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$|\bar{N}, p_n|_k$ SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

S.M. MAZHAR

Abstract. In this note summability $|\bar{N}, p_n|_k$ of Fourier series and its conjugate series are studied. Our results are generalization of a theorem of Mohanty on conjugate series and a theorem of Prem Chandra for Fourier series.

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1. INTRODUCTION

Let $\sum a_n$ be a given series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$, as $n \to \infty$. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if [1]

$$\sum_1^\infty \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty, \tag{1.1}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v.$$

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $[-\pi, \pi]$. Let the Fourier series of $f(t)$ be

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \sum_0^\infty A_n(t).$$

Then the conjugate series of the Fourier series is

$$\sum_1^\infty (b_n \cos nt - a_n \sin nt) = \sum_1^\infty B_n(t).$$

We denote

$$\begin{aligned}
\varphi(t) &= \frac{1}{2}\{f(x+t) + f(x-t)\} \\
\psi(t) &= \frac{1}{2}\{f(x-t) - f(x-t)\} \\
\Lambda(t) &= \frac{1}{t}\int_0^t u \mathrm{d}\varphi(u).
\end{aligned}$$

We usually write $|R, \lambda_n, 1|$ for $|\bar{N}, p_n|$, where $P_n = \lambda_n$, and $0 < \lambda_1 < \cdots < \lambda_n \to \infty,\ n \to \infty$.

2. ABSOLUTE SUMMABILITY OF CONJUGATE SERIES

Concerning absolute summability of conjugate series $\sum_1^\infty B_n(x)$, Mohanty [3] has proved the following theorem.

THEOREM A. *If*

$$\psi(t)\log\frac{C}{t}\in BV[0,\pi], \tag{2.1}$$

$$\frac{|\psi(t)|}{t}\in L[0,\pi], \tag{2.2}$$

then $\sum_1^\infty B_n(x)$ *is summable* $|R,\mathrm{e}^{n^\alpha},1|,\ 0<\alpha<1.$

The conditions (2.1) and (2.2) are equivalent to the conditions [3]:

$$\int_0^\pi \log\frac{C}{t}|\mathrm{d}\psi(t)|<\infty \tag{2.3}$$

$$\psi(+0)=0. \tag{2.4}$$

We give below a simple generalization of this result for $|\bar{N},p_n|_k$ summability.

THEOREM 1. *If* $\psi(+0)=0$ *and*

$$\int_0^\pi \log\frac{C}{t}|\mathrm{d}\psi(t)|^k<\infty,\ k\geq 1, \tag{2.5}$$

then $\sum_1^\infty B_n(x)$ *is summable* $|\bar{N},p_n|_k$, *where*

$$\left\{\frac{P_n}{n}\right\}\uparrow \tag{2.6}$$

$$n^{1-\alpha}p_n=O(P_n),\ 0<\alpha<1. \tag{2.7}$$

For $k=1,\ P_n=\mathrm{e}^{n^\alpha},\ 0<\alpha<1$, Theorem 1 reduces to Theorem A.

Proof. Let $T_n(x)$ denote the $(\bar{N},p_n)$ mean of the series $\sum_1^\infty B_n(x)$ and let C_1 denote a positive constant not necessarily the same at each occurrence. Then

$$T_n(x)-T_{n-1}(x)=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^n P_{v-1}B_v(x)$$

$$=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^n P_{v-1}\frac{2}{\pi}\int_0^\pi \mathrm{d}\psi(t)\int_t^\pi \sin vu\ \mathrm{d}u,\ \text{since } \psi(+0)=0,$$

so that

$$\sum_1^\infty\left(\frac{P_n}{p_n}\right)^{k-1}|T_n(x)-T_{n-1}(x)|^k$$

$$=\sum_1^\infty\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^n P_{v-1}\frac{2}{\pi}\int_0^\pi \mathrm{d}\psi(t)\int_t^\pi \sin vu\ \mathrm{d}u\right|^k$$

$$\leq C_1 \sum_1^\infty \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^n P_{v-1} \int_0^\pi \mathrm{d}\psi(t) \int_t^\pi \sin vu \ \mathrm{d}u \right|^k$$

