z)=\frac{z}{(1-z)^{1+\lambda}}*f(z)=z+\sum_{n=2}^{\infty}\frac{(\lambda+1)_{n-1}}{(n-1)!}a_nz^n$$

is the familiar Ruscheweyh derivative and $(\gamma)_n$ is the Pochhammer symbol defined by

$$(\gamma)_n=\begin{cases}\gamma(\gamma+1)\cdots(\gamma+n-1), & n=1,2,\cdots\\ 1, & n=0\end{cases}$$

Now, with the above notations we define the class $\mathbb{D}^{+}(\alpha,\beta,\lambda)$ as follows:

$$\mathbb{D}^{+}(\alpha,\beta,\lambda)=\left\{f\in A:\mathrm{Re}\frac{z(\mathcal{D}^{\lambda}f(z))'}{\mathcal{D}^{\lambda}f(z)}>\alpha\left|\frac{z(\mathcal{D}^{\lambda}f(z))'}{\mathcal{D}^{\lambda}f(z)}-1\right|+\beta\right\}.$$

Considering the fact that $\mathcal{D}^{\lambda}(zf')(z)=z(\mathcal{D}^{\lambda}f(z))'$ we introduce the following classes:

$$\mathbb{D}_1(\alpha,\beta,\lambda)=\left\{f(z)\in T:zf'\in\mathbb{D}(\alpha,\beta,\lambda)\right\},$$
$$\mathbb{D}_1^{+}(\alpha,\beta,\lambda)=\left\{f(z)\in A:zf'\in\mathbb{D}(\alpha,\beta,\lambda)\right\}.$$

Obviously, putting $\alpha = 0, \lambda = 0$ in both $\mathbb{D}(\alpha,\beta,\lambda), \mathbb{D}^+(\alpha,\beta,\lambda)$ we get the class of functions that are starlike of order β, denoted by $S^*(\beta)$. Replacing $\alpha = 0, \lambda = 1$ in those classes we get the class of functions that are convex of order β, denoted by $K(\beta)$. Also, putting $\beta = \lambda = 0$ we obtain the class of uniformly α-starlike functions introduced by S. Kanas and A. Wiśniowska [4]. Also, the classes $\mathbb{D}_1(\alpha,\beta,\lambda), \mathbb{D}_1^+(\alpha,\beta,\lambda)$ are reduced to the class $K(\beta)$ when $\alpha = 0, \lambda = 0$, and when $\alpha = 1, \beta = 0, \lambda = 1$ we obtain UCV, the class of uniformly convex functions introduced by Goodman [3], and when $\lambda = 0, \alpha = 1, \beta = 0$ we obtain the class S_p introduced by Rønning [6].

During this paper we need the following theorems (see [8]).

Theorem A. *The condition* $\sum_{n=2}^{\infty} \frac{[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)(n-1)!}|a_n| \le 1$ *is a sufficient condition for* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *to be in* $\mathbb{D}^+(\alpha,\beta,\lambda)$. *Also* $f(z) \in T$ *belongs to* $\mathbb{D}(\alpha,\beta,\lambda)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)(n-1)!} \le 1.$$

Theorem B. *The condition* $\sum_{n=2}^{\infty} \frac{n[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)(n-1)!}|a_n| \le 1$ *is a sufficient condition for* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *to be in* $\mathbb{D}_1^+(\alpha,\beta,\lambda)$. *Also the function* $f(z) \in T$ *belongs to* $\mathbb{D}_1(\alpha,\beta,\lambda)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{n[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)(n-1)!} a_n \le 1.$$

Next let a, b, c, d be real numbers such that $c, d \ne 0, -1, -2, \ldots$. Define $E(a,b,c,d,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (d)_n} z^n$. In case $d = 1$ this function reduces to $F(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$, the familiar hypergeometric function and it is well known that $F(a,b,c,1)$ converges for $\mathrm{Re}(c-a-b) > 0$ and $F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, see [2]. Some authors, for instance Ruscheweyh and Singh [7], Merkes and Scott [5], have found sufficient conditions such that $zF(a,b,c,z) \in S^*(\beta), 0 \le \beta < 1$, for different values of parameters a, b, c. Carlson and Shaffer [1] showed how some convolution results about $S^*(\beta)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Also Silverman [9] determined conditions for $zF(a,b,c,z)$ and some related functions to be in $S^*(\beta), S_1^*(\beta), K(\beta), K_1(\beta)$ where $S_1^*(\beta)$ is the subclass of $S^*(\beta)$ consisting of functions f for which we have $|(zf'/f) - 1| < 1 - \beta$, $z \in U$. Also $K_1(\beta)$ includes functions f such that $zf' \in S_1^*(\beta)$.

