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SOME RESEARCH DIRECTIONS IN FIBRE CONTRACTION THEORY AND ITS APPLICATIONS

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Abstract. In this paper, we will discuss the fixed point theory of triangular operators in the setting of generalized metric spaces and for contraction type operators. Global asymptotic stability of the fixed point, well-posedness of the fixed point problem, Ulam-Hyers stability and Ostrowski property are investigated. Some applications of the basic fibre contraction principles are also considered.

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1. INTRODUCTION

In this paper, we will present the fixed point theory of triangular operators in the setting of generalized metric spaces and for contraction type operators. Global asymptotic stability of the fixed point, well-posedness and Ulam-Hyers stability of the fixed point problem, as well as the Ostrowski stability property are investigated. Some applications of the basic fibre contraction principles are also considered. Throughout this paper we follow the notation and terminology in [8]. See also [57, 46, 40].

2. FIBRE CONTRACTION PRINCIPLE

The starting result in fibre contraction theory is the following one, see ([17]).

THEOREM 2.1 (Hirsch-Pigh (1970)). *Let (X_1, d_1) be a metric space and $T_1 : X_1 \to X_1$ be an operator having an attractive fixed point $x_1^* \in X_1$. Let (X_2, d_2) be a complete metric space and $T_2 : X_1 \times X_2 \to X_2$ be an operator such that:*

(i) *There exists $l \in]0, 1[$ such that $T_2(x_1, \cdot) : X_2 \to X_2$ is an l-contraction, $\forall\, x_1 \in X_1$.*

(ii) *The triangle operator, $T : X_1 \times X_2 \to X_1 \times X_2$,*

$$T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2))$$

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is continuous.

Let x_2^* be the fixed point of $T_2(x_1^*, \cdot)$. Then (x_1^*, x_2^*) is an attractive fixed point of T.

Proof. The proof is a standard one in the successive approximations theory.

Let $(x_{1,0}, x_{2,0}) \in X_1 \times X_2$ and $(x_{1,n+1}, x_{2,n+1}) = T(x_{1,n}, x_{2,n})$, $n \in \mathbb{N}$, i.e., $x_{1,n+1} = T_1(x_{1,n})$ and $x_{2,n+1} = T_2(x_{1,n}, x_{2,n})$, $n \in \mathbb{N}$. By the contraction principle, it is clear that $x_{1,n} \to x_1^*$ as $n \to \infty$.

Now, let for all $n \in \mathbb{N}^*$

$$\begin{aligned} d_2(x_{2,n+1}, x_2^*) &= d_2(T_2(x_{1,n}, x_{2,n}), T_2(x_1^*, x_2^*)) \\ &\leq d_2(T_2(x_{1,n}, x_{2,n}), T_2(x_{1,n}, x_2^*)) + d_2(T_2(x_{1,n}, x_2^*), T_2(x_1^*, x_2*)) \\ &\leq l d_2(x_{2,n}, x_2^*) + d_2(T_2(x_)1, n, x_2^*), T_2(x_1^*, x_2^*) \leq \dots \\ &\leq l^{n+1} d_2(x_{2,0}, x_2^*) + l^n d_2(T_2(x_{1,0}, x_2^*), T_2(x_1^*, x_2^*)) + \dots \\ &+ l d_2(T_2(x_{1,n-1}, x_2^*), T_2(x_1^*, x_2^*)) + d_2(T_2(x_{1,n}, x_2^*), T_2(x_1^*, x_2^*)). \end{aligned}$$

By a well-known Cauchy lemma (see e.g. [49]) we have $x_{2,n} \to x_2^*$, $n \to \infty$. □

REMARK 2.2. In the terminology of [8], Theorem 2.1 takes the following form:

Theorem I. *Let (X_i, d_i), $i = 1, 2$, be two metric spaces and $T = (T_1, T_2)$ a triangular operator. We suppose that:*

(i) *(X_2, d_2) is a complete metric space;*
(ii) *the operator $T_1 : X_1 \to X_1$ is a Picard operator;*
(iii) *$T_2(x_1, \cdot) : X_2 \to X_2$ is an l-contraction, $\forall\, x_1 \in X_1$.*
(iv) *The triangle operator, $T : X_1 \times X_2 \to X_1 \times X_2$,*

$$T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2))$$

is continuous.

Then the operator T is a Picard operator.

