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LOEWNER CHAINS AND A MODIFICATION OF THE ROPER-SUFFRIDGE EXTENSION OPERATOR

GABRIELA KOHR

Abstract. In this paper we continue the study of the Roper-Suffridge extension operator. Let f be a locally univalent function on the unit disc and let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. We consider the family of operators extending f to a holomorphic mapping from the unit ball B^n in $\mathbb{C}^n$ into $\mathbb{C}^n$ given by $\Phi_{n,Q}(f)(z) = (f(z_1) + Q(\tilde{z})f'(z_1), \widetilde{z}(f'(z_1))^{1/2})$, where $\widetilde{z} = (z_2, \ldots, z_n)$. This operator was recently introduced by Muir. In the case $Q \equiv 0$, this operator reduces to the well known Roper-Suffridge extension operator. We prove that if $f \in S$ then $\Phi_{n,Q}(f) \in S^0(B^n)$ whenever $\|Q\| \le 1/4$. Our proof yields Muir's result that if $f \in S^*$ then $\Phi_{n,Q}(f)$ is also starlike on B^n. Moreover, if $f \in K$ is imbedded in a convex subordination chain $f(z_1, t)$ over $[0, \infty)$ then $\Phi_{n,Q}(f)$ is also imbedded in a c.s.c. over $[0, \infty)$ on B^n whenever $\|Q\| \le 1/2$.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. For $n \ge 2$, let $\widetilde{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ so that $z = (z_1, \widetilde{z}) \in \mathbb{C}^n$. The unit ball in $\mathbb{C}^n$ is denoted by B^n. In the case of one variable, B^1 is denoted by U. The ball in $\mathbb{C}^n$ of radius $r > 0$ and center 0 is denoted by B^n_r.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm,

$$\|A\| = \sup\{\|A(z)\| : \ \|z\| = 1\}$$

and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. A mapping $Q : \mathbb{C}^n \to \mathbb{C}$ is called a homogeneous polynomial of degree k if there is a mapping $A : \prod_{j=1}^{k} \mathbb{C}^n \to \mathbb{C}$ which is continuous multilinear of degree k and

$$Q(z) = L(\underbrace{z, \cdots, z}_{k\text{-times}}),\ z \in \mathbb{C}^n.$$

Then $Q \in H(\mathbb{C}^n)$ and $DQ(z)(z) = kQ(z)$ for $z \in \mathbb{C}^n$.

If Ω is a domain in $\mathbb{C}^n$, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. Also let $H(B^n, \mathbb{C})$ be the set of holomorphic functions from B^n into

$\mathbb{C}$. A mapping $f \in H(B^n)$ is called normalized if $f(0) = 0$ and $Df(0) = I_n$. If $f \in H(B^n)$ we say that f is locally biholomorphic on B^n if the complex Jacobian matrix $Df(z)$ is nonsingular at each $z \in B^n$. Let $J_f(z) = \det Df(z)$ for $z \in B^n$. Let $\mathcal{L}S_n$ be the set of normalized locally biholomorphic mappings on B^n and let $S(B^n)$ denote the set of normalized biholomorphic mappings on B^n. In the case of one variable, the set $S(B^1)$ is denoted by S and $\mathcal{L}S_1$ is denoted by $\mathcal{L}S$. A mapping $f \in S(B^n)$ is called starlike (respectively convex) if its image is a starlike domain with respect to the origin (respectively convex domain). The classes of normalized starlike (respectively convex) mappings on B^n will be denoted by $S^*(B^n)$ (respectively $K(B^n)$). In the case of one variable, $S^*(B^1)$ (respectively $K(B^1)$) is denoted by S^* (respectively K).

If $f, g \in H(B^n)$ we say that f is subordinate to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$. If g is biholomorphic on B^n, this is equivalent to requiring that $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.

We recall that a mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on B^n, $f(0, t) = 0$, $Df(0, t) = \mathrm{e}^t I_n$ for $t \geq 0$, and $f(z, s) \prec f(z, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in B^n$. We note that the requirement $f(z, s) \prec f(z, t)$ is equivalent to the condition that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B^n,\ t \geq s \geq 0.$$

We also note that the normalization of $f(z, t)$ implies the normalization $Dv(0, s, t) = \mathrm{e}^{s-t} I_n$ for $0 \leq s \leq t < \infty$.

