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INEQUALITIES FOR ONE MAXIMUM OF PARTIAL SUMS OF RANDOM VARIABLES OBTAINED BY USING SUBADDITIVE FUNCTIONS

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Abstract. This note extends the Hájek-Rényi inequality by using a class of subadditive functions. It also extends some results of Kounias and Weng (cf. [2]) and Szynal (cf. [3]).

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1. INTRODUCTION AND NOTATION

In this article we present some inequalities which generalize the well-known Hájek-Rényi inequality and also generalize results obtained by Kounias and Weng (cf. [2]) and Szynal (cf. [3]). We obtain our results by using a family of subadditive functions.

Let us denote by $\mathcal{N}$ the class of all non-decreasing functions $N : [0,\infty) \mapsto [0,\infty)$, $N(0) = 0$ which are subadditive, i.e. $N(a+b) \le N(a)+N(b)$, $a, b \ge 0$. The class $\mathcal{N}$ contains, of course, functions $x \mapsto x^r$, $0 < r \le 1$. However, $\mathcal{N}$ includes also functions increasing slower than each power function.

Let $\{X_i, i \ge 1\}$ be a sequence of random variables and put $S_n = \sum_{i=1}^n X_i$.

2. RESULTS

First, we present a theorem which generalizes Theorem 1 of Kounias and Weng (cf. [2]).

THEOREM 1. *Let $\{X_i, i \ge 1\}$ be a sequence of random variables such that $EN(|X_i|) < \infty$ for some $N \in \mathcal{N}$ and all $i \ge 1$. If $\{c_i, i \ge 1\}$ is a non-decreasing sequence of positive constants, then for every positive integers m, n with $m < n$ and arbitrary $\epsilon > 0$.*

$$(1)\quad P\left(\max_{m\le k\le n} c_k|S_k| \ge \epsilon\right) \le \sum_{i=1}^{m} EN(c_m|X_i|) + \sum_{i=m+1}^{n} EN(c_i|X_i|))/N(\epsilon).$$

Proof. Let us put

$$A_i = \{\omega : c_m|S_m(\omega)| < \epsilon, ..., c_{i-1}|S_{i-1}(\omega)| < \epsilon, c_i|S_i(\omega)| \ge \epsilon\},$$

$i = m, m+1, ..., n$. Then $A_i \cap A_j = \varnothing$ for $i \ne j$ and $A = \bigcup_{i=m}^n A_i$, where $A = \{w : \max_{m\le i\le n} c_i|S_i(\omega)| \ge \epsilon\}$.

Now write

$$Z = N(c_n|S_n|) + \sum_{k=m}^{n-1}(N(c_k|S_k|) - N(c_{k+1}|S_k|))$$

$$+ \sum_{k=m}^{n-1} I_{A_k}(N(c_k|S_k|) - N(c_n|S_n|) - \sum_{i=k}^{n-1}(N(c_i|S_i|) - N(c_{i+1}|S_i|)),$$

where I_{A_k} is the indicator of the event A_k. Observe that $Z \geq 0$ everywhere and $Z \geq N(\epsilon)$ in A. Furthermore, if $F(x_1, ..., x_n)$ is the joint distribution of $X = (X_1, ..., X_n)$, then

$$P\left(\max_{m\leq i\leq n} c_i|S_i| \geq \epsilon\right) = P(X \in A) = \int_A \mathrm{d}F \leq \int_A Z\mathrm{d}F/N(\epsilon) \leq EZ/N(\epsilon).$$

It is easy to see that

$$Z = N(c_m|S_m|) + \sum_{k=m+1}^{n}(N(c_k|S_k|) - N(c_k|S_{k-1}|))(1 - I_{A_{k-1}} - ... - I_{A_m}).$$

As the events A_i are disjoint, then $I_{A_m} + ... + I_{A_n} \leq 1$. Note that

$$N(c_k|S_k|) \leq N(c_k|S_{k-1}| + c_k|X_k|) \leq N(c_k|S_{k-1}|) + N(c_k|X_k|).$$

Thus

$$Z \leq \sum_{k=1}^{m} N(c_m|S_k|) + \sum_{k=m+1}^{n} N(c_k|X_k|),$$

which completes the proof. □

THEOREM 2. *If $\{X_i, i \geq 1\}$ is a sequence of random variables and $N \in \mathcal{N}$, then for every $\epsilon > 0$*

$$P\left(\max_{1\leq i\leq n}|S_i| \geq 2\epsilon\right) \leq 2\sum_{i=1}^{n} E[N(|X_i|)/(N(\epsilon) + N(|X_i|))]. \tag{2}$$

Proof. Put $X_i^* = X_i I_{[|X_i|<\epsilon]}, X_i^{**} = X_i I_{[|X_i|\geq\epsilon]}$. We have

$$P\left(\max_{1\leq i\leq n}|S_i| \geq 2\epsilon\right) \leq P\left(\max_{1\leq i\leq n}|S_i^*| \geq \epsilon\right) + P\left(\max_{1\leq i\leq n}|S_i^{**} \geq \epsilon\right), \tag{3}$$

where $S_i^* = \sum_{j=1}^{i} X_j^*$ and $S_i^{**} = \sum_{j=1}^{i} X_j^{**}$.

