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ON THE INTEGRAL REPRESENTATION OF EXCESSIVE FUNCTIONS UNDER BOCHNER SUBORDINATION

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Abstract. Let $\mathbb{P}$ be a semigroup of kernels on a Lusin space E with associated resolvent $\mathbb{U}$, let β be a Bochner subordinator and let $\mathbb{P}^\beta$ be the subordinate semigroup of $\mathbb{P}$ by means of β. In this paper we give sufficient conditions to have an integral representation of $\mathbb{P}^\beta$-excessive functions in terms of $\mathbb{U}$-exit laws and β. As application, if $\mathbb{P}$ is the semigroup of a transient right Markov process X, we derive a probabilistic representation of $\mathbb{P}^\beta$-excessive functions in terms of additive functionals of X and β.

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1. INTRODUCTION

This paper is devoted to the integral representation of excessive functions under the potential theory defined by a semigroup of kernels, obtained after Bochner subordination. This subordination is a convenient way of transforming semigroup of kernels and their functional energies. A usual problem is to show that regularities properties are transferred from the given semigroup to the subordinated one. Our problem is different but is related to the usual problem, because we must have the stability of some properties such as the properness and the unicity of charges (cf. [16]). The key of the representation in our problem is the notion of *resolvents' exit laws* which is well known in the ergodic theory for resolvents [8, XII-3]. Thanks to this notion, authors in [15] found an integral representation of potentials by additive kernels. Also authors in [19] characterized subordinated exit laws in terms of initial entities. This describes clearly the importance of resolvents' exit laws.

Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markovian semigroup of kernels on a Lusin measurable space $(E, \mathcal{E})$ and let $\mathbb{U} = (U_p)_{p>0}$ be the associated resolvent. An exit law for $\mathbb{U}$ is a family $f := (f_p)_{p>0}$ of non-negative measurable functions on E satisfying

$$(1)\qquad f_p = f_q + (q-p)U_p f_q \quad ; \quad U_q f_p = U_p f_q, \qquad 0 < p < q.$$

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Let $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator. The subordinate semigroup $\mathbb{P}^\beta$ of $\mathbb{P}$ by means of β is defined by

$$P_t^\beta := \int_0^\infty P_s \, \beta_t(\mathrm{d}s), \qquad t > 0 \tag{2}$$

We are interested in Bochner subordinator β of (K)-type, that is $\kappa := \int_0^\infty \beta_s \,\mathrm{d}s$ is absolutely continuous with respect to λ and its density is completely monotone. Let ρ be the associated measure on $[0, \infty[$ given by $\kappa = \mathcal{L}(\rho) \cdot \lambda$ and μ be a reference measure for $\mathbb{U}$. The first aim of the present paper is to prove, under finiteness conditions, that $\int_0^\infty f_s \,\rho(\mathrm{d}s)$ is equal μ-almost everywhere to a $\mathbb{P}^\beta$-excessive function, for each $\mathbb{U}$-exit law (f_p). The natural question that arises is the following: given a $\mathbb{P}^\beta$-excessive function h, can us find a (unique) $\mathbb{U}$-exit law (f_p) such that $h = \int_0^\infty f_s \rho(\mathrm{d}s), \mu$-a.e.? The study of this question is the main goal of this paper and we will solve, under some appropriate assumptions, this converse problem. Precisely, we will suppose that $\mathbb{P}$ admits a dual semigroup $\widehat{\mathbb{P}}$, both are proper and the cones of their μ-a.e. finite excessive functions are inf-stable and generates $\mathcal{E}$. Moreover β will be supposed belonging to a subclass of (K)-type subordinators as described later. Based on [17], the idea is to represent first $\mathbb{U}^\beta$-purely excessive measures by $\mathbb{U}$-entrance laws and next by using Hunt's approximation Theorem, where $\mathbb{U}^\beta$ is the resolvent of $\mathbb{P}^\beta$.

Our integral representation is a generalisation of some result given in [19], without imposing restrictive conditions on excessive functions. Similar integral representation by means of semigroups exit laws was studied in many papers, see for example [1, 9, 10, 13, 14, 18].

Let X and $\widehat{X}$ be transient right Markov processes with associated semigroups $\mathbb{P}$ and $\widehat{\mathbb{P}}$, in duality with respect to μ. As a consequence of the main result we prove, for each μ-a.e. finite $\mathbb{P}^\beta$-excessive function, that there exists a unique additive functional (A_t) for X such that $h(x) = \mathbb{E}^x(\int_0^\infty \mathcal{L}(\rho)(t)\, dA_t)$, μ-a.e.. Integral representation of excessive functions in terms of additive functionals was investigated in [20], for the particular case when β is the trivial subordinator.

2. PRELIMINARIES

Let E be a Lusin measurable space equipped with its Borel σ-field $\mathcal{E}$ (which denotes also the cone of all $\mathcal{E}$-measurable functions). We denote by $p\mathcal{E}$ the cone of positive functions of $\mathcal{E}$ and by $\mathcal{M}$ the cone of σ-finite positive measures on E.

A kernel on E is a mapping $K : E \times \mathcal{E} \to [0, \infty[$ such that $x \to K(x, A)$ is measurable for each $A \in \mathcal{E}$ and $A \to K(x, A)$ is a (positive) measure for each $x \in \mathcal{E}$. In this case, K acts to the right on $p\mathcal{E}$ and to the left on $\mathcal{M}$ by $Kf(x) := \int f(y)\, K(x, \mathrm{d}y)$ for $f \in p\mathcal{E}, x \in E$ and $\mu K(A) := \int K(x, A)\, \mu(\mathrm{d}x)$ for $\mu \in \mathcal{M}, A \in \mathcal{E}$. In the sequel we fix $\mu \in \mathcal{M}$, a property holds μ-a.e.

means that this property holds except on a μ-negligible set. We put $\mathcal{F} := \{u \in \mathcal{E} \ : \ u \,\text{is finite}, \mu\text{-a.e.}\}$. We endow $\mathbb{R}_+$ with its Borel field $\mathcal{A}$ and we denote by λ the Lebesgue measure on $\mathbb{R}_+$. The notation $\mathcal{L}(\tau)$ stands for the Laplace transform of a positive measure τ on $\mathbb{R}_+$ and δ_t stands for the Dirac measure at $t \in [0,\infty]$. We denote by *id* the identity function on $\mathbb{R}_+$. Finally, we abbreviate the expression "the monotone convergence theorem" by MCT.