In connection with this result we present the following questions:

PROBLEM 2.3. *Extend Theorem I to the case of contraction type* ([8]) *conditions (see also* [36, 46, 57, 1, 21]*).*

Commentaries:

Other extensions of the Theorem I can be obtained by replacing condition (iii) by one of the following conditions:

(1) φ-contraction (see [54, 57])
(iii′) $T_2(x_1, \cdot) : X_2 \to X_2$ is a φ-contraction, $\forall\, x_1 \in X_1$.

(2) Kannan condition
(iii″) $T_2(x_1, \cdot) : X_2 \to X_2$ is a Kannan operator, $\forall\, x_1 \in X_1$, i.e. there exist $0 < l < \dfrac{1}{2}$ such that

$$d_2(T_2(x_1, x_2), T_2(x_1, \widetilde{x}_2)) \leq l[d_2(T_2(x_1, x_2), x_2) + d_2(T_2(x_1, \widetilde{x}_2), \widetilde{x}_2)],$$

$\forall\ x_1 \in X_1, x_2, \widetilde{x}_2 \in X_2$.

Related to the Kannan condition we have the following result.

THEOREM 2.4. *Let (X_i, d_i), $i = 1, 2$, be two metric spaces and $T = (T_1, T_2)$ a triangular operator. We suppose that:*

(i) *(X_2, d_2) is a complete metric space;*

(ii) *the operator $T_1 : X_1 \to X_1$ is a Picard operator;*

(iii′′′) *there exist $L > 0$ and $0 < l < \frac{1}{2}$ such that*

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \leq L d_1(x_1, \widetilde{x}_1) + l[d_2(T_2(x_1, x_2), x_2) + d_2(T_2(\widetilde{x}_1, \widetilde{x}_2), \widetilde{x}_2)],$$

$$\forall\ (x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in X_1 \times X_2;$$

Then the operator T is a Picard operator.

Proof. Let $(x_{1,0}, x_{2,0}) \in X_1 \times X_2$ and $(x_{1,n+1}, x_{2,n+1}) = T(x_{1,n}, x_{2,n})$, $n \in \mathbb{N}$, i.e., $x_{1,n+1} = T_1(x_{1,n})$ and $x_{2,n+1} = T_2(x_{1,n}, x_{2,n})$, $n \in \mathbb{N}$.

T_1 is a Picard operator, so $F_{T_1} = \{x_1^*\}$ and $x_{1,n} \to x_1^*$ as $n \to \infty$. From (3"') we have that $T_2(x_1, \cdot) : X_2 \to X_2$ satisfies $(3'\ )$ for all $x_1 \in X_1$ and from Kannan theorem we have that $T_2(x_1, \cdot)$ has an unique fixed point for all $x_1 \in X_1$. Let $F_{T_2(x_1^*, \cdot)} = \{x_2^*\}$. We prove that $x_{2,n} \to x_2^*$ as $n \to \infty$. We have that:

$$\begin{aligned} d_2(x_{2,n+1}, x_2^*) &= d_2(T_2(x_{1,n}, x_{2,n}), T_2(x_1^*, x_2^*)) \leq \\ &\leq L d_1(x_{1,n}, x_1^*) + l d_2(T_2(x_{1,n}, x_{2,n}), x_{2,n}) \\ &\leq L d_1(x_1^n, x_1^*) + l d_2(x_{2,n+1}, x_2^*) + l d(x_{2,n}, x_2^*). \end{aligned}$$

This implies that

$$\begin{aligned} &d_2(x_{2,n+1}, x_2^*) \\ &\leq \frac{L}{1-l} d_1(x_{1,n}, x_1^*) + \frac{l}{1-l} d(x_{2,n}, x_2^*) \\ &\leq \frac{L}{1-l} d_1(x_{1,n}, x_1^*) + \frac{l}{1-l} \left[\frac{L}{1-l} d_1(x_{1,n-1}, x_1^*) + \frac{l}{1-l} d(x_{2,n-1}, x_2^*) \right] \\ &\leq \frac{L}{1-l} d_1(x_{1,n}, x_1^*) + \frac{l}{1-l} \cdot \frac{L}{1-l} d_1(x_{1,n-1}, x_1^*) + \ldots \\ &+ \left(\frac{l}{1-l} \right)^n \frac{L}{1-l} d_1(x_{1,0}, x_1^*) + \left(\frac{l}{1-l} \right)^{n+1} d_2(x_{2,0}, x_2^*) \to 0 \end{aligned}$$

as $n \to \infty$, as in Theorem 2.1, by the Cauchy lemma. □

For other results in terms of admissible perturbation of an operator see: [55].