Certain subclasses of $S(B^n)$ can be characterized in terms of Loewner chains. In particular, $f \in S^*(B^n)$ if and only if $f(z, t) = \mathrm{e}^t f(z)$ is a Loewner chain.

The authors [4], [10] (see also [8, Theorem 8.1.6]; cf. [16] and [17]) obtained the following sufficient condition for a mapping to be a Loewner chain.

Lemma 1.1. *Let $h_t(z) = h(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) *$h(\cdot, t)$ is a normalized holomorphic mapping on B^n and* $\operatorname{Re} \langle h(z, t), z\rangle \geq 0$ *for $z \in B^n$, $t \geq 0$.*

(ii) *$h(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in B^n$.*

Let $f = f(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping such that $f(\cdot, t) \in H(B^n)$, $f(0, t) = 0$, $Df(0, t) = \mathrm{e}^t I_n$ for $t \geq 0$, and $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Assume that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t) h(z, t) \quad \text{a.e.} \quad t \geq 0, \forall z \in B^n.$$

Further, assume that there exists an increasing sequence $\{t_m\}_{m \in \mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and

$$\lim_{m \to \infty} \mathrm{e}^{-t_m} f(z, t_m) = F(z)$$

locally uniformly on B^n*. Then* $f(z,t)$ *is a Loewner chain.*

Graham, Hamada and Kohr [5] have recently introduced the notion of a convex subordination chain in $\mathbb{C}^n$. In the case of one variable, see [19].

Definition 1.2. Let J be an interval in $\mathbb{R}$. A mapping $f = f(z,t)$ is called a convex subordination chain (c.s.c.) over J if the following conditions hold:
(i) $f(0,t) = 0$ and $f(\cdot,t)$ is convex for $t \in J$.
(ii) $f(\cdot,t_1) \prec f(\cdot,t_2)$ for $t_1, t_2 \in J$, $t_1 \leq t_2$.

Definition 1.3. (see [11], [4]) We say that a normalized mapping $f \in H(B^n)$ has parametric representation if there exists a mapping $h : B^n \times [0,\infty) \to \mathbb{C}^n$ which satisfies the following conditions:
(i) $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$, $Dh(0,t) = I_n$, $t \geq 0$, $\mathrm{Re}\, \langle h(z,t), z\rangle \geq 0$, for $z \in B^n$, $t \geq 0$;
(ii) $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B^n$,
such that $f(z) = \lim\limits_{t\to\infty} \mathrm{e}^t v(z,t)$ locally uniformly on B^n, where $v = v(z,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t) \quad a.e. \quad t \geq 0,\, v(z,0) = z,$$

for all $z \in B^n$.

In [10] (see also [8]) it is proved that a mapping $f \in H(B^n)$ has parametric representation if and only if there exists a Loewner chain $f(z,t)$ such that $\{\mathrm{e}^{-t} f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n and $f = f(\cdot,0)$.

Let $S^0(B^n)$ be the set of mappings which have parametric representation on B^n.

Definition 1.4. (see [18]) The Roper-Suffridge extension operator $\Phi_n : \mathcal{L}S \to \mathcal{L}S_n$ is defined by

$$\Phi_n(f)(z) = \left(f(z_1), \widetilde{z}\sqrt{f'(z_1)}\right), \quad z = (z_1, \widetilde{z}) \in B^n.$$

We choose the branch of the power function such that

$$\sqrt{f'(z_1)}\Big|_{z_1=0} = 1.$$

Roper and Suffridge [18] proved that if f is convex on U then $\Phi_n(f)$ is also convex on B^n. Graham and Kohr [7] proved that if f is starlike on U then so is $\Phi_n(f)$ on B^n, and in [9] (see also [8]) it is shown that if $f \in S$ then $\Phi_n(f) \in S^0(B^n)$. On the other hand, Gong and Liu (see [2] and [3]) studied a number of properties of the Roper-Suffridge extension operator on some Reinhardt domains in $\mathbb{C}^n$.