In the following section we will introduce some definitions which will be useful in the remainder of this paper, for more details see [8, Chap. VII], [6, Sec. II-1,2,3] and [22, Sec. 1].

2.1. SEMIGROUPS AND RESOLVENTS OF KERNELS

A (sub-Markovian) semigroup $\mathbb{P} := (P_t)_{t>0}$ on E is a family of kernels on $(E,\mathcal{E})$ such that

(1) $(t,x) \to P_t f(x)$ is $\mathcal{A}\otimes\mathcal{E}$-measurable for each $f \in \mathcal{E}$
(2) $P_t 1 \leq 1$ and $P_s P_t = P_{s+t}$ for all $s,t > 0$

Two semigroups are said to be in duality with respect to $\mu \in \mathcal{M}$ provided $\int P_t u\, v \,\mathrm{d}\mu = \int u\, \widehat{P}_t v \,\mathrm{d}\mu$ for each $u, v \in p\mathcal{E}$ and all $t > 0$.

Let $\mathbb{P} := (P_t)_{t>0}$ be a semigroup on E, then the family $\mathbb{U} := (U_p)_{p>0}$ defined by

$$U_p = \int_0^\infty \exp(-pt)\, P_t \,\mathrm{d}t, \qquad t > 0$$

is called the resolvent of $\mathbb{P}$. It satisfies $pU_p 1 \leq 1$ for each $p > 0$ and

$$U_p = U_q + (q-p) U_p U_q \quad ; \quad U_q U_p = U_p U_q, \qquad 0 < p < q$$

Since the mapping $p \to U_p$ is decreasing then we may define the initial kernel U of the resolvent $\mathbb{U}$ by $U := U_0 := \sup_{p>0} U_p = \int_0^\infty P_s \,\mathrm{d}s$, which is called the *potential kernel* of $\mathbb{P}$. The resolvent equation may be extended to $p = 0$:

$$U = U_q + qU_qU, \qquad q > 0$$

For a given $q > 0$, the family $\mathbb{U}^q := (U_{p+q})_{p>0}$ is the resolvent of $\mathbb{Q}^q = (e^{-qt} P_t)_{t>0}$. Following [8, VII, p. 7], we say that $\mathbb{P}$ is proper if there exists a strictly positive function l such that Ul is bounded.

Remember that a set $N \in \mathcal{E}$ is called of potential zero if $U_p 1_N = 0$ for some $p > 0$. By using the resolvent equation we have the same property for all $p > 0$.

If $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are in duality then their resolvents are also in duality, that is $\int U_t u\, v \,\mathrm{d}\mu = \int u\, \widehat{U}_t v \,\mathrm{d}\mu$ for every $u, v \in p\mathcal{E}$ and all $p > 0$.

The resolvent $\mathbb{U}$ is said to be μ-basic if there exists a mesurable function $G :]0,\infty[\times E \times E \to [0,\infty]$ such that

$$U_p u(x) = \int G_p(x,y)\, u(y) \mu(\mathrm{d}y), \qquad x \in E.$$

Following [22, p. 271], a proper semigroup $\mathbb{P}$ is said to satisfy the *principle unicity of charges* (UC), if for all positive measures ν_1, ν_2 on E.

$$\nu_1 U = \nu_2 U \in \mathcal{M} \Rightarrow \nu_1 = \nu_2.$$

2.2. EXCESSIVE STRUCTURE

A function $h \in p\mathcal{E}$ is called $\mathbb{P}$-excessive (resp.$\mathbb{U}$-excessive) if $P_t h \leq h$ for all $t > 0$ (supermedianity) and $P_t h \to h$ as $t \to 0$ (resp. $pU_p h \uparrow h$ as $p \to \infty$). In the same way, a measure $m \in \mathcal{M}$ is called $\mathbb{P}$-excessive (resp. $\mathbb{U}$-excessive) provided $mP_t \uparrow m$ as $t \to 0$ (resp. $pmU_p \uparrow m$ as $p \to \infty$). We say that $m \in \mathcal{M}$ is $\mathbb{U}$-purely excessive if it is $\mathbb{U}$-excessive and $pmU_p \downarrow 0$ as $p \downarrow 0$. For $m \in \mathcal{M}$ satisfying $mU \in \mathcal{M}$, it is known that mU is $\mathbb{U}$-purely excessive. According to [8, XII 18], $\mathbb{P}$-excessive functions are exactly $\mathbb{U}$-excessive functions. Analogously, we can prove that there is identity between excessive measures for $\mathbb{P}$ and $\mathbb{U}$. We denote by $\mathrm{Exc}(\mathbb{P})$ the cone of $\mathbb{P}$-excessive measures and by $\mathcal{S}(\mathbb{P})$ the cone of $\mathbb{P}$-excessive functions belonging to $\mathcal{F}$. If $\mathbb{P}$ admits a dual $\widehat{\mathbb{P}}$ with respect to μ, it is well known that the set $\{h \cdot \mu : h \in \mathcal{S}(\mathbb{P})\} \subset \mathrm{Exc}(\widehat{\mathbb{P}})$. Let $\mathbb{P}$ be a proper resolvent, the function $L : \mathcal{S}(\mathbb{P}) \times \mathrm{Exc}(\mathbb{P}) \to [0, \infty]$ defined by

$$L(h, l) := \sup\{\nu(h) : \nu U \in \mathcal{M},\, \nu U \leq l\}$$

was introduced by Meyer [8, p. 23-24] and called *the energy functional* associated to $\mathbb{P}$.