PROBLEM 2.5. *Extend Theorem* 2.1 *to the case of generalized metric spaces* (*see* [12, 30, 34, 46, 23, 15, 18, 42, 1, 21, 61]).

Commentaries:

(1) The case of $\mathbb{R}^p_+$-metric spaces

If (X,d) is a generalized metric space in the sense that the metric takes vector values (i.e., $d : X \times X \to \mathbb{R}^p_+$), we can use the notion of Perov contraction. More precisely, an operator $T : X \to X$ is called a Q-contraction if there exists a matrix $Q \in M_{pp}(\mathbb{R}_+)$ such that Q is a matrix convergent to zero (i.e., Q^k converges to the zero matrix as $k \to +\infty$) and

$$d(T(x),T(y)) \leq Qd(x,y),\ \forall x,y \in X.$$

Theorem 2.6 (I.A. Rus (1999) [36]). *Let (X_1,d_1) be a metric space and (X_2,d_2) be a generalized metric space. Let $T = (T_1,T_2)$ be a triangular operator. We suppose that:*

(i) *(X_2,d_2) is a complete generalized metric space;*
(ii) *the operator $T_1 : X_1 \to X_1$ is a Picard operator;*
(iii) *$T_2(x_1,\cdot) : X_2 \to X_2$ is an Q-contraction, $\forall\ x_1 \in X_1$.*
(iv) *The triangle operator, $T : X_1 \times X_2 \to X_1 \times X_2$, $T(x_1,x_2) := (T_1(x_1), T_2(x_1,x_2))$ is continuous.*

Then the operator T is a Picard operator.

(2) The case of dislocated metric spaces (see [23, 15, 12])

Let X be a nonempty set. Then $d : X \times X \to \mathbb{R}_+$ is a dislocated metric on X if the following axioms hold:

(i) $d(x,y) = d(y,x) = 0 \Longrightarrow x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(iii) $d(x,y) \leq d(x,z) + d(z,y)$.

Problem 2.7. *Extend Theorem 2.1 to the case when (X_2,d_2) is a complete dislocated metric space and $T_2(x_1,\cdot) : X_2 \to X_2$ is an l-contraction, $\forall\ x_1 \in X_1$.*

Problem 2.8. *Extend Theorem 2.1 to the case of generalized metric spaces and contraction type conditions.*

Commentaries:

(1) The Kannan type condition and the framework of a $\mathbb{R}^p_+$-metric space

To study the above problem under the assumptions that (X_2,d_2) *is a complete generalized metric space with* $d_2 : X_2 \times X_2 \to \mathbb{R}^p_+$ and $T_2(x_1,\cdot) : X_2 \to X_2$ *is a Q-Kannan operator,* $\forall\ x_1 \in X_1$, *i.e.* there exists a matrix $Q \in M_{pp}(\mathbb{R}_+)$ with $(I-Q)^{-1}Q$ a matrix convergent to zero, such that:

$$d_2(T_2(x_1,x_2),T_2(x_1,\widetilde{x}_2)) \leq Q[d_2(T_2(x_1,x_2),x_2) + d_2(T_2(x_1,\widetilde{x}_2),\widetilde{x}_2)],$$

$\forall\ x_1 \in X_1, x_2, \widetilde{x}_2 \in X_2$.

(2) The Φ-contraction condition and the framework of a $\mathbb{R}^p_+$ metric space

To study the above problem if (X_2,d_2) *is a complete generalized metric space with* $d_2 : X_2 \times X_2 \to \mathbb{R}^p_+$ and $T_2(x_1,\cdot) : X_2 \to X_2$ *is an Φ-contraction,* $\forall\ x_1 \in X_1$, *i.e.*:

$$d_2(T_2(x_1,x_2),T_2(x_1,\widetilde{x}_2)) \leq \Phi\left(d_2(x_2,\widetilde{x}_2)\right),$$

$\forall\ x_1 \in X_1, x_2, \widetilde{x}_2 \in X_2$, where $\Phi : \mathbb{R}^p_+ \to \mathbb{R}^p_+$ such that:

(i) Φ is increasing;
(ii) $\Phi^n(t) \to 0$ as $n \to +\infty$, $\forall t \in \mathbb{R}^p_+$.