In this paper we will indicate some conditions on $a, b, c, d, \alpha, \beta, \lambda$, in such a way that $zE(a,b,c,d,z)$ and several relevant functions to be in $\mathbb{D}(\alpha,\beta,\lambda)$, $\mathbb{D}_1(\alpha,\beta,\lambda)$, $\mathbb{D}^+(\alpha,\beta,\lambda)$, $\mathbb{D}_1^+(\alpha,\beta,\lambda)$.

THEOREM 1. *Let $a, b > 0, c > a+b+2, \lambda < d$. Then a sufficient condition for $zE(a,b,c,d,z) \in \mathbb{D}^{+}(\alpha,\beta,\lambda)$ is*

$$\frac{\Gamma(c)\Gamma(c-a-b-2)}{\alpha(1-\beta)\Gamma(c-a)\Gamma(c-b)} N \leq 2\,, \tag{1}$$

where $N = (1+\alpha)(a)_2(b)_2 + ab(c-a-b-2)[(1+\alpha)(1+d)+1-\beta] + d(1-\beta)(c-a-b-2)_2$. This condition is necessary and sufficient for $E_1(a,b,c,d,z) \in \mathbb{D}(\alpha,\beta,\lambda)$, where $E_1(a,b,c,d,z) = z(2 - E(a,b,c,d,z))$.

Proof. Since $zE(a,b,c,d,z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} z^n$, by Theorem A we must show

$$\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} \frac{(\lambda+1)_{n-1}}{(n-1)!} \leq 1-\beta\,. \tag{2}$$

Therefore by making use of (1) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} \frac{(\lambda+1)_{n-1}}{(n-1)!} \\
&< \sum_{n=1}^{\infty} [n(1+\alpha) + (1-\beta)] \frac{(a)_n(b)_n}{(c)_n n!} \frac{d+n}{d} \\
&= \frac{1+\alpha}{d} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n-2)!} + \frac{(1+\alpha)(1+d)+1-\beta}{d} \\
&\quad \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n-1)!} + (1-\beta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \\
&= \frac{1+\alpha}{d} \frac{(a)_2(b)_2}{(c)_2} \sum_{n=2}^{\infty} \frac{(a+2)_{n-2}(b+2)_{n-2}}{(c+2)_{n-2}(n-2)!} \\
&\quad + \frac{(1+\alpha)(1+d)+1-\beta}{d} \frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(n-1)!} \\
&\quad + (1-\beta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b-2)}{d\Gamma(c-a)\Gamma(c-b)} N - (1-\beta) \leq (1-\beta).
\end{aligned}$$

Finally, since $E_1(a,b,c,d,z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} z^n$, by Theorem A the condition (1) is necessary and sufficient for $E_1(a,b,c,d,z) \in \mathbb{D}(\alpha,\beta,\lambda)$. □

THEOREM 2. (i) *Let $f \in \mathbb{D}(\alpha,\beta,\lambda)$, $a,b,c,d > 0$, $ab \leq cd$. Then $f(z) * (zE(a,b,c,d,z)) \in \mathbb{D}(\alpha,\beta,\lambda)$.*

(ii) *Let $f \in \mathbb{D}(\alpha,\beta,\lambda)$, $0 < d$. Then $f(z) * (zE(a,b,c,d,z)) \in \mathbb{D}(\alpha,\beta,\lambda_1)$ if*

$$\lambda_1 \leq \inf_n \left[\frac{(\lambda+1)_{n-1}(d)_n \Gamma(c-a)\Gamma(c-b)}{n![\Gamma(c)\Gamma(c-a-b) - \Gamma(c-a)\Gamma(c-b)]} \right]^{1/n} - 1.$$