In the sequel we suppose that μ is a reference measure for $\mathbb{U}$ that is $\mathbb{U}$ is μ-basic and μ is $\mathbb{U}$-excessive. In this case sets of potential zero are exactly μ-negligeable sets. We index by "$\widehat{\ }$" all entities associated to $\widehat{\mathbb{P}}$.

3. EXIT AND ENTRANCE LAWS FOR RESOLVENTS

The following notions of exit laws and entrance laws are taken from [8, p. 38-40].

A $\mathbb{U}$-entrance law is a family $m := (m_p)_{p>0} \subset \mathcal{M}$ such that for all $0 < p < q$:

$$m_p = m_q + (q-p)\, m_q U_p \quad ; \quad m_p U_q = m_q U_p$$

Let m be a $\mathbb{U}$-entrance law, then the mapping $p \mapsto m_p$ is increasing as $p \downarrow 0$ and $m_0 := \sup_{p>0} m_p$ is a positive measure.

A $\mathbb{U}$-exit law is a family $f := (f_p)_{p>0}$ of nonnegative functions of $\mathcal{F}$ satisfying the functional equation (1).

If (1) holds μ-a.e. we say that (f_p) is a μ-exit law for $\mathbb{U}$. Let f be a $\mathbb{U}$-exit law, then the mapping $p \mapsto f_p$ is increasing as $p \downarrow 0$ and $f_0 := \sup_{p>0} f_p$ is $\mathbb{P}$-supermedian. Moreover $f_\infty = \inf_{p>0} f_p$ is finite and satisfies $f_\infty = 0$, μ-a.e.. So the function f_p is equal μ-a.e. to some $\mathbb{Q}^p$-excessive function for each $p \geq 0$. For more examples of $\mathbb{U}$-exit laws we refer the reader to [15, p. 125]. Note that the family $(f_{p+q})_{p>0}$ is a $\mathbb{U}^q$-exit law for each $q > 0$.

Lemma 3.1. *Let f be a $\mathbb{U}$ exit law. Then $U_{p+s}f_{q+s} \uparrow U_p f_q$ as $s \to 0$, for each $p, q > 0$.*

Lemma 3.2. *Let $\mathbb{P}$ and $\widehat{\mathbb{P}}$ be semigroups in duality with respect to μ. Let (m_p) be a $\mathbb{U}$-entrance law such that m_p is absolutely continuous with respect to μ for each $p > 0$ and let $\widehat{f}_p := dm_p/d\mu$. Then $(\widehat{f}_p)$ is a μ-exit law for $\widehat{\mathbb{U}}$.*

Proof. Since $m_p \in \mathcal{M}$, then $f_p \in \mathcal{F}$ for each $p > 0$. For $0 < p < q$, we have from the entrance law equation

$$(\widehat{f}_p - \widehat{f}_q) \cdot \mu = \widehat{f}_p \cdot \mu - \widehat{f}_q \cdot \mu = (q-p)(\widehat{f}_q \cdot \mu)U_p = (q-p)(\widehat{U}_p \widehat{f}_q) \cdot \mu$$

and $(\widehat{U}_p \widehat{f}_q) \cdot \mu = (\widehat{f}_q \cdot \mu)U_p = (\widehat{f}_p \cdot \mu)U_q = (\widehat{U}_q \widehat{f}_p) \cdot \mu$. Which yields the result. □

Lemma 3.3. *Let (g_p) be a μ-exit law for $\mathbb{U}$, then there exists a $\mathbb{U}$-exit law (f_p) such that $f_p = g_p, \mu$-a.e.*

Proof. Define for $n \in \mathbb{N}^*$: $g_p^n(x) := nU_n g_p(x)$. Let $p > 0, n > p$ and $r = n-p$ then

$$g_p^n = (r+p)\, U_{r+p} g_p = \frac{r+p}{r}\, rU_{r+p} g_p$$

Since g_p is $\mathbb{Q}^p$-supermedian then $r \to rU_{r+p}g_p$ is increasing as $r \uparrow \infty$. Hence $f_p(x) := \lim_{n\to\infty} g_p^n(x)$ exists and belongs to $\mathcal{F}$ for each $p > 0$ and $x \in E$. By (1) and the fact that $\mathbb{U}$ is μ-basic we get for all $0 < p < q$

$$nU_n g_p(x) - nU_n g_q(x) = (q-p)\, U_p(nU_n g_q)(x), \qquad x \in E \tag{3}$$

Letting $n \to \infty$ in (3) and using MCT we deduce that (f_p) is a $\mathbb{U}$-exit law. In the other hand

$$nU_n g_p = r\, U_{r+p} g_p = g_p - g_{r+p}$$

By letting $r \to \infty$ we obtain $f_p = g_p, \mu$-a.e. □

4. BOCHNER SUBORDINATION AND INTEGRAL REPRESENTATION

4.1. BOCHNER SUBORDINATION

For the following notion we refer the reader to [5, Chap. II-9], [6, Sec. V-3], [11] and [12].

A *Bochner subordinator* $\beta = (\beta_t)_{t>0}$ is a family of sub-probability measures on $(\mathbb{R}_+, \mathcal{A})$ such that that

(1) $\beta_t * \beta_s = \beta_{s+t}$ for all $s, t > 0$.
(2) $\lim_{t\to 0} \beta_t = \delta_0$ vaguely.

For each $p > 0$, we put $\kappa_p := \int_0^\infty e^{-ps}\beta_s\, \mathrm{d}s$ and $\kappa := \kappa_0 := \sup_p \kappa_p = \int_0^\infty \beta_s\, \mathrm{d}s$.

The associated Bernstein function ϕ is given by the relation $\mathcal{L}\beta_t(s) = \exp(-t\phi(s))$ for each $s, t > 0$.

Let $\mathbb{P}$ be a semigroup on E and β be a Bochner subordinator. Then the subordinate semigroup of $\mathbb{P}$ by means of β is defined by (2).