3. SATURATED FIBRE CONTRACTION PRINCIPLE

Following the Saturated Contraction Principle from [44] we can give the corresponding Saturated Fibre Contraction Principle:

THEOREM 3.1 (Şerban (2017), [56]). *Let (X_1, d_1) be a metric space, (X_2, d_2) a complete metric space and $T = (T_1, T_2) : X_1 \times X_2 \to X_1 \times X_2$ a triangular operator. We suppose that:*

(i) *$T_1 : X_1 \to X_1$ is a Picard operator ($F_{T_1} = \{x_1^*\}$);*
(ii) *$T_2(x_1, \cdot) : X_2 \to X_2$ is an l-contraction, for all $x_1 \in X_1$;*
(iii) *$T_2(\cdot, x_2) : X_1 \to X_2$ is L-Lipschitz, for all $x_2 \in X_2$.*

Then:

(a) *T is a Picard operator;*
(b) *$F_T = F_{T^n} = \{(x_1^*, x_2^*)\}$, where $\{x_2^*\} = F_{T_2(x_1^*,\cdot)}$.*

If in addition, T_1 is ψ_1-Picard operator, then:

(c_1) *T is ψ-Picard operator in $(X_1 \times Y_2, d_\infty)$, where*

$$\psi(t) = \max\left\{\psi_1(t), \frac{1}{1-l}[t + L\psi_1(t)]\right\},\ t \in \mathbb{R}_+,$$

$$d_\infty\left((x_1, x_2), (\tilde{x}_1, \tilde{x}_2)\right) = \max\left\{d_1(x_1, \tilde{x}_1), d_2(x_2, \tilde{x}_2)\right\};$$

(c_2) *the fixed point problem for T is well-posed;*
(c_3) *the fixed point equation for T is generalized Ulam-Hyers stable.*

If in addition T_1 is an α-quasicontraction then:

(d_1) *T is an l_T-quasicontraction in $(X_1 \times Y_2, \rho_\infty)$, where*

$$\rho_\infty\left((x_1, x_2), (\tilde{x}_1, \tilde{x}_2)\right) = \max\left\{r \cdot d_1(x_1, \tilde{x}_1), d_2(x_2, \tilde{x}_2)\right\},$$

with $r > \frac{L}{1-l}$ and

$$l_T = \max\left\{\alpha, \frac{L}{r} + l\right\}.$$

(d_2) *T has the Ostrowski property.*

Proof. The proof of (a) and (b) is the same as in Theorem 2.1.
(c_1) Let $(x_1, x_2) \in X_1 \times X_2$. If T_1 is ψ_1-Picard operator, then

$$d_1(x_1, x_1^*) \le \psi_1\left(d_1(x_1, T_1(x_1))\right),\ \forall x_1 \in X_1,$$

where $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 with $\psi_1(0) = 0$. We know that $F_T = \{(x_1^*, x_2^*)\}$, where $\{x_2^*\} = F_{T_2(x_1^*,\cdot)}$, then we have:

$$\begin{aligned} & d_2(x_2, x_2^*) \\ \leq \; & d_2(x_2, T_2(x_1, x_2)) + d_2(T_2(x_1, x_2), T_2(x_1^*, x_2)) \\ & + d_2(T_2(x_1^*, x_2), T_2(x_1^*, x_2^*)) \\ \leq \; & d_2(x_2, T_2(x_1, x_2)) + L d_1(x_1, x_1^*) + l d_2(x_2, x_2^*), \end{aligned}$$

so

$$\begin{aligned} d_2(x_2, x_2^*) \;\leq\; & \frac{1}{1-l}\left[d_2(x_2, T_2(x_1, x_2)) + L d_1(x_1, x_1^*)\right] \\ & \leq \frac{1}{1-l}\left[d_2(x_2, T_2(x_1, x_2)) + L\psi_1(d_1(x_1, T_1(x_1)))\right]. \end{aligned}$$

This implies that

$$\begin{aligned} & d_\infty((x_1, x_2), (x_1^*, x_2^*)) \leq \max\{\psi_1(d_1(x_1, T_1(x_1))), \\ & \tfrac{1}{1-L}\left[d_2(x_2, T_2(x_1, x_2)) + L\psi_1(d_1(x_1, T_1(x_1)))\right]\} \\ \leq \; & \psi(d_\infty((x_1, x_2), T(x_1, x_2))), \end{aligned}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\psi(t) = \max\left\{\psi_1(t), \frac{1}{1-l}\left[t + L\psi_1(t)\right]\right\}.$$

It is easy to check that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous at 0 with $\psi(0) = 0$.