Proof. (i) Since $f \in \mathbb{D}(\alpha, \beta, \lambda)$, we have:

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} \frac{(\lambda+1)_{n-1}}{(n-1)!} a_n \leq 1 \tag{3}$$

and hence $f(z) * (zE(a,b,c,d,z)) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} a_n z^n$. Therefore, by considering the condition $ab \leq cd$, we have:

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} \frac{(\lambda+1)_{n-1}}{(n-1)!} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} a_n \\ &\leq \sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} \frac{(\lambda+1)_{n-1}}{(n-1)!} a_n \leq 1 \end{aligned}$$

and this completes the proof of this part.

(ii) By Theorem A we must show

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} \frac{(\lambda_1+1)_{n-1}}{(n-1)!} a_n \leq 1\,. \tag{4}$$

We have:

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} \frac{(\lambda_1+1)_{n-1}}{(n-1)!} a_n \\ &\leq \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} \frac{(\lambda_1+1)_{n-1}}{(\lambda+1)_{n-1}} = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(d)_n} \frac{(\lambda_1+1)_{n-1}}{(\lambda+1)_{n-1}}. \end{aligned}$$

However, the inequality (4) holds true if

$$\frac{(\lambda_1+1)_{n-1}}{(\lambda+1)_{n-1}(d)_n} \leq \frac{1}{n!} \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)-\Gamma(c-a)\Gamma(c-b)}.$$

Equivalently, we must have

$$(\lambda_1+1)^n \leq \frac{(\lambda+1)_{n-1}(d)_n}{n!} \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)-\Gamma(c-a)\Gamma(c-b)}.$$

This completes the proof. □

Theorem 3. *If* $c > 0$, $a, b > -1$, $ab < 0$, $c > a+b+2$ *and* $0 < \lambda < d$, *then* $zE(a,b,c,d,z) \in \mathbb{D}(\alpha,\beta,\lambda)$ *if and only if*

$$\frac{|ab|(c-a-b-2)_2\Gamma(c+1)\Gamma(c-a-b-2)}{cd\Gamma(c-a)\Gamma(c-b)} A \leq 2,$$

where $A = \frac{(a+1)(b+1)(1+\alpha)}{(c-a-b-2)_2} + \frac{(d+1)(1+\alpha)+1-\beta}{c-a-b-1} + \frac{1-\beta}{ab}$.

Proof. Since $zE(a,b,c,d,z) = z - \frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(d)_{n-1}}$, by Theorem A we must show:

$$\sum_{n=2}^{\infty}\frac{[n(1+\alpha)-(\alpha+\beta)](a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}(n-1)!} \le \frac{c(1-\beta)}{|ab|}\,. \tag{5}$$

Therefore we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{[n(1+\alpha)-(\alpha+\beta)](a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}(n-1)!}\\
&< \frac{1}{d}\sum_{n=0}^{\infty}\frac{[(n+1)(1+\alpha)+1-\beta](n+1+d)(a+1)_n(b+1)_n}{(c+1)_n(n+1)!}\\
&= \frac{(a+1)(b+1)(1+\alpha)}{d(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n n!}\\
&\quad+\frac{(d+1)(1+\alpha)+1-\beta}{d}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}+\frac{c(1-\beta)}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\
&= \frac{(c-a-b-2)_2\Gamma(c+1)\Gamma(c-a-b-2)}{d\Gamma((c-a)\Gamma(c-b)}\\
&\quad\cdot\left[\frac{(a+1)(b+1)(1+\alpha)}{(c-a-b-2)_2}+\frac{(d+1)(1+\alpha)+1-\beta}{c-a-b-1}+\frac{1-\beta}{ab}\right]\\
&\quad-\frac{c(1-\beta)}{|ab|}\le\frac{c(1-\beta)}{|ab|}.
\end{aligned}$$

□

THEOREM 4. *Under the conditions of* Theorem 1, *let* $E_1(a,b,c,d,z)$ *belong to* $D(\alpha,\beta,\lambda)$. *Then it is close-to-convex of order* γ $(0\le\gamma<1)$ *in* $|z|<r$, *where*

$$r \le \inf_n\left\{\frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)n!}\right\}^{\frac{1}{n-1}}\,. \tag{6}$$