By Theorem 1 we get

$$P\left(\max_{1\leq i\leq n}|S_i^*| \geq \epsilon\right) \leq \left(\sum_{i=1}^{n} EN(|X_i^*|)\right)/N(\epsilon).$$

Note that $EN(|X_i^*|)/N(\epsilon) \leq 2E(N(|X_i^*|)/(N(\epsilon) + N(|X_i^*|)))$. Thus

$$P\left(\max_{1\leq i\leq n}|S_i^*| \geq \epsilon\right) \leq 2\sum_{i=1}^{n} E[(N(|X_i|)/(N(\epsilon) + N(|X_i|))I_{[|X_i[<\epsilon]}]. \tag{4}$$

On the other hand

$$P\left(\max_{1\le i\le n}|S_i^{**}|\ge\epsilon\right)\le\sum_{i=1}^n P(|X_i|\ge\epsilon)$$

(5)

$$\le 2\sum_{i=1}^n E[N(|X_i|)/(N(\epsilon)+N(|X_i|))I_{[|X_i|>\epsilon]}].$$

Taking into account (4) and (5) we get (2). □

If we put in Theorem 2 $N(x)=x^r$, $0<r\le 1$, then we get [3, Lemma 1] (in the case $0<r\le 1$ and $s=1$).

THEOREM 3. *If* $\{X_i, i\ge 1\}$ *is a sequence of random variables and* $\{c_i, i\ge 1\}$ *is a non-decreasing sequence of positive integers, then for all* $m,n\in\mathbb{N}$ *with* $m<n$ *and every* $\epsilon>0$

(6)

$$P\left(\max_{m\le i\le n}c_i|S_i|\ge 3\epsilon\right)\le 2\Bigg(\sum_{i=1}^m E(N(c_m|X_i|)/(N(\epsilon)+N(c_m|X_i|))) + \sum_{i=m+1}^n E(N(c_i|X_i|)/(N(\epsilon)+N(c_i|X_i|)))\Bigg).$$

Proof. Let us put $X_i^*=X_iI_{[c_i|X_i|<\epsilon]}$, $X_i^{**}=X_iI_{[c_i|X_i|>\epsilon]}$, $S_i^*=\sum_{j=1}^i X_j^*$ and $S_i^{**}=\sum_{j=1}^i X_j^{**}$.

Define $Y_i=N(c_i|X_i|)/(N(\epsilon)+N(c_i|X_i|))$. Thus, by Theorem 1, we get

(7)

$$P\left(\max_{m\le i\le n}c_i|S_i^*|\ge\epsilon\right) \le 2\Bigg(\sum_{i=1}^m E(N(c_m|X_i|)/(N(\epsilon)+N(c_m|X_i|)))I_{[c_i|X_i|<\epsilon]} + \sum_{i=m+1}^n EY_iI_{[c_i|X_i|<\epsilon]}\Bigg).$$

Furthermore, we have

(8)

$$P\left(\max_{m\le i\le n}c_i|S_i^{**}|\ge 2\epsilon\right)=P(c_m|S_m^{**}|\ge 2\epsilon) + \sum_{i=m+1}^n P\left(\bigcap_{j=m}^{i-1}([c_j|S_j^{**}|<2\epsilon]\cap[c_i|S_i^{**}|\ge 2\epsilon])\right)$$

$$\le P(c_m|S_m^{**}|\ge 2\epsilon)+\sum_{i=m+1}^n P(c_i|X_i|\ge\epsilon).$$

By (2) we have

$$P(c_m|S_m^{**}| \geq 2\epsilon) \leq 2\left(\sum_{i=1}^{m} E[N(c_m|X_i|)/(N(\epsilon)+N(c_m|X_i|))I_{[c_i|X_i|\geq\epsilon]}]\right).$$

Thus, by (8) we get

$$\begin{aligned} (9)\qquad & P\left(\max_{m\leq i\leq n} c_i|S_i^{**}| \geq 2\epsilon\right) \\ & \leq 2\Bigg(\sum_{i=1}^{m} E[N(c_m|X_i|)/(N(\epsilon)+N(c_m|X_i|)]I_{[c_i|X_i|\geq\epsilon]} \\ & \quad + \sum_{i=m+1}^{n} EY_i I_{[c_i|X_i|\geq\epsilon]}\Bigg). \end{aligned}$$

Therefore, taking into account (7) and (9), we get (6). □

Theorem 3 is an extension of Lemma 3 in [3] (in the case $0 < r \leq 1$ and $s = 1$).

Corollary 4. *Under the assumptions of* Theorem 3 *we get*

$$(10)\qquad P\left(\max_{m\leq i\leq n} c_i|S_i| \geq 3\epsilon\right) \leq 2\left(\sum_{i=1}^{n} E[N(c_m|X_i|)/(N(\epsilon+N(c_m|X_i|))]\right).$$

Corollary 5. *Under the assumption of* Theorem 2 *we have*

$$(11)\qquad P(c_n|S_n| \geq 2\epsilon) \leq 2\left(\sum_{i=1}^{n} E[N(c_n|X_i|)/(N(\epsilon)+N(c_n|X_i|))]\right).$$

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