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NEW CRITERIA FOR MEROMORPHIC P-VALENT CONVEX FUNCTIONS

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Abstract. Let $G_n(\alpha)$ be the class of functions of the form $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k$ $(a_{-p} \neq 0,\ p \in N = \{1, 2, ...\})$ which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$ and satisfying $\operatorname{Re}\left\{ \frac{(D^{n+1} f(z))'}{(D^n f(z))'} - (p+1) \right\} < -\alpha$ $(n \in N_0 = \{0, 1, 2, ...\}, |z| < 1, 0 \leq \alpha < p)$, where $D^n f(z) = \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n$ $a_{m-1} z^{m-1}$. It is proved that $G_{n+1}(\alpha) \subset G_n(\alpha)$. Since $G_0(\alpha)$ is the class of meromorphically p-valent convex functions of order α, $0 \leq \alpha < p$, all functions in $G_n(\alpha)$ are p-valent convex. A property preserving integrals is also considered.

MSC 2000. 30C45.
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1. INTRODUCTION

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0,\ p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$\begin{aligned} D^1 f(z) &= \frac{a_{-p}}{z^p} + (p+1) a_0 + (p+2) a_1 z + (p+3) a_2 z^2 + ... \\ &= \frac{\left(z^{p+1} f(z)\right)'}{z^p}, \end{aligned} \tag{1.3}$$

$$D^2 f(z) = D\left(D^1 f(z)\right), \tag{1.4}$$

and for $n = 1, 2, ...$

$$\begin{aligned} D^n f(z) = D\left(D^{n-1} f(z)\right) &= \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1} \\ &= \frac{\left(z^{p+1} D^{n-1} f(z)\right)'}{z^p}. \end{aligned} \tag{1.5}$$

In this paper, we shall show that a function $f(z)$ in $\sum_p$, which satisfies the conditions

$$\text{(1.6)}\quad \operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'}-(p+1)\right\} < -\alpha \quad (z\in U=\{z:|z|<1\}),$$

for some $\alpha\,(0\le\alpha<p)$ and $n\in N_0=\{0,1,2,...\}$, is meromorphically p-valent convex in U^*. More precisely, it is proved that, for the classes $G_n(\alpha)$ of functions in $\sum_p$ satisfying (1.6),

$$\text{(1.7)}\qquad G_{n+1}(\alpha) \subset G_n(\alpha)$$

holds. Since $G_0(\alpha)$ equals $\sum_k^*(\alpha)$ (the class of meromorphically p-valent convex functions of order $\alpha, 0\le\alpha<p$), the convexity of members of $G_n(\alpha)$ is a consequence of (1.7). Further for $c>0$, let

$$\text{(1.8)}\qquad F(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,\mathrm{d}t.$$

It is shown that $F(z)\in G_n(\alpha)$ whenever $f(z)\in G_n(\alpha)$. Some known results of Bajpai [2], Goel and Sohi [3] and Uralegaddi and Somanatha [6] are extended. In [5] Rusheweyh obtained the new criteria for univalent functions.

In [1] Aouf and Hossen obtained a new criteria for meromorphic p-valent starlike functions via the basic inclusion relationship $B_{n+1}(\alpha)\subset B_n(\alpha)$, $0\le\alpha<p$ and $n\in N_0$, where $B_n(\alpha)$ is the class of functions $f(z)\in\sum_p$ satisfying

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}-(p+1)\right\}<-\alpha,$$

$0\le\alpha<p$, $n\in N_0$ and $|z|<1$.

2. PROPERTIES OF THE CLASS $G_n(\alpha)$

In proving our main results [Theorem 2 and Theorem 3 below], we shall need the following lemma due to Jack [4].

LEMMA 1. *Let $w(z)$ be non-constant regular in $U=\{z:|z|<1\}$, $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0, we have $z_0w'(z_0)=kw(z_0)$, where k is a real number, $k\ge1$.*

THEOREM 2. *$G_{n+1}(\alpha)\subset G_n(\alpha)$ for each integer $n\in N_0$.*

Proof. Let $f(z)\in G_{n+1}(\alpha)$. Then

$$\text{(2.1)}\qquad \operatorname{Re}\left\{\frac{\left(D^{n+2}f(z)\right)'}{\left(D^{n+1}f(z)\right)'}-(p+1)\right\} < -\alpha, \quad |z|<1.$$

We have to show that (2.1) implies the inequality

$$\text{(2.2)}\qquad \operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'}-(p+1)\right\} < -\alpha.$$

Define a regular function $w(z)$ in U by

$$\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'}-(p+1) = -\frac{p+(2\alpha-p)\,w(z)}{1+w(z)}. \tag{2.3}$$

Clearly $w(0)=0$. Equation (2.3) may be written as

$$\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'} = \frac{1+(2p+1-2\alpha)\,w(z)}{1+w(z)}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity (easy to verify)

$$z\left(D^{n}f(z)\right)' = D^{n+1}f(z)-(p+1)\,D^{n}f(z), \tag{2.5}$$

and

$$z\left(D^{n}f(z)\right)'' = \left(D^{n+1}f(z)\right)'-(p+2)\left(D^{n}f(z)\right)', \tag{2.6}$$

we obtain

$$\begin{aligned}&\frac{\frac{\left(D^{n+2}f(z)\right)'}{\left(D^{n+1}f(z)\right)'}-(p+1)+\alpha}{p-\alpha}\\ &= \frac{2zw'(z)}{(1+w(z))\left[1+(2p+1-2\alpha)\,w(z)\right]}-\frac{1-w(z)}{1+w(z)}.\end{aligned} \tag{2.7}$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's lemma) there exists a point z_o in U such that

$$z_o w'(z_o) = kw(z_o), \tag{2.8}$$

where $|w(z_o)|=1$ and $k\geq 1$. From (2.7) and (2.8), we obtain

$$\begin{aligned}&\frac{\frac{\left(D^{n+2}f(z_o)\right)'}{\left(D^{n+1}f(z_o)\right)'}-(p+1)+\alpha}{p-\alpha}\\ &= \frac{2kw(z_0)}{(1+w(z_0))\left[1+(2p+1-2\alpha)\,w(z_0)\right]}-\frac{1-w(z_0)}{1+w(z_0)}.\end{aligned} \tag{2.9}$$