Proof. Since $E_1(a,b,c,d,z)\in\mathbb{D}(\alpha,\beta,\lambda)$, we have:

$$\sum_{n=2}^{\infty}\frac{[n(1+\alpha)-(\alpha+\beta)](a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(1-\beta)(c)_{n-1}(d)_{n-1}(n-1)!}\le 1\,. \tag{7}$$

Now we must show

$$|E_1'(a,b,c,d,z)-1|<1-\gamma\,. \tag{8}$$

Therefore, we have $|E_1'(a,b,c,d,z)-1|\le\sum_{n=2}^{\infty}n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}}|z|^{n-1}$. However, in view of (7) the inequality (8) holds true if

$$n|z|^{n-1}\le\frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)](\lambda+1)_{n-1}}{(1-\beta)(n-1)!},$$

and this gives (6). □

THEOREM 5. *Under the conditions of* Theorem 1, *suppose* $E_1(a,b,c,d,z) \in \mathbb{D}(\alpha,\beta,\lambda)$. *Then* $H(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}E_1(a,b,c,d,t)\mathrm{d}t$, $\gamma > -1$, *also belongs to* $\mathbb{D}(\alpha,\beta,\lambda)$.

Proof. After an easy calculation we get $H(z) = z - \sum_{n=2}^\infty \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}}\frac{\gamma+1}{n+\gamma}z^n$. Also, since $E_1(a,b,c,d,z) \in \mathbb{D}(\alpha,\beta,\lambda)$, we have:

$$\sum_{n=2}^\infty \frac{[n(1+\alpha)-(\alpha+\beta)](\gamma+1)(a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(1-\beta)(n+\gamma)(c)_{n-1}(d)_{n-1}(n-1)!}$$
$$< \sum_{n=2}^\infty \frac{[n(1+\alpha)-(\alpha+\beta)](a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(1-\beta)(c)_{n-1}(d)_{n-1}(n-1)!} \le 1.$$

and this completes the proof. □

THEOREM 6. (i) *If* $a,b > 0$, $\lambda < d$ *and* $c > a+b+1$, *then a sufficient condition for* $H_1(z) = \int_0^z E(a,b,c,d,t)\mathrm{d}t$ *to be in* $\mathbb{D}^+(0,\beta,\lambda)$ *is that*

$$\frac{\Gamma(c)\Gamma(c-a-b-1)}{d(1-\beta)\Gamma(c-a)\Gamma(c-b)}B - \frac{(1-d)(c-1)\beta}{d(a-1)(b-1)(1-\beta)} \le 2\,, \tag{9}$$

where $B = ab + (d-\beta)(c-a-b-1) + \frac{\beta(1-d)(c-a-b-1)_2}{(a-1)(b-1)}$.

(ii) *If* $a,b > -1$, $c > 0$, $ab < 0$, $c > a+b+1$ *and* $\lambda < d$, *then* $H_1(z) \in \mathbb{D}(0,\beta,\lambda)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{d\Gamma(c-a)\Gamma(c-b)}D + \frac{\beta(d-1)(c-1)_2}{d(a-1)_2(b-1)_2} \le 0\,, \tag{10}$$

where $D = 1 + \frac{(d-\beta)(c-a-b-1)}{ab} - \frac{\beta(d-1)(c-a-b-1)_2}{(a-1)_2(b-1)_2}$.

Proof. (i) It is easy to see that $H_1(z) = z + \sum_{n=2}^\infty \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}}\frac{z^n}{n}$. Now we must show $\sum_{n=2}^\infty \frac{(n-\beta)(a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}n!} \le 1-\beta$. Considering (9) we have:

$$\sum_{n=2}^\infty \frac{(n-\beta)(a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}n!} < \frac{1}{d}\sum_{n=1}^\infty \frac{(n+1-\beta)(d+n)(a)_n(b)_n}{(c)_n(n+1)!}$$
$$= \frac{1}{d}\sum_{n=1}^\infty \frac{(a)_n(b)_n}{(c)_n(n-1)!} + \frac{d-\beta}{d}\sum_{1}^\infty \frac{(a)_n(b)_n}{(c)_n n!} + \frac{(1-d)\beta}{d}\sum_{n=1}^\infty \frac{(a)_n(b)_n}{(c)_n(n+1)!}$$
$$= \frac{\Gamma(c)\Gamma(c-a-b-1)}{d\Gamma(c-a)\Gamma(c-b)}B - \frac{(1-d)(c-1)\beta}{d(a-1)(b-1)} - (1-\beta) \le 1-\beta.$$