Motivated by recent results concerning extreme points of the family $K(B^n)$, $n \geq 2$ (see [13] and [14]), Muir [12] introduced the following new extension operator that under certain conditions takes extreme points of K into extreme points of $K(B^n)$.

Definition 1.5. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. The modification Roper-Suffridge extension operator $\Phi_{n,Q} : \mathcal{L}S \to \mathcal{L}S_n$ is defined by

$$\Phi_{n,Q}(f)(z) = \left(f(z_1) + Q(\widetilde{z})f'(z_1), \widetilde{z}\sqrt{f'(z_1)}\right), \quad z = (z_1, \widetilde{z}) \in B^n.$$

We choose the branch of the power function such that

$$\sqrt{f'(z_1)}\Big|_{z_1=0} = 1.$$

Muir [12] proved that if $\|Q\| \le 1/2$ then the operator $\Phi_{n,Q}$ preserves convexity and if $\|Q\| \le 1/4$ then $\Phi_{n,Q}$ preserves starlikeness. In this paper we prove that if $f \in S$ and $\|Q\| \le 1/4$ then $\Phi_{n,Q} \in S^0(B^n)$. In particular, if $f \in S^*$ then $\Phi_{n,Q} \in S^*(B^n)$ whenever $\|Q\| \le 1/4$. Moreover, if $f \in K$ is imbedded in a convex subordination chain $f(z_1, t)$ over $[0, \infty)$ then $\Phi_{n,Q}(f)$ is also imbedded in a convex subordination chain over $[0, \infty)$ on B^n whenever $\|Q\| \le 1/2$.

2. LOEWNER CHAINS AND THE OPERATOR $\Phi_{N,Q}$

We begin this section with the following result. In the case $Q \equiv 0$, see [8] and [9].

Theorem 2.1. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/4$ and let $f(z_1, t) : U \times [0, \infty) \to \mathbb{C}$ be a Loewner chain. Also let $F(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be the mapping given by*
(2.1)

$$F(z,t) = \left(f(z_1,t) + Q(\widetilde{z})f'(z_1,t), \widetilde{z}e^{t/2}(f'(z_1,t))^{1/2}\right), z = (z_1, \widetilde{z}) \in B^n, t \ge 0.$$

We choose the branch of the power function such that $(f'(z_1,t))^{1/2}|_{z_1=0} = e^{t/2}$ for $t \ge 0$. Then $F(z,t)$ is a Loewner chain.

Proof. Clearly $F(0,t) = 0$ and since Q is a homogeneous polynomial of degree 2, it follows that $DF(0,t) = e^t I_n$ for $t \ge 0$. It is easily seen that $e^{-t}F(z,t) = \Phi_{n,Q}(e^{-t}f(\cdot,t))(z)$ for $z \in B^n$ and $t \ge 0$. Also it is not difficult to deduce that $F(\cdot,t)$ is biholomorphic on B^n. On the other hand, since $f(z_1,t)$ is a Loewner chain, $f(z_1,\cdot)$ is locally absolutely continuous on $[0,\infty)$, locally uniformly with respect to $z_1 \in U$, and there is a function $p(z_1,t)$ such that $p(\cdot,t) \in H(U)$, $p(0,t) = 1$, $\mathrm{Re}\, p(z_1,t) > 0$, $|z_1| < 1$, $t \ge 0$, and

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t)p(z_1,t) \quad a.e. \quad t \ge 0, \forall z_1 \in U.$$

Moreover, the limit

$$\lim_{t\to\infty} e^{-t} f(z_1,t) = g(z_1)$$

exists locally uniformly on U (see e.g. [8]). Clearly g is a holomorphic function on U and since $g(0) = 0$, $g'(0) = 1$, we deduce by Hurwitz's theorem that

$g \in S$. Then $F(z,\cdot)$ is also locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$ and

$$\lim_{t\to\infty} \mathrm{e}^{-t}F(z,t) = \Phi_{n,Q}(g)(z)$$

locally uniformly on B^n.

