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ON PRECONNECTED SPACES

ERDAL EKICI

Abstract. In this paper, properties of preconnected spaces, preseparated subsets and p_s-connected subsets are studied.

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1. INTRODUCTION

In 1990, Noiri and Popa [5] introduced the concept of preconnected spaces. This form is a strong form of connected spaces. In this paper, properties of preconnected spaces are investigated.

Throughout the present paper, X and Y are topological spaces. Let A be a subset of X. We denote the interior and the closure of the set A by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$, respectively. A subset A of a space X is said to be preopen [2] if $A \subset \mathrm{int}(\mathrm{cl}(A))$. The complement of a preopen set is called preclosed [2]. The intersection of all preclosed sets containing A is called the preclosure [1] of A and is denoted by $\mathrm{pcl}(A)$. The preinterior of A is defined by the union of all preopen sets contained in A and is denoted by $\text{p-int}(A)$. A subset A is said to be α-open [4] if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. The family of all α-open (resp. preopen, preclosed, preclopen) sets of X is denoted by $\alpha O(X)$ (resp. $PO(X)$, $PC(X)$, $PCO(X)$). The family of all preopen (resp. preclosed) sets of X containing a point x is denoted by $PO(X, x)$ (resp. $PC(X, x)$).

2. PRECONNECTED SPACES

Definition 1. ([5]) A topological space X is called preconnected if X can not be expressed as the union of two nonempty disjoint preopen sets of X.

Definition 2. A subset A of a topological space X is called preconnected if A is preconnected as a subspace of X.

Definition 3. Nonempty subsets A, B of a topological space X are said to be preseparated if $A \cap \mathrm{pcl}(B) = \emptyset = \mathrm{pcl}(A) \cap B$.

Lemma 1. ([3]) *Let A and Y be subsets of a topological space X.*
(1) *If $Y \in \alpha O(X)$ and $A \in PO(X)$, then $A \cap Y \in PO(Y)$,*
(2) *If $A \subset Y \subset X$, $A \in PO(Y)$ and $Y \in PO(X)$, then $A \in PO(X)$.*

Lemma 2. *Let X be a topological space and A, Y subsets of X such that $A \subset Y \subset X$ and $Y \in \alpha O(X)$. Then $A \in PO(Y)$ if and only if $A \in PO(X)$.*

Proof. Let $A \in PO(Y)$. Since $Y \in \alpha O(X) \subset PO(X)$, by Lemma 1, we have $A \in PO(X)$.

Conversely, let $A \in PO(X)$. By Lemma 1, $A = A \cap Y \in PO(Y)$. □

THEOREM 1. *Let X be a topological space. If A and B are preseparated sets of X and $A \cup B \in \alpha O(X)$, then A, $B \in PO(X)$.*

Proof. Since A and B are preseparated in X, then we have $(A \cup B) \cap (X \backslash \mathrm{pcl}(B)) = A$. Since $A \cup B \in \alpha O(X)$ and $\mathrm{pcl}(B)$ is preclosed in X, we have $A \in PO(X)$ by Lemmas 1 and 2. In a similar way we obtain $B \in PO(X)$. □

LEMMA 3. *Let X be a topological space and A, Y subsets of X such that $A \subset Y \subset X$ and $Y \in \alpha O(X)$. Then $\mathrm{pcl}(A) \cap Y = \mathrm{pcl}_Y(A)$, where $\mathrm{pcl}_Y(A)$ denotes the preclosure of A in the subspace Y.*

Proof. Let $x \in \mathrm{pcl}(A) \cap Y$ and $V \in PO(Y, x)$. Then by Lemma 1, $V \in PO(X, x)$ and hence $V \cap A \neq \emptyset$. Therefore, $x \in \mathrm{pcl}_Y(A)$.