Thus

$$\operatorname{Re}\left\{\frac{\frac{\left(D^{n+2}f(z_0)\right)'}{\left(D^{n+1}f(z_0)\right)'}-(p+1)+\alpha}{p-\alpha}\right\} \geq \frac{1}{2(1+p-\alpha)}>0, \tag{2.10}$$

which contradicts (2.1). Hence $|w(z)|<1$ in U and from (2.3) it follows that $f(z)\in G_n(\alpha)$. □

THEOREM 3. *Let $f(z)\in\sum_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'}-(p+1)\right\}<-\alpha+\frac{p-\alpha}{2(p-\alpha+c)}\quad (z\in U), \tag{2.11}$$

for a given $n \in N_0$ *and* $c > 0$. *Then*

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)\, \mathrm{d}t,$$

belongs to $G_n(\alpha)$.

Proof. From the definition of $F(z)$, we have

$$z\left(D^n F(z)\right)'' = c\left(D^n f(z)\right)' - (c+p+1)\left(D^n F(z)\right)', \tag{2.12}$$

and also

$$z\left(D^n F(z)\right)'' = \left(D^{n+1} F(z)\right)' - (p+2)\left(D^n F(z)\right)'. \tag{2.13}$$

Using (2.12) and (2.13), the condition (2.11) may be written as

$$\operatorname{Re}\left\{ \frac{\dfrac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'} + c - 1}{1 + (c-1)\dfrac{\left(D^{n}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}} - (p+1) \right\} < -\alpha + \frac{p-\alpha}{2(p-\alpha+c)}. \tag{2.14}$$

We have to prove that (2.14) implies the inequality

$$\operatorname{Re}\left\{ \frac{\left(D^{n+1}F(z)\right)'}{\left(D^{n}F(z)\right)'} - (p+1) \right\} < -\alpha. \tag{2.15}$$

Define $w(z)$ in U by

$$\frac{\left(D^{n+1}F(z)\right)'}{\left(D^{n}F(z)\right)'} - (p+1) = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \tag{2.16}$$

Clearly $w(z)$ is regular and $w(0) = 0$. The equation (2.16) may be written as

$$\frac{\left(D^{n+1}F(z)\right)'}{\left(D^{n}F(z)\right)'} = \frac{1 + (2p+1-2\alpha)w(z)}{1 + w(z)}. \tag{2.17}$$

Differentiating (2.17) logarithmically and using (2.12), we obtain

$$\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'} - \frac{\left(D^{n+1}F(z)\right)'}{\left(D^{n}F(z)\right)'} = \frac{2(p-\alpha)zw'(z)}{(1+w(z))\left[1 + (2p+1-2\alpha)w(z)\right]}. \tag{2.18}$$

The above equation may be written as

$$\frac{\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'} + (c-1)}{1 + (c-1)\frac{\left(D^{n}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}} - (p+1) = \frac{\left(D^{n+1}F(z)\right)'}{\left(D^{n}F(z)\right)'} - (p+1)$$

$$+ \left[\frac{2(p-\alpha)zw'(z)}{(1+w(z))\left[1 + (2p+1-2\alpha)w(z)\right]}\right]\left[\frac{1}{1 + (c-1)\frac{\left(D^{n}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}}\right],$$

which by using (2.16) and (2.17) reduces to

$$\frac{\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}+(c-1)}{1+(c-1)\frac{\left(D^{n}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}}-(p+1) = -\left[\alpha+(p-\alpha)\frac{1-w(z)}{1+w(z)}\right] + \frac{2(p-\alpha)zw'(z)}{(1+w(z))\left[c+(c+2(p-\alpha))w(z)\right]}.$$

The remaining part of the proof is similar to that of Theorem 2. □

REMARK 1. (i) Putting $p=1, a_{-1}=1, n=0$ and $\alpha=0$ in Theorem 3, we get the result of Goel and Sohi [3,Corollary 2].

(ii) For $p=1, a_{-1}=1, n=0, \alpha=0$ and $c=1$ the above theorem extends a result of Bajpai [2, Theorem 1].

THEOREM 4. *$f(z)\in G_n(\alpha)$ if and only if*

$$F(z)=\frac{1}{z^{1+p}}\int_0^z t^p f(t)\,\mathrm{d}t\in G_{n+1}(\alpha).$$

Proof. From the definition of $F(z)$, we have

$$D^n\left(zF'(z)\right)+(1+p)D^nF(z)=D^nf(z),$$

that is

$$z\left(D^nF(z)\right)''+(2+p)\left(D^nF(z)\right)' = \left(D^nf(z)\right)'. \tag{2.19}$$

By using the identity (2.13), (2.19) reduces to

$$\left(D^nf(z)\right)'=\left(D^{n+1}F(z)\right)'.$$

Hence

$$\left(D^{n+1}f(z)\right)'=\left(D^{n+2}F(z)\right)'.$$

Therefore

$$\frac{\left(D^{n+1}f(z)\right)'}{\left(D^nf(z)\right)'}=\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}$$

and the result follows. □

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