$$= C_1 \sum_1^\infty \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^n \frac{P_{v-1}}{v} \int_0^\pi \mathrm{d}\psi(t)(\cos vt - \cos v\pi) \right|^k$$

$$\leq C_1 \sum_1^\infty \frac{p_n}{P_n P_{n-1}^k} \left(\int_0^\pi |\mathrm{d}\psi(t)| \left| \sum_{v=1}^n \frac{P_{v-1}}{v} (\cos vt - \cos v\pi) \right| \right)^k$$

$$\leq C_1 \sum_1^\infty \frac{p_n}{P_n P_{n-1}^k} \left(\int_0^\pi |\mathrm{d}\psi(t)|^k \left| \sum_{v=1}^n \frac{P_{v-1}}{v} (\cos vt - \cos v\pi) \right| \right)$$

$$\times \left(\int_0^\pi |\mathrm{d}\psi(t)| \left| \sum_{v=1}^n \frac{P_{v-1}}{v} (\cos vt - \cos v\pi) \right| \right)^{k-1},$$

$$\begin{aligned} &\leq C_1 \sum_1^\infty \frac{p_n}{P_n P_{n-1}} \int_0^\pi |\mathrm{d}\psi(t)|^k \left| \sum_{v=1}^n \frac{P_{v-1}}{v} (\cos vt - \cos v\pi) \right| \\ &= C_1 \int_0^\pi |\mathrm{d}\psi(t)|^k L_n(t), \end{aligned}$$

where

$$L_n(t) = \sum_1^\infty \frac{p_n}{P_n P_{n-1}} \left| \sum_{v=1}^n \frac{P_{v-1}}{v} (\cos vt - \cos v\pi) \right|$$

$$= \sum_1^T + \sum_{T+1}^\infty = L_1 + L_2, \text{ where } T = \left(\frac{1}{t}\right)^{1/1-\alpha}, \ 0 < \alpha < 1.$$

Now

$$\begin{aligned} L_1 &\leq C_1 \sum_1^T \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \frac{P_{v-1}}{v} \\ &\leq C_1 \sum_{v=1}^T \frac{P_{v-1}}{v} \sum_{n=v}^\infty \frac{p_n}{P_n P_{n-1}} \\ &\leq C_1 \sum_{v=1}^T \frac{1}{v} = O(\log T) = O\left(\log \frac{C}{t}\right). \end{aligned}$$

Also

$$\begin{aligned} L_2 &\leq \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{v=1}^{n} \frac{P_{v-1}}{v} \cos vt \right| \\ &+ \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{v=1}^{n} \frac{P_{v-1}}{v} \cos v\pi \right| \\ &\leq C_1 \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-1}}{n} \frac{1}{t} \\ &+ C_1 \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-1}}{n} \\ &\leq C_1 \sum_{n=T+1}^{\infty} \frac{p_n}{n P_n} t^{-1} \leq C_1 \sum_{n=T+1}^{\infty} \frac{t^{-1}}{n^{2-\alpha}} \leq C_1. \end{aligned}$$

Therefore $L_n(t) = O(\log \frac{C}{t})$. Thus

$$\sum_{1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n(x) - T_{n-1}(x)|^x \leq C_1 \int_0^{\pi} \log \frac{C}{t} |\mathrm{d}\psi(t)|^k < \infty.$$

This proves the theorem.

3. ABSOLUTE SUMMABILITY OF FOURIER SERIES

Concerning $\sum_{\partial}^{\infty} A_n(x)$, Prem Chandra [4] obtained the following result.

THEOREM B. *If*

$$\varphi(t) \in BV[0, \pi] \tag{3.1}$$

$$\Lambda(t) \left(\log \frac{C}{t} \right)^{1+\varepsilon} \in BV[0, \pi], \tag{3.2}$$

where $\varepsilon > 0$ *and* $C \geq \pi \mathrm{e}^2$, *then* $\sum_0^{\infty} A_n(x)$ *is summable* $|R, \mathrm{e}^{n^{\alpha}}, 1|$, $0 < \alpha < 1$.