Conversely, let $x \in \mathrm{pcl}_Y(A)$ and $V \in PO(X, x)$. Then $x \in V \cap Y \in PO(Y)$ and hence $\emptyset \neq A \cap (V \cap Y) \subset A \cap V$. Therefore, we obtain $x \in \mathrm{pcl}(A) \cap Y$. □

THEOREM 2. *Let X be a topological space. If X is preconnected and $Y \in PO(X)$, then Y is preconnected.*

Proof. Suppose that Y is not preconnected. Then there exists a preclopen set A of the subspace Y such that $A \neq \emptyset$ and $A \neq Y$. Since $Y \in PO(X)$, by Lemma 1, $A \in PCO(X)$. Thus X is not preconnected, which is a contradiction. □

THEOREM 3. *Let X be a topological space, Y an α-open set of X and A, B be subsets of Y. Then A, B are preseparated in Y if and only if A, B are preseparated in X.*

Proof. By Lemma 3, we have $\mathrm{pcl}_Y(A) \cap B = \emptyset = A \cap \mathrm{pcl}_Y(B)$ if and only if $\mathrm{pcl}(A) \cap B = \emptyset = A \cap \mathrm{pcl}(B)$. □

DEFINITION 4. A subset G of a topological space X is said to be p_s-connected if G is not the union of two preseparated sets in X.

THEOREM 4. *If A is a p_s-connected set of a topological space X and U, V are preseparated sets of X such that $A \subset U \cup V$, then either $A \subset U$ or $A \subset V$.*

Proof. Since $A = (A \cap U) \cup (A \cap V)$, we have

$$(A \cap U) \cap \mathrm{pcl}(A \cap V) \subset U \cap \mathrm{pcl}(V) = \emptyset.$$

In a similar way, we obtain $(A \cap V) \cap \mathrm{pcl}(A \cap U) = \emptyset$. If $A \cap U$ and $A \cap V$ are nonempty, then A is not p_s-connected, which is a contradiction. Hence either $A \cap U = \emptyset$ or $A \cap V = \emptyset$. It follows that either $A \subset U$ or $A \subset V$. □

THEOREM 5. *Let Y be an α-open set of a topological space X. Then Y is p_s-connected in X if and only if Y is preconnected in X.*

Proof. ($\Rightarrow$) Suppose that Y is not preconnected. Then there exist nonempty disjoint A, $B \in PO(Y)$ such that $A \cup B = Y$. Since $Y \in \alpha O(X)$, by Lemma 1, A, $B \in PO(X)$. Since A and B are disjoint, we have $\mathrm{pcl}(A) \cap B = \emptyset = A \cap \mathrm{pcl}(B)$. This shows that A, B are preseparated sets in X. Hence Y is not p_s-connected in X. This is a contradiction.

($\Leftarrow$) Suppose that Y is not p_s-connected in X. Then there exist preseparated sets A, B such that $Y = A \cup B$. By Theorem 1, A, $B \in PO(X)$ and by Lemma 1, A, $B \in PO(Y)$. Since A and B are preseparated in X, they are nonempty disjoint. Hence Y is not preconnected. This is a contradiction. □

Corollary 1. *If A is an α-open and preconnected set of a topological space X and U, V are preseparated sets of X such that $A \subset U \cup V$, then either $A \subset U$ or $A \subset V$.*

Proof. It can be obtained from Theorems 4 and 5. □

Theorem 6. *If A is a p_s-connected set of a topological space X and $A \subset S \subset \mathrm{pcl}(A)$, then S is p_s-connected.*

Proof. Suppose that S is not p_s-connected. Then there exist preseparated sets U and V such that $S = U \cup V$. Hence U and V are nonempty and $U \cap \mathrm{pcl}(V) = \emptyset = V \cap \mathrm{pcl}(U)$. By Theorem 4, we obtain either $A \subset U$ or $A \subset V$.

(1) Suppose that $A \subset U$. Then $\mathrm{pcl}(A) \subset \mathrm{pcl}(U)$ and $V \cap \mathrm{pcl}(A) = \emptyset$. We have $V \subset S \subset \mathrm{pcl}(A)$ and $V = \mathrm{pcl}(A) \cap V = \emptyset$. Hence V is an empty set. This is a contradiction since V is nonempty.

(2) Suppose that $A \subset V$. In a similar way, we obtain that U is empty. This is a contradiction.