Let $\mathbb{U}^\beta$ be the resolvent of $\mathbb{P}^\beta$ then we can write for all $p > 0$

$$U_p^\beta = \int_0^\infty P_s \, \kappa_p(\mathrm{d}s) \tag{4}$$

A Bochner subordinator is said to be of (K)-*type* if there exists a completely monotone function ψ on $]0, \infty[$ such that $\kappa = \psi \cdot \lambda$.

Let β be a subordinator of (K)-type then $\psi = \mathcal{L}(\rho)$ for some non-negative measure ρ on $[0, \infty[$ due to the Bernstein Theorem. According to [12, Proposition 11], ψ is integrable at 0 and $\kappa_p(\mathrm{d}t) = \psi_p(t) \cdot \mathrm{d}t$ where ψ_p is also a completely monotone and integrable function on $]0, \infty[$, for each $p > 0$. Therefore ψ_p is also the Laplace transform of a non-negative measure ρ_p on $[0, \infty[$. Following [12, p. 157], we have $\rho_p(\{0\}) = 0$ and $\int_0^\infty \frac{1}{s} \rho_p(\mathrm{d}s) \leq \frac{1}{p}$ for all $p > 0$. Moreover from [11, p. 240], it was affirmed that

$$\lim_{p \to 0} \frac{1}{1+t} \, \rho_p(\mathrm{d}t) = \frac{1}{1+t} \, \rho(\mathrm{d}t) \quad \text{weakly.} \tag{5}$$

We give now the most important subordinator η^α, defined by its Bernstein function $\phi^\alpha(x) = x^\alpha$ for $\alpha \in]0, 1[$. It is called the one sided stable subordintor. Following [6, p. 187] we have $\kappa^\alpha = \psi^\alpha \cdot \lambda = \mathcal{L}(\rho^\alpha) \cdot \lambda$ where

$$\psi^\alpha(s) = \frac{s^{\alpha-1}}{\Gamma(\alpha)} \, 1_{]0,\infty[}(s) \quad \text{and} \quad \rho^\alpha(\mathrm{d}s) = \frac{s^\alpha}{\Gamma(\alpha) \, \Gamma(1-\alpha)} \, 1_{]0,\infty[}(s)$$

Let β be a Bochner subordinator of (K)-type, then we have

$$U_p^\beta = \int_0^\infty U_s \, \rho_p(\mathrm{d}s), \qquad p > 0 \tag{6}$$

Since $\kappa_p \uparrow \kappa$ as $p \to 0$ then (4) and (6) may be extended to $p = 0$. We denote by L^β the energy functional associated to $\mathbb{U}^\beta$. In the sequel subordinators are considered of (K)-type. We denote by $\mathcal{H}$ the set of Bochner subordinators β of (K)-type such that $\rho([0, \varepsilon[) > 0$ for all $\varepsilon > 0$. Note that the trivial subordinator $\varepsilon = (\varepsilon_t)_{t>0} \in \mathcal{H}$. Also, if ρ is absolutely continuous with respect to λ then $\beta \in \mathcal{H}$, in particular $\eta^\alpha \in \mathcal{H}$.

Let $f := (f_p)_{p>0}$ be a $\mathbb{U}$-exit law, we denote by $f^\beta := (f_p^\beta)_{p>0}$ the family defined by $f_p^\beta = \int_0^\infty f_s \, \rho_p(\mathrm{d}s)$. According to [19, Proposition 4.3], f^β is a μ-exit law for $\mathbb{U}^\beta$ whenever $f_0 \in \mathcal{F}$.

Proposition 4.1. *If $\mathbb{P}$ is proper then $\mathbb{P}^\beta$ is proper. Moreover $\mathcal{S}(\mathbb{P}) \subset \mathcal{S}(\mathbb{P}^\beta)$ and* $\mathrm{Exc}(\mathbb{P}) \subset \mathrm{Exc}(\mathbb{P}^\beta)$.

Theorem 4.2. *Let $f = (f_p)_{p>0}$ be a $\mathbb{U}$-exit law such that $f_0 \in \mathcal{F}$ and $f_q \in L^1(\mu)$ for some $q > 0$, then the function $h := \int_0^\infty f_s \, \rho(\mathrm{d}s)$ is equal μ-a.e. to some $\mathbb{P}^\beta$-excessive function.*

Proof. Suppose first that $f_0 \in \mathcal{F}$. According to Proposition 4.1, μ is also a reference measure for $\mathbb{U}^\beta$. Taking into account that f^β is a $\mathbb{U}^\beta$-exit law then $f_\infty^\beta = 0$, μ-a.e. Therefore f_0^β is equal μ-a.e. to a $\mathbb{P}^\beta$-excessive function. We shall prove that $f_0^\beta = h$, μ-a.e.. Making use of the relation $U_q f_p^\beta = U_p^\beta f_q$ for each $p, q > 0$ together with MCT, we get

$$U_q f_0^\beta = U_q \left(\lim_{p\to 0} f_p^\beta \right) = \lim_{p\to 0} U_q f_p^\beta = U^\beta f_q = \int_0^\infty U_s f_q \, \rho(\mathrm{d}s) = U_q h. \tag{7}$$

Moreover

$$U_q h \le \left(\frac{1}{q} + 1) \, f_0 \int_0^\infty \frac{1}{1+s} \, \rho(\mathrm{d}s \right) < \infty, \quad \mu\text{-a.e.} \tag{8}$$

Suppose first that $f_p \in L^1(\mu)$ for all $p > 0$. Denote by $\mathcal{B}$ the σ-field generated by functions of the form $U_r l$ for $l \in L^1(\mu)$ and $r > 0$. The fact that $f_\infty = 0$, $\mu - a.e$ implies that f_p is equal $\mu - a.e$ to some $\mathbb{Q}^p$-excessive function. Without loss of generality we can suppose that f_p is $\mathbb{Q}^p$-excessive so f_p is $\mathcal{B}$-measurable for each $p > 0$. The continuity of the mapping $p \to f_p(x)$ on $[0, \infty[$ yields the $\mathcal{A}\otimes\mathcal{B}$-measurability of $(p, x) \to f_p(x)$. In view of the boundedness of measures $(1+s)^{-1}\rho(\mathrm{d}s)$ and $(1+s)^{-1}\rho_p(\mathrm{d}s)$, we affirm by Tonelli's Theorem that h and f_0^β are $\mathcal{B}$-measurable. From [8, XII 57], (7) and (8) we claim that $\lim_{q\to\infty} qU_q f_0^\beta = f_0^\beta$, $\mu - a.e.$ and $\lim_{q\to\infty} qU_q h = h$, μ-a.e. Consequently $f_0^\beta = h$, $\mu - a.e$ due to (7).