(c_2) Let $((x_{1,n}, x_{2,n}))_{n\in\mathbb{N}} \subset X_1 \times X_2$ such that $d_\infty((x_{1,n}, x_{2,n}), T(x_{1,n}, x_{2,n})) \to 0$ as $n \to +\infty$. Then, we have:

$$d_\infty((x_{1,n}, x_{2,n}), (x_1^*, x_2^*)) \leq \psi(d_\infty((x_{1,n}, x_{2,n}), T(x_{1,n}, x_{2,n}))) \to 0 \text{ as } n \to \infty.$$

(c_3) Let $\varepsilon > 0$ and $(y_1^*, y_2^*) \in X_1 \times X_2$ be a solution of the inequation

$$d_\infty((y_1, y_2), T(y_1, y_2)) \leq \varepsilon.$$

Since T is ψ-Picard operator, then

$$d_\infty((y_1^*, y_2^*), (x_1^*, x_2^*)) \leq \psi(d_\infty((y_1^*, y_2^*), T(y_1^*, y_2^*))) \leq \psi(\varepsilon),$$

so the fixed point equation for T is generalized Ulam-Hyers stable.

(d_1) Let $(x_1, x_2) \in X_1 \times X_2$ and $r > \frac{L}{1-l}$. If T_1 is an α-quasicontraction then

$$\begin{aligned} r \cdot d_1(T_1(x), x_1^*) \;\leq\; & \alpha \cdot r \cdot d_1(x_1, x_1^*) \leq \\ \leq\; & \alpha \cdot \rho_\infty((x_1, x_2), (x_1^*, x_2^*)). \end{aligned}$$

$r > \frac{L}{1-\alpha} \Longleftrightarrow \frac{L}{r} + \alpha < 1$ and from (iii) and (iv) we have

$$\begin{aligned} d_2(T_2(x_1, x_2), x_2^*) \;\leq\; & \frac{L}{r} \cdot r \cdot d_1(x_1, x_1^*) + l \cdot d_2(x_2, x_2^*) \leq \\ \leq\; & \left(\frac{L}{r} + l\right) \rho_\infty((x_1, x_2), (x_1^*, x_2^*)), \end{aligned}$$

therefore

$$\begin{aligned}\rho_\infty\left(T\left(x_1,x_2\right),\left(x_1^*,x_2^*\right)\right) &= \max\left\{r\cdot d_1\left(T_1\left(x\right),x_1^*\right),d_2\left(T_2\left(x_1,x_2\right),x_2^*\right)\right\}\leq\\ &\leq \max\left\{\alpha,\frac{L}{r}+l\right\}\cdot\rho_\infty\left(\left(x,y\right),\left(x^*,y^*\right)\right),\end{aligned}$$

for all $(x_1,x_2)\in X_1\times X_2$, so T is an l_T-quasicontraction in $(X_1\times Y_2,\rho_\infty)$ with $l_T=\max\left\{\alpha,\frac{L}{r}+l\right\}$.

(d_2) Let $((x_{1,n},x_{2,n}))_{n\in\mathbb{N}}\subset X_1\times X_2$ such that $\rho_\infty((x_{1,n+1},x_{2,n+1}),T(x_{1,n},x_{2,n}))\to 0$ as $n\to+\infty$. Then, we have:

$$\begin{aligned}&\rho_\infty\left((x_{1,n+1},x_{2,n+1}),(x_1^*,x_2^*)\right)\leq\\ \leq\ &\rho_\infty\left((x_{1,n+1},x_{2,n+1}),T(x_{1,n},x_{2,n})\right)+\rho_\infty\left(T(x_{1,n},x_{2,n}),(x_1^*,x_2^*)\right)\leq\\ \leq\ &\rho_\infty\left((x_{1,n+1},x_{2,n+1}),T(x_{1,n},x_{2,n})\right)+l_T\rho_\infty\left((x_{1,n+1},x_{2,n+1}),(x_1^*,x_2^*)\right)\leq\dots\\ \leq\ &\sum_{j=0}^{n}l_T^j\cdot\rho_\infty\left((x_{1,n+1-j},x_{2,n+1-j}),T(x_{1,n-j},x_{2,n-j})\right)\\ &+l_T^n\cdot\rho_\infty\left((x_{1,0},x_{2,0}),(x_1^*,x_2^*)\right).\end{aligned}$$