This implies that S is p_s-connected. □

Corollary 2. *Let X be a topological space and $K \subset X$. If K is a p_s-connected set, then the preclosure of K is p_s-connected.*

Theorem 7. *Let A and B be subsets of a topological space X. If A and B are α-open, preconnected and not preseparated in X, then $A \cup B$ is preconnected.*

Proof. Suppose that $A \cup B$ is not preconnected. Since $A \cup B \in \alpha O(X)$, by Theorem 5, $A \cup B$ is not p_s-connected. There exist preseparated sets M, N in X such that $A \cup B = M \cup N$. Since A is α-open preconnected in X and $A \subset M \cup N$, by Corollary 1, we have either $A \subset M$ or $A \subset N$. Similarly, we obtain that either $B \subset M$ or $B \subset N$. If $A \subset M$ and $B \subset M$, then $A \cup B \subset M$ and hence N is empty. This is a contradiction. Hence $A \subset M$ and $B \subset N$. Similarly, $A \subset N$ and $B \subset M$. Hence we obtain $\mathrm{pcl}(A) \cap B \subset \mathrm{pcl}(M) \cap N = \emptyset$ and $\mathrm{pcl}(B) \cap A \subset \mathrm{pcl}(M) \cap N = \emptyset$. Therefore, A, B are preseparated in X. This is a contradiction. Hence $A \cup B$ is preconnected. □

Theorem 8. *If $\{B_i : i \in I\}$ is a nonempty family of p_s-connected sets of a topological space X such that $\bigcap_{i\in I} B_i \neq \emptyset$, then $\bigcup_{i\in I} B_i$ is p_s-connected.*

Proof. Suppose that $A = \bigcup_{i\in I} B_i$ and A is not p_s-connected. Then $A = U \cup V$, where U and V are preseparated sets in X. Since $\bigcap_{i\in I} B_i \neq \emptyset$, we can choose a point x in $\bigcap_{i\in I} B_i$. Since $x \in A$, either $x \in U$ or $x \in V$.

(1) Suppose that $x \in U$. Since $x \in B_i$ for each $i \in I$, B_i and U intersect for each $i \in I$. By Theorem 4, B_i must be in either U or V. Since U and V are disjoint, $B_i \subset U$ for all $i \in I$ and hence $A \subset U$. This means that V is empty, which is a contradiction.

(2) Suppose that $x \in V$. Then, in a similar way, we obtain that U is empty, which is a contradiction.

Hence $\bigcup_{i\in I} B_i$ is p_s-connected. □

Corollary 3. *If $\{B_i : i \in I\}$ is a nonempty family of preconnected α-open sets of a topological space X such that $\bigcap_{i\in I} B_i \neq \emptyset$, then $\bigcup_{i\in I} B_i$ is p_s-connected.*

Proof. It can be obtained from Theorem 5 and Theorem 8. □

Theorem 9. *If $\{A_n : n \in N\}$ is an infinite sequence of preconnected α-open sets of a topological space X and $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in N$, then $\bigcup_{n\in N} A_n$ is preconnected.*

Proof. By induction on the natural number n, the set $B_n = \bigcup_{k\leq n} A_k$ is a preconnected α-open set for each $n \in N$ by Corollary 3. The sets B_n have a nonempty intersection and hence $\bigcup_{n\in N} A_n$ is preconnected by Corollary 3. □

Definition 5. Let X be a topological space and x a point of X. The precomponent of X containing x is the union of all p_s-connected subsets of X containing x.

Remark 1. Since the union of any family of p_s-connected subsets of X containing a point $x \in X$ has nonempty intersection, by Theorem 8, the precomponent of X containing x is p_s-connected.

Theorem 10. *Let X be a topological space. Then each precomponent of X is a maximal p_s-connected set of X.*

Proof. Obvious. □

Theorem 11. *Let X be a topological space. Then the set of all distinct precomponents of X forms a partition of X.*

Proof. Suppose that U and V are two distinct precomponents of X. If U and V intersect, then $U \cup V$ is p_s-connected in X by Theorem 8. Since $U \subset U \cup V$, then U is not maximal. Hence U and V are disjoint. □

THEOREM 12. *Let X be a topological space. Then each precomponent of X is preclosed in X.*

Proof. Let V be any precomponent of X. By Theorem 6, $\mathrm{pcl}(V)$ is p_s-connected and $V = \mathrm{pcl}(V)$. Hence V is preclosed in X. □

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Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus
17020 Canakkale, Turkey
E-mail: eekici@comu.edu.tr