Now suppose that there exists $q > 0$ such that $f_q \in L^1(\mu)$ then $(f_p^q)_{p>0}$ is a $\mathbb{U}^q$-exit law included in $L^1(\mu)$ and $f_0^q = f_q < \infty$, μ-a.e.. According to the first case we have μ-a.e.:

$$\sup_{p>0} p \int_0^\infty U_r^q \left(\int_0^\infty f_s^q \, \rho(\mathrm{d}s) \right) \rho_p(\mathrm{d}r) = \int_0^\infty f_s^q \, \rho(\mathrm{d}s) \tag{9}$$

Using Fubini's Theorem, Lemma 3.1, (9) and MCT we get μ-a.e.

$$\begin{aligned}
\sup_{p>0} pU_p^\beta \int_0^\infty f_s \, \rho(\mathrm{d}s) &= \sup_{p>0} p \int_0^\infty U_r f_s \, \rho(\mathrm{d}s) \, \rho_p(\mathrm{d}r) \\
&= \sup_{p>0} \sup_{q>0} p \int_0^\infty U_{r+q} \, f_{s+q} \, \rho(\mathrm{d}s) \, \rho_p(\mathrm{d}r) \\
&= \sup_{q>0} \sup_{p>0} p \int_0^\infty U_r^q \int_0^\infty f_s^q \, \rho(\mathrm{d}s) \, \rho_p(\mathrm{d}r) \\
&= \sup_{q>0} \int_0^\infty f_s^q \, \rho(\mathrm{d}s) = \int_0^\infty f_s \, \rho(\mathrm{d}s)
\end{aligned}$$

□

4.2. INTEGRAL REPRESENTATION

Consider two semigroups $\mathbb{P}$ and $\widehat{\mathbb{P}}$ in duality with respect to μ. Suppose, until the end of this section that $\mathbb{P}$ and $\widehat{\mathbb{P}}$ verify the following condition (C):

(1) $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are proper and satisfy the principle uniqueness of charges.
(2) The cones $\mathcal{S}(\mathbb{P})$ and $\mathcal{S}(\widehat{\mathbb{P}})$ are inf-stable and generates $\mathcal{E}$.

Remark 4.3. We cite two situations when the condition (C) is satisfied:

(1) E is locally compact space with countable base and $\mathbb{P}$ together with $\widehat{\mathbb{P}}$ are proper strong feller semigroups on E. In this case excessive functions are lower-semi-continuous functions. The properness of $\mathbb{P}$ and $\widehat{\mathbb{P}}$ implies that $\mathcal{E}$ is generated by $\mathcal{S}(\mathbb{P})$ and $\mathcal{S}(\widehat{\mathbb{P}})$ as well.

(2) $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are associated to transient right Markov processes. This situation will be focused later.

The idea of the proof of the following proposition is adapted from the proofs of [17, Proposition 11 and Theorem 12].

Proposition 4.4. *Let Λ be a $\mathbb{U}^\beta$-purely excessive measure such that $L^\beta(\Lambda, v) < \infty$ for some $\mathbb{U}$-excessive function $v > 0$. Then there exists a unique $\mathbb{U}$-entrance law (m_p) such that $\Lambda = \int_0^\infty m_s\,\rho(\mathrm{d}s)$.*

Proof. According to [17, Theorem 6 and Remark 13] there exists a unique $\mathbb{U}$-purely excessive measure l such that $L^\beta(\Lambda, v) = L(l, v)$. Let $m_q := l - l(qU_q)$ for $q > 0$, then (m_q) is a $\mathbb{U}$-entrance law and $m_0 := \lim_{q\to 0} m_q = l$. From [17, Theorem 6] again, we have $L^\beta(\Lambda p U_p, v) = L(lpU_p, v)$. By reason of [8, XII 39.1] and the entrance law equation we have

$$L^\beta(m_pU^\beta, v) = m_p(v) = L(m_pU, v) = L(m_0\,U_p, v) = \frac{1}{p}L(lpU_p, v) =$$
$$= \frac{1}{p}L^\beta(\Lambda pU_p, v) = L^\beta(\Lambda U_p, v)$$

Hence, by [17, Proposition 9], we conclude that

$$m_pU^\beta = \Lambda U_p \tag{10}$$

Using the resolvent equation again and (10), we get for each $0 < p < q$

$$(m_p - m_q)\,U^\beta = \Lambda\,U_p - \Lambda\,U_q = (q-p)\,\Lambda U_pU_q = (q-p)m_pU^\beta U_q =$$
$$= (q-p)m_pU_q\,U^\beta$$

Following [15, Theorem 1], $\mathbb{P}^\beta$ satisfies also (UC) and consequently (m_p) is a $\mathbb{U}$-entrance law. Using (10) we get $\Lambda U^\beta = \int_0^\infty m_s\,\rho(\mathrm{d}s)\,U^\beta$. Put $\Upsilon = \int_0^\infty m_s\,\rho(\mathrm{d}s)$. Then for each $h \in \mathcal{S}(\mathbb{P})$ we obtain

$$\Lambda(h) = L^\beta(\Lambda U^\beta, h) = L^\beta(\Upsilon\,U^\beta, h) = \Upsilon(h)$$

So $\Lambda = \Upsilon$. For the uniqueness, suppose that there exists some $\mathbb{U}$-entrance law $(\widetilde{m}_p)$ such that $\Lambda = \int_0^\infty \widetilde{m}_s\rho(\mathrm{d}s)$, μ-a.e. Then we get $m_pU^\beta = \Lambda U_p = \widetilde{m}_pU^\beta$ for each $p > 0$ and the proof is achieved by using (UC). □

Lemma 4.5. *Let ϕ be the Bernstein function associated to β, then id/ϕ is a Bernstein function and*

$$U = U^\beta\,U^{\tilde{\beta}} \tag{11}$$

where $\widetilde{\beta}$ is the Bochner subordinator associated to id/ϕ.