(ii) We can write $H_1(z) = z - \frac{|ab|}{c}\sum_{n=2}^\infty \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(d)_{n-1}}\frac{z^n}{n}$. According to Theorem A, we know that $f \in \mathbb{D}(0,\beta,\lambda)$ if and only if

$$\sum_{n=2}^\infty \frac{(n-\beta)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}n!} \le \frac{c(1-\beta)}{|ab|}\,. \tag{11}$$

Therefore by making use of (10) for showing (11) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(n-\beta)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}n!}\\
&<\sum_{n=2}^{\infty}\frac{(n-\beta)(d+n-1)(a+1)_{n-2}(b+1)_{n-2}}{d(c+1)_{n-2}n!}\\
&=\frac{1}{d}\Bigg[\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}+\frac{c(d-\beta)}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\
&\quad-\frac{\beta c(c-1)(d-1)}{ab(a-1)(b-1)}\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n n!}\Bigg]\\
&=\frac{c}{|ab|}(1-\beta)+\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{d\Gamma(c-a)\Gamma(c-b)}D\\
&\quad+\frac{\beta(d-1)(c-1)_2}{d(a-1)_2(b-1)_2}\le\frac{cd}{|ab|}(1-\beta)
\end{aligned}$$

This completes the proof. □

Theorem 7. (i) *If $a,b>0$, $c>a+b+2$ and $\lambda<d$, then a sufficient condition for $H_2(z)=\frac{\gamma+1}{z^{\gamma}}\int_0^z t^{\gamma}E(a,b,c,d,t)\mathrm{d}t\in\mathbb{D}^{+}(0,\beta,\lambda)$ is that*

$$\frac{(c-a-b-2)_2\Gamma(c)\Gamma(c-a-b-2)}{d(1-\beta)\Gamma(c-a)\Gamma(c-b)}E\le 2\,, \tag{12}$$

where $E=(1-\beta)+\frac{ab(d+2-\beta)}{c-a-b-1}+\frac{(a)_2(b)_2}{(c-a-b-2)_2}$.

(ii) *If $a,b>-1$, $ab<0<c$, $\lambda<d$, then $H_2(z)\in\mathbb{D}(0,\beta,\lambda)$ if and only if*

$$\frac{(a+1)(b+1)+(d+2-\beta)(c-a-b-2)}{d}+\frac{(1-\beta)(c-a-b-2)}{ab}\le 0\,. \tag{13}$$

Proof. (i) An easy computation gives $H_2(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(\gamma+1)}{(c)_{n-1}(d)_{n-1}(n+\gamma)}z^n$. By Theorem A we must show $\sum_{n=2}^{\infty}\frac{(n-\beta)(a)_{n-1}(b)_{n-1}(\gamma+1)(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}(n+\gamma)(n-1)!}\le 1-\beta$, so with respect to (12) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(n-\beta)(a)_{n-1}(b)_{n-1}(\gamma+1)(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}(n+\gamma)(n-1)!}\\
&<(1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}+\frac{ab(d+2-\beta)}{cd}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}\\
&\quad+\frac{(a)_2(b)_2}{d(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n n!}+\frac{ab}{cd}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}\\
&=\frac{(c-a-b-2)_2\Gamma(c)\Gamma(c-a-b-2)}{d\Gamma(c-a)\Gamma(c-b)}E-(1-\beta)\le 1-\beta.
\end{aligned}$$