From Cauchy Lemma we get

$$\rho_\infty\left((x_{1,n+1},x_{2,n+1}),(x_1^*,x_2^*)\right)\to 0 \text{ as } n\to+\infty,$$

so T has the Ostrowski property. □

As in the case of Theorem 2.1, the following questions emerge from Theorem 3.1:

Problem 3.2. *Extend Theorem* 3.1 *to the case of contraction type conditions.*

Problem 3.3. *Extend Theorem* 3.1 *to the case of generalized metric spaces.*

Problem 3.4. *Extend Theorem* 3.1 *to the case of generalized metric spaces and contraction type conditions.*

Commentaries:

(1) The Kannan condition and the context of a $\mathbb{R}_+^p$-metric space

To study the above problem if (X_2,d_2) *is a complete generalized metric space with* $d_2:X_2\times X_2\to\mathbb{R}_+^p$ and $T_2(x_1,\cdot):X_2\to X_2$ *is an* Q*-Kannan operator,* $\forall\ x_1\in X_1$, *i.e.* there exists a matrix $Q\in M_{pp}(\mathbb{R}_+)$ with $(I-Q)^{-1}\cdot Q$ a matrix convergent to zero, such that:

$$d_2(T_2(x_1,x_2),T_2(x_1,\widetilde{x}_2))\leq Q[d_2(T_2(x_1,x_2),x_2)+d_2(T_2(x_1,\widetilde{x}_2),\widetilde{x}_2)],$$

$\forall\ x_1\in X_1,x_2,\widetilde{x}_2\in X_2$.

(2) The Φ-contraction condition and the context of a $\mathbb{R}_+^p$ metric space

To study the above problem if (X_2,d_2) *is a complete generalized metric space with* $d_2:X_2\times X_2\to\mathbb{R}_+^p$ and $T_2(x_1,\cdot):X_2\to X_2$ *is an* Φ*-contraction,* $\forall\ x_1\in X_1$, *i.e.*:

$$d_2(T_2(x_1,x_2),T_2(x_1,\widetilde{x}_2))\leq\Phi\left(d_2(x_2,\tilde{x}_2)\right),$$

$\forall\ x_1\in X_1,x_2,\widetilde{x}_2\in X_2$, where $\Phi:\mathbb{R}_+^p\to\mathbb{R}_+^p$ such that:

(i) Φ is increasing;
(ii) $\Phi^n(t) \to 0$ as $n \to +\infty$, $\forall t \in \mathbb{R}^p_+$.

4. HYBRID FIBRE CONTRACTION PRINCIPLE

Another basic fibre contraction principle is the following theorem (see [37, 40]).

THEOREM 4.1 (Rus (1999, 2003)). *Let $(X_1, \xrightarrow{F})$ be an L-space, (X_2, d_2) a complete metric space and $T = (T_1, T_2) : X_1 \times X_2 \to X_1 \times X_2$ a triangular operator. We suppose that:*

(i) *$T_1 : X_1 \to X_1$ is weakly Picard operator;*
(ii) *$T_2(x_1, \cdot) : X_2 \to X_2$ is an l-contraction;*
(iii) *if $(x_1^*, x_2^*) \in F_T$, then $T_2(\cdot, x_2^*)$ is continuous in x_1^*.*

Then the operator T is a weakly Picard operator. If T_1 is Picard operator, then T is a Picard operator.

The proof is similar with the proof of Theorem 2.1.
In the case of the above result we have the following questions:

PROBLEM 4.2. *Extend Theorem* 4.1 *to the case of contraction type conditions.*

PROBLEM 4.3. *Extend Theorem* 4.1 *to the case of generalized metric spaces.*

PROBLEM 4.4. *Extend Theorem* 4.1 *to the case of generalized metric spaces and contraction type conditions.*

As starting references for these problems see: [46, 57, 53, 49, 2, 3, 12, 36, 38, 39, 40, 47, 8] etc.