Proof. The fact that $1/\phi = \mathcal{L}(\kappa) = \mathcal{L}(\mathcal{L}\rho)$ yields $1/\phi$ is a Stieltjes function (see [21, Definition 2.1]). According to [21, Proposition 7.1 and Theorem 7.3], id/ϕ is also a Bernstein function. We have

$$\mathcal{L}(\lambda) = \frac{1}{id} = \frac{1}{\phi}\,\frac{1}{\frac{id}{\phi}} = \mathcal{L}(\kappa) * \mathcal{L}(\widetilde{\kappa}) = \mathcal{L}(\kappa * \widetilde{\kappa})$$

where $\widetilde{\kappa} = \int_0^\infty \widetilde{\beta}_s \, \mathrm{d}s$. Thus $\kappa * \widetilde{\kappa} = \lambda$ and consequently

$$U = \int_0^\infty P_s \, \mathrm{d}s = \int_0^\infty \int_0^\infty P_{s+r}\, \kappa(\mathrm{d}s)\, \widetilde{\kappa}(\mathrm{d}r) = \int_0^\infty P_s U^{\widetilde{\beta}}\, \kappa(\mathrm{d}s) = U^\beta\, U^{\widetilde{\beta}}.$$

□

THEOREM 4.6. *Suppose that $\beta \in \mathcal{H}$. Then for each $h \in \mathcal{S}(\mathbb{P}^\beta)$, there exists a unique $\mathbb{U}$-exit law (f_p) such that $h = \int_0^\infty f_s \, \rho(\mathrm{d}s)$, μ-a.e.*

Proof. The fact that $\widehat{\mathbb{P}}$ is proper yields the existence of a positive function l such that $\widehat{U}l$ is bounded. Since $h \cdot \mu \in \mathrm{Exc}(\widehat{\mathbb{P}}^\beta)$ and $\widehat{\mathbb{P}}^\beta$ is proper, then there exists a sequence of bounded measures $(\nu_n) \subset \mathcal{M}$ such that $\nu_n \widehat{U}^\beta \uparrow h \cdot \mu$, due to Hunt's approximation Theorem [8, XII 38]. Let $\widehat{L}^\beta$ be the energy functional of $\widehat{\mathbb{P}}^\beta$. By virtue of [8, XII 39.1], we have for each $n \in \mathbb{N}$

$$\widehat{L}^\beta(\nu_n \widehat{U}^\beta, \widehat{U}l) = \int \widehat{U}l \, \mathrm{d}\nu_n < \infty$$

According to Proposition 4.4, there exists a $\widehat{\mathbb{U}}$-entrance law $(\widehat{m}_p^n)_{p>0}$ such that

$$\nu_n \widehat{U}^\beta = \int_0^\infty \widehat{m}_s^n \, \rho(\mathrm{d}s), \qquad n \in \mathbb{N} \tag{12}$$

From (10) we have $\nu_n \widehat{U}^\beta\, \widehat{U}_p = \widehat{m}_p^n\, \widehat{U}^\beta$, which implies that the sequence $(\widehat{m}_p^n\, \widehat{U}^\beta)_n$ is increasing for each $p > 0$. By reason of [8, XII 17], the properness of $\widehat{\mathbb{P}}$ leads to the existence of a sequence $(\varphi_k)_k \subset p\mathcal{E}$ such that $\widehat{U}\varphi_k \uparrow \widehat{h}$ for every $\widehat{h} \in \mathcal{S}(\widehat{\mathbb{U}})$. Let $n_1, n_2 \in \mathbb{N}$ such that $n_1 < n_2$, by using (11) we obtain

$$\widehat{m}_p^{n_1} \widehat{U}\varphi_k = \widehat{m}_p^{n_1} \widehat{U}^\beta (\widehat{U}^{\widetilde{\beta}} \varphi_k) \le \widehat{m}_p^{n_2} \widehat{U}^\beta (\widehat{U}^{\widetilde{\beta}} \varphi_k) = \widehat{m}_p^{n_2} \widehat{U}\varphi_k$$

letting $k \to \infty$ and using MCT we get $\widehat{m}_p^{n_1}(\widehat{h}) \le \widehat{m}_p^{n_2}(\widehat{h})$, which affirm that $(\widehat{m}_p^n)_n$ is increasing for each $p > 0$. Consequently $\widehat{m}_p := \lim_{n\to\infty} \widehat{m}_p^n$ is a positive measure on E. Letting $n \to \infty$ in (12) and applying MCT again we obtain

$$h \cdot \mu = \int_0^\infty \widehat{m}_s \, \rho(\mathrm{d}s) \tag{13}$$

From (13) we deduce that $(\widehat{m}_p) \subset \mathcal{M}$ and we can show easily that $(\widehat{m}_p)$ is a $\widehat{\mathbb{U}}$-entrance law. Let $A \in \mathcal{E}$ such that $\mu(A) = 0$ then $\int_0^\infty \widehat{m}_s(A)\, \rho(\mathrm{d}s) = 0$ due to (13). For each $s > 0$, since $\rho([0, s[) > 0$, then there exists $0 < r < s$ such that $\widehat{m}_r(A) = 0$ and consequently $\widehat{m}_s(A) = 0$ because $q \to \widehat{m}_q(A)$ is decreasing.