(ii) It is obvious that $H_2(z) = z - \frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}(\gamma+1)}{(c+1)_{n-2}(d)_{n-1}(n+\gamma)}z^n$. According to theorem A, $H_2(z)\in\mathbb{D}^+(0,\beta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}\frac{(n-\beta)(\gamma+1)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(n+\gamma)(c+1)_{n-2}(d)_{n-1}(n-1)!}\le\frac{c}{|ab|}(1-\beta)\,. \tag{14}$$

For showing (14) by making use of (13) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(n-\beta)(\gamma+1)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(n+\gamma)(c+1)_{n-2}(d)_{n-1}(n-1)!}\\
&<\frac{1}{d}\sum_{n=0}^{\infty}\frac{(n+2-\beta)(d+n+1)(a+1)_n(b+1)_n}{(c+1)_n(n+1)!}\\
&=\frac{(a+1)(b+1)}{d(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_{n-1}(b+2)_{n-1}}{(c+2)_{n-1}(n-1)!}\\
&\quad+\frac{d+2-\beta}{d}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}+\frac{(1-\beta)c}{ab}\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(n+1)!}\\
&=\frac{(1-\beta)c}{|ab|}+\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[\frac{(a+1)(b+1)}{d}\\
&\quad+\frac{(d+2-\beta)(c-a-b-2)}{d}+\frac{(1-\beta)(c-a-b-2)_2}{ab}\Bigg]\le\frac{(1-\beta)c}{|ab|}
\end{aligned}$$

and the proof is complete. □

THEOREM 8. *If $a,b>0$, $c>a+b+3$ and $\lambda<d$, then a sufficient condition for $zE(a,b,c,d,z)\in\mathbb{D}_1^+(\alpha,\beta,\lambda)$ is that:*

$$\frac{(c-a-b-3)_3\Gamma(c)\Gamma(c-a-b-3)}{d(1-\beta)\Gamma(c-a)\Gamma(c-b)}F\le 2\,, \tag{15}$$

where $F=(1-\beta)d+\frac{ab[2(d+1)(1+\alpha)+(d+2)(1-\beta)]}{c-a-b-1}+\frac{(a)_2(b)_2[(d+4)(1+\alpha)+1-\beta]}{(c-a-b-2)_2}+\frac{(a)_3(b)_3(1+\alpha)}{c-a-b-3}$. This condition is necessary and sufficient for $zE(a,b,c,d,z)\in\mathbb{D}_1(\alpha,\beta,\lambda)$.

Proof. According to Theorem B we must show

$$\sum_{n=2}^{\infty}\frac{n[n(1+\alpha)-(\alpha+\beta)](a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}(n-1)!}\le 1-\beta\,. \tag{16}$$

Making use of (15) we can write

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n(1+\alpha)-(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}(\lambda+1)_{n-1}}{(c)_{n-1}(d)_{n-1}(n-1)!}\\
&<\frac{1}{d}\sum_{n=2}^{\infty}n[n(1+\alpha)-(\alpha+\beta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}(d+n-1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+\alpha}{d}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-3)!} + \frac{(d+4)(1+\alpha)+1-\beta}{d}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-2)!} \\
&\quad + \frac{(d+2)(1+\alpha)+1-\beta}{d}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-1)!} \\
&\quad + \frac{d(1+\alpha)+(d+1)(1-\beta)}{d}\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!} \\
&\quad + (1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!} = \frac{(1+\alpha)(a)_3(b)_3\Gamma(c+3)\Gamma(c-a-b-3)}{d(c)_3\Gamma(c-a)\Gamma(c-b)} \\
&\quad + \frac{(a)_2(b)_2[(d+4)(1+\alpha)+1-\beta]\Gamma(c+2)\Gamma(c-a-b-2)}{d(c)_2\Gamma(c-a)\Gamma(c-b)} \\
&\quad + \frac{[2(d+1)(1+\alpha)+(d+2)(1-\beta)]ab\Gamma(c+1)\Gamma(c-a-b-1)}{dc\Gamma(c-a)\Gamma(c-b)} \\
&\quad + \frac{(1-\beta)\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - (1-\beta) \\
&= \frac{(c-a-b-3)_3\Gamma(c)\Gamma(c-a-b-3)}{d\Gamma(c-a)\Gamma(c-b)}F - (1-\beta) \\
&\le 1-\beta,
\end{aligned}
$$