5. FIBRE CONTRACTION PRINCIPLE ON A SUBSET OF THE CARTESIAN PRODUCT

Let (X_i, d_i), $i = 1, 2$, be two metric spaces, $U \subset X_1 \times X_2$ be a nonempty subset such that

$$U_{x_1} := \{x_2 \in X_2 \mid (x_1, x_2) \in U\} \neq \emptyset,\ \forall\, x_1 \in X_1.$$

For the operators $T_1 : X_1 \to X_1$, $T_2 : U \to X_2$, we consider the operator defined by

$$T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)).$$

We have the following result ([35]):

THEOREM 5.1 (Petruşel-Rus-Şerban (2021)). *We suppose that:*

(i) *(X_2, d_2) is a complete metric space and U is a closed subset of $X_1 \times X_2$;*
(ii) *$T(U) \subset U$;*
(iii) *T_1 is weakly Picard operator;*

(iv) *there exist* $L > 0$ *and* $0 < l < 1$ *such that:*

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \leq Ld_1(x_1, \widetilde{x}_1) + ld_2(x_2, \widetilde{x}_2),$$

for all $(x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U$.

Then T *is a weakly Picard operator. If* T_1 *is a Picard operator, then* T *is a Picard operator too.*

Problem 5.2. *Give a saturated variant of Theorem 5.1 (see* [8, 35]*).*

6. TRIANGULAR OPERATOR ON $\prod_{i=1}^{m} X_i$

Let $X = \prod_{i=1}^{m} X_i$. We consider the operators $T_j : X_1 \times \ldots \times X_j \to X_j$, $j = 1, 2, \ldots, m$, and

$$T : X \to X$$
$$T(x_1, \ldots, x_m) = (T_1(x_1), T_2(x_1, x_2), \ldots, T_m(x_1, \ldots, x_m))$$

Theorem 6.1 (Rus (1999), [37]). *Suppose that:*

(i) $(X_1, \stackrel{F}{\longrightarrow})$ *be an L-space;*
(ii) $T_1 : X_1 \to X_1$ *is weakly Picard operator;*
(iii) (X_j, d_j) *is a complete metric space,* $j = 2, \ldots, m$*;*
(iv) $T_j(x_1, \ldots, x_{j-1}, \cdot) : X_j \to X_j$ *is* l_j-*contraction,* $j = 2, \ldots, m$*;*
(v) *If* $(x_1^*, \ldots, x_m^*) \in F_T$*, then* $T_j\left(\cdot, \ldots, \cdot, x_j^*\right)$ *is continuous in* $(x_1^*, \ldots, x_{j-1}^*)$*,* $j = 2, \ldots, m$.

Then T *is weakly Picard operato. Moreover, if* T_1 *is Picard operator, then* T *is Picard operator.*

Proof. The proof can be obtained by induction using Theorem 5.1. □

Theorem 6.2 (Şerban (1999), [54]). *Suppose that:*

(i) $(X_1, \stackrel{F}{\longrightarrow})$ *be an L-space;*
(ii) $T_1 : X_1 \to X_1$ *is weakly Picard operator;*
(iii) (X_j, d_j) *is a complete metric space,* $j = 2, \ldots, m$*;*
(iv) $T_j(x_1, \ldots, x_{j-1}, \cdot) : X_j \to X_j$ *is* φ_j*-contraction, where* φ_j *is a subadditive strong comparison function,* $j = 2, \ldots, m$*;*
(v) *If* $(x_1^*, \ldots, x_m^*) \in F_T$*, then* $T_j(\cdot, \ldots, \cdot, x_j^*)$ *is continuous in* $(x_1^*, \ldots, x_{j-1}^*)$*,* $j = 2, \ldots, m$.