Therefore there exists a measurable function g_s such that $\widehat{m}_s = g_s \cdot \mu$ for each $s > 0$, by reason of Radon-Nikodym Theorem. According to Lemma 3.2, (g_p) is a μ-exit law for $\mathbb{U}$ and from Lemma 3.3, the integral representation of h holds for some $\mathbb{U}$-exit law (f_p). Now, let us prove the uniqueness. Suppose that there exists some $\mathbb{U}$-exit law $\tilde{f}$ such that $h = \int_0^\infty \tilde{f}_s \, \rho(\mathrm{d}s)$ μ-a.e., then we have for all $p > 0$

$$(14) \quad U^\beta f_p = \int_0^\infty U_s f_p \, \rho(\mathrm{d}s) = U_p \int_0^\infty f_s \, \rho(\mathrm{d}s) = U_p \int_0^\infty \tilde{f}_s \, \rho(\mathrm{d}s) = U^\beta \tilde{f}_p,$$

for all $p > 0$. Since h is supermedian for $\mathbb{U}^\beta$ and $\rho_p \neq 0$, then there exists $r > 0$ such that $U_r h < \infty$ and so $U_p h < \infty$ for all $p \geq r$. The duality property together with (14) yields

$$(f_p \cdot \mu)\widehat{U}^\beta = (U^\beta f_p) \cdot \mu = (U^\beta \widetilde{f}_p) \cdot \mu = (\widetilde{f}_p \cdot \mu)\, \widehat{U}^\beta, \qquad p \geq r$$

It follows that $\widetilde{f}_p = f_p$, μ-a.e. for all $p \geq r$, because $U^\beta f_p = U_p h < \infty$ and $\widehat{\mathbb{P}}^\beta$ satisfies (UC). By using (1) we get for $p < r$

$$\widetilde{f}_p - \widetilde{f}_r = (r-p) U_p \widetilde{f}_r = (r-p) U_p f_r = f_p - f_r$$

which implies $\widetilde{f}_p = f_p$ for all $p > 0$. □

COROLLARY 4.7. *Suppose that $\beta \in \mathcal{H}$. Then for each $\mathbb{U}^\beta$-exit law (g_p) satisfying $g_0 \in \mathcal{F}$, there exists a unique $\mathbb{U}$-exit law (f_p) such that $g_p = f_p^\beta$, μ-a.e. for each $p > 0$.*

Proof. We know that there exists $h \in \mathcal{S}(\mathbb{P}^\beta)$ such that $g_0 = h, \mu$-a.e.. From Theorem 4.6, there exists a unique $\mathbb{U}$-exit law (f_p) such that $g_0 = \int_0^\infty f_s \, \rho(\mathrm{d}s)$, μ-a.e.. Using (1) and (6) we obtain for each $p > 0$

$$U^\beta g_p = U_p^\beta g_0 = \int_0^\infty U_s f_p^\beta \, \rho(\mathrm{d}s) = U^\beta f_p^\beta \leq \frac{1}{p} g_0,$$

which implies $(g_p \cdot \mu)\widehat{U}^\beta = (f_p^\beta \cdot \mu)\widehat{U}^\beta \in \mathcal{M}$. The result is a consequence from (UC). □

5. APPLICATION

Let $X := (\Omega, \mathcal{F}, \mathcal{F}_t, (X_t), (\Theta_t), \mathbf{P}^x)$ be a right Markov process with state space $(E, \mathcal{E})$ (see [2, p. 306-307]). The associated semigroup $\mathbb{P} := (P_t)_{t>0}$ is given by

$$P_t f(x) = \mathbb{E}^x(f(X_t)), \qquad t > 0,\, x \in E,\, f \in p\mathcal{E}$$

If $\mathbb{P}$ is proper then X is called transient. It is known that h is $\mathbb{Q}^p$-excessive if and only if the process $(e^{-pt} h(X_t))$ is a right continuous $(\mathcal{F}_t)$-supermartingale with respect to $\mathbf{P}^x$ for all $x \in E$ (for more details we refer the reader to [4, Appendix p. 418-419]).

An additive functional (A_t) for X is an increasing right continuous process, $(\mathcal{F}_t)$-adapted, satisfying $A_0 = 0$ and for all $s, t > 0$: $A_{s+t} = A_s + A_t \circ$

Θ_s, $\mathbf{P}^x$-a.e.. We put $e_p(A)(x) := \mathbb{E}^x[\int_0^\infty \exp(-pt)\,\mathrm{d}A_t]$. According to [8, XV 29], the family $(e_p(A))$ is a $\mathbb{U}$-exit law when it is included in $\mathcal{F}$.

Let β be a subordinator of (K)-type and let Y be the right Markov process whose semigroup is $\mathbb{P}^\beta$. The process Y is called the subordinate of X by means of β. Now, let (A_t) be an additive functional of X. It follows from Theorem 4.2 that the function h defined by

(15)
$$h(x) = \mathbb{E}^x\left(\int_0^\infty \psi(t)\,\mathrm{d}A_t\right) = \mathbb{E}^x\left(\int_0^\infty \mathcal{L}\rho(t)\,\mathrm{d}A_t\right) = \int_0^\infty e_s(A)(x)\,\rho(\mathrm{d}s)$$

is equal μ-a.e. to a $\mathbb{P}^\beta$-excessive function whenever $\mathbb{E}^x(A_\infty) < \infty, \mu$-a.e. and $e_q(A) \in L^1(\mu)$ for some $q > 0$.

In the next Theorem we will prove the converse while supposing that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, (X_t), (\Theta_t), \mathbf{P}^x)$ and $\widehat{X} = (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, (\widehat{X}_t), (\widehat{\Theta}_t), \widehat{\mathbf{P}^x})$ are two right transient Markov processes on $(E, \mathcal{E})$ and their associated semigroups are in duality with respect to μ. According to [2, Proposition 1.8.2], $\mathbb{P}$ and $\widehat{\mathbb{P}}$ satisfy the condition (C).