Later on [5] he proved a more general result for summability $|R, \lambda_n, 1|$.

THEOREM C. *Let, for* $0 < \alpha < 1$, *the strictly increasing sequences* $\{\lambda_n\}$ *and* $\{g(n)\}$ *of non-negative terms, tending to infinity with* n, *satisfy the following conditions.*

$$\left\{ \frac{\lambda_n}{n+1} \right\} \uparrow, n \geq n_0. \tag{3.3}$$

$$n^{1-\alpha} \Delta \lambda_n = O(\lambda_{n+1}), \ n \to \infty \tag{3.4}$$

$$\log \frac{\pi}{t} = O\left(g(C/t)\right), t \to 0 \tag{3.5}$$

(3.6)
$$\frac{x}{g(x)} \uparrow$$

(3.7)
$$x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(C/x)} \uparrow \text{ with } x$$

(3.8)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(\frac{C}{x})}\right) \downarrow \text{ with } x$$

(3.9)
$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{g(C/t)}\right)\right]_{t=1/n} = O\left(\frac{n}{g(n)}\right)$$

(3.10)
$$\sum_{1}^{\infty}\frac{1}{ng(n)} < \infty.$$

If $\varphi(t) \in BV[0,\pi]$ *and* $\Lambda(t)g(\frac{C}{t}) \in BV[0,\pi]$*, then the series* $\sum_0^\infty A_n(x)$ *is summable* $|R,\lambda_n,1|$*, where* C *is a positive constant such that* $g(C/t) > 0$ *for* $t > 0$*.*

In this section we prove the corresponding result for summability $|\bar{N},p_n|_k$, for $k \geq 1$.

THEOREM 2. *Let*

(3.11)
$$\left\{\frac{P_n}{n}\right\} \uparrow$$

and

(3.12)
$$\frac{p_n}{P_n} = O\left(\frac{1}{n^{1-\alpha}}\right), 0 < \alpha < 1.$$

Then under the conditions (3.5)–(3.10)*,* $\varphi(t) \in BV[0,\pi]$ *and* $\Lambda(t)g(\frac{C}{t}) \in BV[0,\pi]$*, the series* $\sum_0^\infty A_n(x)$ *is summable* $|\bar{N},p_n|_k$*.*

For $k = 1$ and $P_n = \lambda_n$ we obtain Theorem C of Prem Chandra.

Proof. Let $t_n(x)$ denote the $(\bar{N},p_n)$ mean of $\sum_0^\infty A_n(x)$. Then

$$t_n(x) - t_{n-1}(x) = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}A_v(x)$$

so that

$$\sum_{1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|t_n(x)-t_{n-1}(x)|^k$$
$$\leq \sum_{1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n}P_{v-1}\frac{2}{\pi}\int_0^\pi \Lambda(t)g\left(\frac{C}{t}\right)\cdot\frac{t}{g(C/t)}\cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin vt}{vt}\right)\mathrm{d}t\right|^k$$

Since $\Lambda(t)g(\frac{C}{t}) \in BV[0,\pi]$ it is enough to prove that

$$\sum_1^\infty \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}\int_0^t \frac{u}{g(C/u)}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\sin vu}{u}\right)\mathrm{d}u\right|^k < \infty$$

uniformly in $0 < t \le \pi$.

Now

$$\begin{aligned}&\int_0^t \frac{u}{g(\frac{C}{u})}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\sin vu}{u}\right)\mathrm{d}u\\ &= \frac{\sin vt}{g(C/t)} - \int_0^t \frac{\sin vu}{ug(C/u)}\mathrm{d}u\\ &- \int_0^t \sin vu\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{g(C/u)}\right)\mathrm{d}u\\ &= M_1(t) + M_2(t) + M_3(t),\end{aligned}$$

say. Using the estimates [5] uniformly in $0 < t \le \pi$

$$\int_0^t \frac{\sin vu}{ug(C/u)}\mathrm{d}u = O\left(\frac{1}{g(v)}\right), \tag{3.13}$$

$$\int_0^t \sin vu\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{g(C/u)}\right)\mathrm{d}u = O\left(\frac{1}{g(v)}\right), \tag{3.14}$$

we have to show that

$$\sum_1^\infty \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}M_1(t)\right|^k < \infty, \tag{3.15}$$

$$\sum_1^\infty \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}M_2(t)\right|^k < \infty, \tag{3.16}$$

$$\sum_1^\infty \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}M_3(t)\right|^k < \infty. \tag{3.17}$$