and this completes the proof. □

THEOREM 9. *Let $a, b > -1$, $ab < 0$, $c > a+b+3$ and $\lambda < d$. Then $zE(a,b,c,d,z) \in \mathbb{D}_1(\alpha,\beta,\lambda)$ if and only if*

$$
\begin{aligned}
&\frac{(1+\alpha)(a+1)_2(b+1)_2}{d(c-a-b-3)_3} + \frac{(a+1)(b+1)[d(1+\alpha)+4\alpha-\beta+5]}{d(c-a-b-2)_2} \\
&+ \frac{2(d+1)(1+\alpha)+(d+2)(1-\beta)}{d(c-a-b-1)} - \frac{1-\beta}{|ab|} \le 0.
\end{aligned}
\tag{17}
$$

Proof. According to Theorem B we must show

$$
\sum_{n=2}^{\infty}\frac{n[n(1+\alpha)-(\alpha+\beta)](a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}(n-1)!} < \frac{c}{|ab|}(1-\beta)\,. \tag{18}
$$

For showing (18) we have:

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{n[n(1+\alpha)-(\alpha+\beta)](a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(c+1)_{n-2}(d)_{n-1}(n-1)!} \\
&< \sum_{n=2}^{\infty}\frac{n[n(1+\alpha)-(\alpha+\beta)](a+1)_{n-2}(b+1)_{n-2}(d+n-1)}{d(c+1)_{n-2}(n-1)!} \\
&= \sum_{n=0}^{\infty}\frac{(n+2)[(n+1)(1+\alpha)+1-\beta](a+1)_n(b+1)_n(d+n+1)}{d(c+1)_n(n+1)!}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{[n(1+\alpha)+2+\alpha-\beta](d+1+n)(a+1)_n(b+1)_n}{d(c+1)_n n!} \\
&\quad + \sum_{n=0}^{\infty} \frac{[n(1+\alpha)+2+\alpha-\beta](d+1+n)(a+1)_n(b+1)_n}{d(c+1)_n (n+1)!} \\
&= \frac{(1+\alpha)(a+1)_2(b+1)_2}{d(c+1)_2} \sum_{n=0}^{\infty} \frac{(a+3)_n(b+3)_n}{(c+3)_n n!} \\
&\quad + \frac{[d(1+\alpha)+5+4\alpha-\beta](a+1)(b+1)}{d(c+1)} \sum_{n=0}^{\infty} \frac{(a+2)_n(b+2)_n}{(c+2)_n n!} \\
&\quad + \frac{[2(d+1)(1+\alpha)+(d+2)(1-\beta)]}{d} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n n!} \\
&\quad - \frac{c(1-\beta)}{|ab|} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \\
&= \frac{\Gamma(c+1)\Gamma(c-a-b-3)(c-a-b-3)_3}{\Gamma(c-a)\Gamma(c-b)} G + \frac{(1-\beta)c}{|ab|} \le \frac{c(1-\beta)}{|ab|},
\end{aligned}
$$

where G is the left side of expression (17) and the proof is complete. □

Theorem 10. (i) *If $a, b, > 0$, $\lambda < d$ and $c \ge a+b+3$, then a sufficient condition for $H_2(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^\gamma E(a,b,c,d,t)\mathrm{d}t \in \mathbb{D}_1^+(0,\beta,\lambda)$ is that*

$$
\frac{\Gamma(c)\Gamma(c-a-b-3)(c-a-b-3)_3}{(1-\beta)\Gamma(c-a)\Gamma(c-b)} R \le 2\,, \tag{19}
$$

where $R = (1-\beta) + \frac{ab[(3-\beta)d+4-2\beta]}{d(c-a-b-1)} + \frac{(a)_2(b)_2(5+d-\beta)}{d(c-a-b-2/2)} + \frac{(a)_3(b)_3}{d(c-a-b-3)_3}$.