THEOREM 5.1. *If $\beta \in \mathcal{H}$, then for each $h \in \mathcal{S}(\mathbb{P}^\beta)$, there exists an additive functional (A_t) for X such that*

$$h(x) = \mathbb{E}^x\left(\int_0^\infty \psi(t)\,\mathrm{d}A_t\right), \quad \mu\text{-a.e.} \tag{16}$$

The uniqueness holds whenever X is continuous.

Proof. By Theorem 4.6 we have $h = \int_0^\infty f_s\,\rho(\mathrm{d}s)$ for some $\mathbb{U}$-exit law (f_p). The fact that f_p is equal μ-a.e. to some $\mathbb{Q}^p$-excessive function and based on [8, XV 7-b)], there exists an additive functional (A_t) for X and a $\mathbb{Q}^p$-excessive function ℓ_p such that $(\mathrm{e}^{-pt}\ell_p(X_t))$ is a local martingale and

$$f_p = e_p(A) + \ell_p, \quad \mu\text{-a.e.}, \qquad p > 0 \tag{17}$$

It is clear that $(e_p(A))$ is a $\mathbb{U}$-exit law. In the other hand, the random variable $T_n := \inf\{s > 0 : \ell_p(X_t) > n\}$ is a stopping time for each $n \in \mathbb{N}$, because the mapping $s \to \mathrm{e}^{-ps}\ell_p(X_s)$ is right continuous μ-a.e.. Let (S_n) be a sequence of stopping times such that $(\mathrm{e}^{-pt}\ell_p(X_{t\wedge S_n}))$ is a martingale then $(\mathrm{e}^{-pt}\ell_p(X_{t\wedge S_n\wedge T_n}))$ is also a martingale, for the reason that $T_n \uparrow \infty$. Taking into account that

$$\mathrm{e}^{-pt}(\ell_p \wedge n)(X_{S_n\wedge T_n\wedge t}) = \mathrm{e}^{-pt}\ell_p(X_{S_n\wedge T_n\wedge t})$$

for all $n \in \mathbb{N}$ and $t > 0$, then $(\mathrm{e}^{-pt}(\ell_p \wedge n)(X_t))$ is a bounded locale martingale and therefore it is a martingale. Hence $\ell_p \wedge n$ is $\mathbb{Q}^p$-invariant, meaning that $qU_{q+p}(\ell_p \wedge n) = \ell_p \wedge n$ for each $q > 0$. By letting $n \to \infty$ and applying MCT we get $qU_{q+p}\ell_p = \ell_p$. From (17) we affirm that (ℓ_p) is also a $\mathbb{U}$-exit law, therefore, for every $p > 0$, $\ell_p = \lim_{q\to 0} qU_{q+p}\ell_p = \lim_{q\to 0}(\ell_p - \ell_{p+q}) = 0$. Consequently $f_p = e_p(A)$, μ-a.e., and from (15) we get (16). To prove the uniqueness, suppose that there exists some additive functional (B_t) for X such that $h(x) = \mathbb{E}^x(\int_0^\infty \psi(t)\,\mathrm{d}B_t)$, μ-a.e.. Then we obtain

$$h = \int_0^\infty e_s(A)\,\rho(\mathrm{d}s) = \int_0^\infty e_s(B)\rho(\mathrm{d}s), \quad \mu\text{-a.e.} \tag{18}$$

Since $\beta \in \mathcal{H}$, it follows from (18) that $e_p(B) \in \mathcal{F}$ for each $p > 0$, and so $(e_p(B))$ is a $\mathbb{U}$-exit law. According to the uniqueness in Theorem 4.6, we affirm that $e_p(A) = e_p(B)$, μ-a.e. for all $p > 0$. Consequently $\mathbb{E}^x(A_t) = \mathbb{E}^x(B_t)$, μ-a.e. Thanks to [7, p. 159], we get $A_t = B_t$ due to the above and the continuity of X. □

Corollary 5.2. *For each $h \in \mathcal{S}(\mathbb{P})$, there exists an additive functional (A_t) for X such that $h(x) = \mathbb{E}^x(A_\infty)$, μ-a.e.. The uniqueness holds whenever X is continuous.*

Corollary 5.3. *Let $\alpha \in]0,1[$ and $h \in \mathcal{S}(\mathbb{P}^{\eta^\alpha})$. Then there exists some additive functional (A_t) such that*

$$h(x) = \frac{1}{\Gamma(\alpha)}\mathbb{E}^x\left(\int_0^\infty t^{\alpha-1}\,\mathrm{d}A_t\right), \quad \mu\text{-a.e.}$$

If X is continuous, the uniqueness holds.

Example 5.4. Let X be a Brownian motion on $\mathbb{R}^d$ and let Y be the subordinate of X by means of η^α. For a bounded domain $D \subset \mathbb{R}^d$, the process Y^D is obtained by killing Y upon leaving D. The process Z is defined as the result of first killing X upon leaving D, and then subordinating the killed Brownian motion X^D using η^α.

Let h be a quasimartingale function for Y^D. According to [3, Corollary 3.7], h is also a quasimartingale function for Z. Furthermore, by [3, Corollary 2.7], there exist two excessive functions h_1, h_2 for the semigroup of Z, such that $h = h_1 - h_2$. Using Theorem 4.6, h can be represented in terms of two exit laws f and g for the resolvent of X^D:

$$h = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty (f_s - g_s)s^\alpha \mathrm{d}s, \quad \mu\text{-a.e.}$$

Morover, Theorem 5.1 guarantees the existence of two additive functionals A and B for X^D such that:

$$h(x) = \frac{1}{\Gamma(\alpha)}\mathbb{E}^x\left[\int_0^\infty s^{\alpha-1}\mathrm{d}A_s - \int_0^\infty s^{\alpha-1}\mathrm{d}B_s\right], \quad \mu\text{-a.e.}$$

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