Now, let

$$h(z,t) = \left(z_1 p(z_1,t) - Q(\widetilde{z}), \frac{\widetilde{z}}{2}\left(1 + p(z_1,t) + z_1 p'(z_1,t) + Q(\widetilde{z})\frac{f''(z_1,t)}{f'(z_1,t)}\right)\right),$$

for all $z \in B^n$ and $t \geq 0$. Then $h(\cdot,t)$ is a normalized holomorphic mapping on B^n for $t \geq 0$ and $h(z,\cdot)$ is measurable on $[0,\infty)$ for all $z \in B^n$. Using elementary computations and the equality (see e.g. [8, Chapter 11])

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z_1}\right)(z_1,t) = \frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial t}\right)(z_1,t) \quad a.e. \quad t \geq 0, \forall z_1 \in U,$$

we obtain that

$$\frac{\partial F}{\partial t}(z,t) = DF(z,t)h(z,t) \quad a.e. \quad t \geq 0, \forall z \in B^n.$$

On the other hand, since $\mathrm{e}^{-t}f(\cdot,t) \in S$, $t \geq 0$, it is well known that

$$\left|\frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1\right| \leq 2,\ |z_1| < 1,\ t \geq 0. \tag{2.2}$$

Next, using the fact that $\|Q\| \leq 1/4$, the above inequality and arguments similar to those in the proof of [6, Theorem 2.1], we obtain that $\mathrm{Re}\,\langle h(z,t), z\rangle \geq 0$ for $z \in B^n$ and $t \geq 0$. Indeed, if $\widetilde{z} = 0$ then

$$\mathrm{Re}\,\langle h(z,t), z\rangle = |z_1|^2 \mathrm{Re}\, p(z_1,t) \geq 0,\ |z_1| < 1.$$

Next, we assume that $\widetilde{z} \neq 0$. Then it is easy to see that $h(\cdot,t)$ is holomorphic in a neighborhood of each point $z = (z_1,\widetilde{z}) \in \overline{B}^n$ with $\widetilde{z} \neq 0$, and in view of the minimum principle for harmonic functions, it suffices to prove that

$$\mathrm{Re}\,\langle h(z,t), z\rangle \geq 0,\ z = (z_1,\widetilde{z}) \in \partial B^n,\ \widetilde{z} \neq 0,\ t \geq 0.$$

Since $p(0,t) = 1$ and $\mathrm{Re}\, p(z_1,t) > 0$, it follows that (see e.g. [8])

$$|p'(z_1,t)| \leq \frac{2}{1-|z_1|^2}\mathrm{Re}\, p(z_1,t),\ |z_1| < 1,\ t \geq 0. \tag{2.3}$$

Fix $t \geq 0$ and let $z = (z_1,\widetilde{z}) \in \partial B^n$ with $\widetilde{z} \neq 0$. In view of the relations (2.2) and (2.3), we obtain

$$\begin{aligned}\mathrm{Re}\,\langle h(z,t), z\rangle &= \frac{1+|z_1|^2}{2}\mathrm{Re}\, p(z_1,t) + \frac{1-|z_1|^2}{2}\mathrm{Re}\,[z_1 p'(z_1,t)] \\ &+ \frac{1-|z_1|^2}{2} + \mathrm{Re}\,\left[Q(\widetilde{z})\left\{\frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1\right\}\right] \\ &\geq \frac{(1-|z_1|)^2}{2}\mathrm{Re}\, p(z_1,t) + \frac{1-|z_1|^2}{2} - 2(1-|z_1|^2)\|Q\| \geq 0,\end{aligned}$$

whenever $\|Q\| \le 1/4$. Taking into account Lemma 1.1, we deduce that $F(z,t)$ is a Loewner chain. This completes the proof. □

We next obtain the following consequences of Theorem 2.1.

COROLLARY 2.2. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/4$ and let $f \in S$. Also let $F = \Phi_{n,Q}(f)$. Then $F \in S^0(B^n)$.*

Proof. Since $f \in S$ there is a Loewner chain $f(z_1,t)$ such that $f = f(\cdot,0)$. In view of Theorem 2.1, $F(z,t)$ given by (2.1) is a Loewner chain. Since $\{e^{-t}F(\cdot,t)\}_{t\ge 0}$ is a normal family on B^n by the proof of Theorem 2.1 and $F = F(\cdot,0)$, we deduce that $F = \Phi_{n,Q}(f) \in S^0(B^n)$, as desired. This completes the proof. □