(ii) *If $a, b > -1$, $c > 0$, $ab < 0$ and $\lambda < d$, then $H_2(z)$ belongs to $\mathbb{D}(0,\beta,\lambda)$ if and only if*

$$
\frac{(c-a-b-3)_3\Gamma(c+1)\Gamma(c-a-b-3)}{d\Gamma(c-a)\Gamma(c-b)} W \le \frac{2(d-\beta+2)}{d}\,, \tag{20}
$$

where $W = \frac{(a+1)_2(b+1)_2}{c-a-b-3} + \frac{(a+1)(b+1)(5+d-\beta)}{(c-a-b-2)_2} + \frac{4-2\beta+(3-\beta)d}{c-a-b-1} + 1 - \beta$.

Proof. (i) It is sufficient to show that

$$
\sum_{n=2}^{\infty} \frac{n(n-\beta)(\gamma+1)(\lambda+1)_{n-1}(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}(n+\gamma)(n-1)!} \le 1-\beta\,. \tag{21}
$$

Making use of (19) we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{n(n-\beta)(\gamma+1)(\lambda+1)_{n-1}(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}(n+\gamma)(n-1)!}\\
&<\sum_{n=2}^{\infty}\frac{n(n-\beta)(a)_{n-1}(b)_{n-1}(d+n-1)}{d(c)_{n-1}(n-1)!}\\
&=\frac{1}{d}\left[\sum_{n=1}^{\infty}\frac{(n+1-\beta)(d+n)(a)_n(b)_n}{(c)_n(n-1)!}+\sum_{n=1}^{\infty}\frac{(n+1-\beta)(d+n)(a)_n(b)_n}{(c)_n n!}\right]\\
&=\frac{1}{d}\sum_{n=3}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-3)!}+\frac{5+d-\beta}{d}\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-2)!}\\
&\quad+\frac{(3-\beta)d+4-2\beta}{d}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(n-1)!}+(1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\
&=\frac{(a)_3(b)_3}{d(c)_3}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+3)_n}{(c+3)_n n!}\\
&\quad+\frac{(5+d-\beta)(a)_2(b)_2}{d(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n n!}\\
&\quad+\frac{ab[(3-\beta)d+4-2\beta]}{cd}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!}+(1-\beta)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\\
&=\frac{\Gamma(c)\Gamma(c-a-b-3)(c-a-b-3)_3}{\Gamma(c-a)\Gamma(c-b)}R-(1-\beta)\le 1-\beta,
\end{aligned}$$

(ii) Since $H_2(z)=z-\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}(\gamma+1)}{(c+1)_{n-2}(d)_{n-1}(n+\gamma)}z^n$ then by Theorem B we must show

$$\sum_{n=2}^{\infty}\frac{n(n-\beta)(\gamma+1)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(n+\gamma)(c+1)_{n-2}(d)_{n-1}(n-1)!}\le\frac{c(1-\beta)}{|ab|}\,. \tag{22}$$

Making use of (20) and doing some calculations we obtain:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{n(n-\beta)(\gamma+1)(a+1)_{n-2}(b+1)_{n-2}(\lambda+1)_{n-1}}{(n+\gamma)(c+1)_{n-2}(d)_{n-1}(n-1)!}\\
&<\frac{1}{d}\sum_{n=1}^{\infty}\frac{(n+2-\beta)(a+1)_n(b+1)_n(d+n+1)}{(c+1)_n n!}\\
&\quad+\frac{1}{d}\sum_{n=1}^{\infty}\frac{(n+2-\beta)(a+1)_n(b+1)_n(d+n+1)}{(c+1)_n(n+1)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(a+1)_2(b+1)_2}{d(c+1)_2}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+3)_n}{(c+3)_n n!} \\
&\quad + \frac{(a+1)(b+1)(5+d-\beta)}{d(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n n!} \\
&\quad + \frac{4-2\beta+(3-\beta)d}{d}\sum_{n=1}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n n!} + \frac{(1-\beta)c}{ab}\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!} \\
&= \frac{(c-a-b-3)_3\Gamma(c+1)\Gamma(c-a-b-3)}{d\Gamma(c-a)\Gamma(c-b)}W - \frac{2(d-\beta+2)}{d} + \frac{c(1-\beta)}{|ab|} \\
&\le \frac{c(1-\beta)}{|ab|}. \qquad \square
\end{aligned}
$$

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