Writing (3.15) as

$$\sum_1^T + \sum_{n=T+1}^\infty = N_1 + N_2,\ T = \left(\frac{1}{t}\right)^{\frac{1}{1-\alpha}},$$

$$\begin{aligned}N_1 &\le C_1\sum_{n=1}^T \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}\frac{\sin vt}{g(C/t)}\right|^k\\ &\le \frac{C_1}{(g(C/t))^k}\sum_{n=1}^T \frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^n \frac{P_{v-1}}{v}\sin vt\right|\end{aligned}$$

$$\times \left|\left(\sum_{v=1}^{n} \frac{P_{v-1}}{v} \sin vt\right)\right|^{k-1}$$

$$\leq \frac{C_1}{g(C/t)^k} \sum_{1}^{T} \frac{p_n}{P_n P_{n-1}} \left(\sum_{v=1}^{n} \frac{P_{v-1}}{v}\right)$$

$$\leq \frac{C_1}{(g(C/t))^k} \sum_{n=1}^{T} \frac{P_{v-1}}{v} \sum_{n=v}^{T} \frac{p_n}{P_n P_{n-1}}$$

$$\leq \frac{C_1}{(g(C/t))^k} \sum_{n=1}^{T} \frac{1}{v} = O(1) \frac{(\log C/t)}{g(C/t)^k}$$

$$= O(1) \frac{1}{(g(C/t))^{k-1}} = O(1).$$

Again in view of (3.11)

$$N_2 \leq \frac{1}{(g(C/t))^k} \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left|\sum_{v=1}^{n} \frac{P_{v-1}}{v} \sin vt\right| \left|\sum_{v=1}^{n} \frac{P_{v-1}}{v} \sin vt\right|^{k-1}$$

$$= \frac{1}{(g(C/t))^k} \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left|\sum_{v=1}^{n} \frac{P_{v-1}}{v} \sin vt\right|$$

$$\leq \frac{C_1 t^{-1}}{(g(C/t))^k} \sum_{n=T+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-1}}{n}$$

$$= \frac{C_1 t^{-1}}{(g(C/t))^k} \sum_{n=T+1}^{\infty} \frac{p_n}{n p_n} = O(1) \frac{t^{-1}}{(g(C/t))^k} \sum_{n=T+1}^{\infty} \frac{1}{n^{2-\alpha}}$$

$$= O(1) \left(\frac{1}{(g(C/t))^k}\right) = O(1).$$

This proves (3.15).

Since $M_2(t) = O(1/g(v))$, the left hand side of (3.16) is less or equal than

$$\begin{aligned} & C_1 \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{v=1}^{n} \frac{P_{v-1}}{v} \frac{1}{g(v)} \right)^k \\ \leq \; & C_1 \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \sum_{v=1}^{n} \frac{P_{v-1}}{v g(v)} \left(\sum_{v=1}^{n} \frac{P_{v-1}}{v g(v)} \right)^{k-1} \\ \leq \; & C_1 \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}}{v g(v)} \\ = \; & C_1 \sum_{v=1}^{\infty} \frac{P_{v-1}}{v g(v)} \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ \leq \; & C_1 \sum_{v=1}^{\infty} \frac{1}{v g(v)} < \infty \end{aligned}$$

in view of (3.10). Similarly $M_3 = O(\frac{1}{g(v)})$ and so as in the case of (3.16), (3.17) holds. This establishes Theorem 2.

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Dept. of Math. and Computer Science
Kuwait University
Box 5969, Kuwait-13060
E-mail: mazhar@mcs.sci.kuniv.edu.kw