The following result is due to Muir [12]. In the case $Q \equiv 0$, see [7]. We have

COROLLARY 2.3. *Let $f \in S^*$ and $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/4$. Then $\Phi_{n,Q}(f) \in S^*(B^n)$.*

Proof. Since $f \in S^*$ it follows that $f(z_1,t) = e^t f(z_1)$ is a Loewner chain. With this choice of $f(z_1,t)$, we deduce that $F(z,t)$ given by (2.1) is a Loewner chain by Theorem 2.1 and the fact that $\|Q\| \le 1/4$. On the other hand, since

$$F(z,t) = \Big(e^t f(z_1) + Q(\widetilde{z})e^t f'(z_1), \widetilde{z}e^t\sqrt{f'(z_1)}\Big) = e^t\Phi_n(f)(z),\ z \in B^n,\ t \ge 0,$$

we deduce that $\Phi_n(f) \in S^*(B^n)$. This completes the proof. □

Another consequence of Theorem 2.1 is given in the following growth result for mappings in the class $\Phi_{n,Q}(S)$.

COROLLARY 2.4. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/4$. If $f \in S$ then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|\Phi_{n,Q}(f)(z)\| \le \frac{\|z\|}{(1-\|z\|)^2},\ z \in B^n.$$

This result is sharp.

Proof. It suffices to apply Theorem 2.1 and [8, Corollary 8.3.9]. □

In the next result we prove that if $f(z_1,t)$ is a c.s.c. over $[0,\infty)$ then $F(z,t)$ given by (2.1) is also a c.s.c. whenever $\|Q\| \le 1/2$. Muir [12] proved that $\Phi_{n,Q}(K) \subseteq K(B^n)$ if and only if $\|Q\| \le 1/2$.

THEOREM 2.5. *If $f(z_1,t) : U \times [0,\infty) \to \mathbb{C}$ is a c.s.c. over $[0,\infty)$ with $f'(0,t) = e^t$, $t \ge 0$, and if $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that $\|Q\| \le 1/2$, then the mapping $F(z,t)$ given by (2.1) is a convex subordination chain over $[0,\infty)$.*

Proof. Since $f(z_1,t)$ is a c.s.c. and $\|Q\| \leq 1/2$, we may use similar arguments to those in the proof of Theorem 2.1 and the fact that (see e.g. [8])

$$\left|\frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1\right| \leq 1,\ |z_1|<1,\ t\geq 0,$$

to deduce that $F(z,t)$ is also a Loewner chain. Next, let $q_t(z_1) = \mathrm{e}^{-t} f_t(z_1)$. Then $q_t \in K$ and since

$$\mathrm{e}^{-t}F(z,t) = \Big(q_t(z_1) + Q(\widetilde{z})q_t'(z_1), \widetilde{z}(q_t'(z_1))^{1/2}\Big) = \Phi_{n,Q}(q_t)(z),\ z\in B^n,\ t\geq 0,$$

we conclude by [12, Theorem 3.1] that $\mathrm{e}^{-t}F(\cdot,t) \in K(B^n)$, $t \geq 0$. Hence $F(z,t)$ is a c.s.c. over $[0,\infty)$, as desired. □

Remark 2.6. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. Also, let $\Lambda[\Phi_{n,Q}(K)]$ be the linear invariant family (L.I.F.) generated by the set $\Phi_{n,Q}(K)$ and $\mathrm{ord}\Lambda[\Phi_{n,Q}(K)]$ be the order of this L.I.F. (see for details [15] and [8, Chapter 10]). Using arguments similar to those in the proofs of [1, Theorem 1] and [8, Theorem 10.3.8], it is possible to prove that $\mathrm{ord}\Lambda[\Phi_{n,Q}(K)] = (n+1)/2$ which is the minimum order of L.I.F.'s in $\mathbb{C}^n$. If $\|Q\| > 1/2$ then $\Phi_{n,Q}(K) \not\subseteq K(B^n)$, and thus the operator $\Phi_{n,Q}$ provides an example of a L.I.F. in $\mathbb{C}^n$ of minimum order which is not a subset of $K(B^n)$ for $n \geq 2$.

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Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
E-mail: gkohr@math.ubbcluj.ro