Then T *is weakly Picard operator. Moreover, if* T_1 *is Picard operator, then* T *is Picard operator.*

7. APPLICATIONS OF FIBRE CONTRACTION PRINCIPLES

7.1 EXISTENCE OF SOLUTIONS

In what follow we apply saturated fibre contraction principle to study the following system of integral equations:

$$(1)\qquad \begin{cases} x(t) = \int_a^t K(t,s,x(s))\,\mathrm{d}s + k(t), & t \in [a,b] \\ y(t) = \int_a^b H(t,s,x(s),y(s))\,\mathrm{d}s + h(t), & t \in [a,b] \end{cases}$$

We consider $X_1 = (C[a,b], \|\cdot\|_\tau)$ and $X_2 = (C[a,b], \|\cdot\|_\infty)$ where

$$\|x\|_\tau = \max_{t\in[a,b]} \left(\left|x(t)\cdot \mathrm{e}^{-\tau(t-a)}\right|\right),\ \tau > 0,$$

$$\|y\|_\infty = \max_{t\in[a,b]} (|y(t)|).$$

From (1) we define the triangular operator $T = (T_1, T_2) : X_1 \times X_2 \to X_1 \times X_2$, where $T_1 : X_1 \to X_1$

$$T_1(x)(t) = \int_a^t K(t,s,x(s))\,\mathrm{d}s + k(t),$$

and $T_2 : X_1 \times X_2 \to X_2$

$$T_2(x,y)(t) = \int_a^b H(t,s,x(s),y(s))\,\mathrm{d}s + h(t).$$

We have:

THEOREM 7.1. *We suppose that:*

(i) $K \in C([a,b]\times[a,b]\times\mathbb{R})$, $H \in C\left([a,b]\times[a,b]\times\mathbb{R}^2\right)$, $k, h \in C[a,b]$;

(ii) *there exists* $L_K > 0$ *such that*

$$|K(t,s,u_1) - K(t,s,u_2)| \le L_K \cdot |u_1 - u_2|$$

for all $t, s \in [a,b]$ *and* $u_1, u_2 \in \mathbb{R}$;

(iii) *there exists* $l_H, L_H > 0$ *such that*

$$|H(t,s,u_1,v_1) - H(t,s,u_2,v_2)| \le l_H \cdot |u_1 - u_2| + L_H \cdot |v_1 - v_2|,$$

for all $t, s \in [a,b]$ *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$;

(iv) $L_H(b-a) < 1$.

Then:

(a) *the system (1) has an unique solution* $(x^*, y^*) \in X_1 \times X_2$.

(b) *the sequence* (x_n, y_n) *given by*

$$\begin{aligned} x_{n+1} &= T_1(x_n) \\ y_{n+1} &= T_2(x_n, y_n) \end{aligned}$$

with $(x_0, y_0) \in X_1 \times X_2$*, converges uniformly to* (x^*, y^*) *for all* $(x_0, y_0) \in X_1 \times X_2$*;*

(c) *we have:* $\|x_0 - x^*\|_\tau \leq \frac{1}{1-l_1} \|x_0 - T_1(x_0)\|_\tau$,

$$\|y_0 - y^*\|_\infty \leq \frac{1}{1-l_2} \left(\|y_0 - T_2(x_0, y_0)\|_\infty + \frac{l}{1-l_1} \|x_0 - T_1(x_0)\|_\tau \right),$$

for any $(x_0, y_0) \in X_1 \times X_2$*, where* $l_1 = \frac{L_K}{\tau}$*, with* τ *chosen such that* $\tau > L_K$*,* $l_2 = L_H(b-a)$*,* $l = \frac{l_H}{\tau}\left(e^{\tau(b-a)} - 1\right)$*;*

(d) *the fixed point problem for* T *is well-posed;*

(e) *the fixed point equation for* T *is generalized Ulam-Hyers stable;*

(f) *the operator* T *has the Ostrowski property.*

Proof. By standard technique, the operator $T_1 : X_1 \to X_1$

$$T_1(x)(t) = \int_a^t K(t, s, x(s))\,ds + k(t),$$

is a l_1−contraction with $l_1 = \frac{L_K}{\tau}$ for suitable choice of $\tau > 0$, $(\tau > L_K)$, thus T_1 is a Picard operator and we denote by $x^* \in X$ the unique fixed point of T_1.

From condition (3) we get

$$\begin{aligned} |T_2(x, y_1)(t) - T_2(x, y_2)(t)| &\leq \int_a^b |H(t, s, x(s), y_1(s)) - H(t, s, x(s), y_1(s))|\,ds \\ &\leq L_H \cdot \int_a^b |y_1(s) - y_2(s)|\,ds \leq L_H(b-a)\|y_1 - y_2\|_\infty \end{aligned}$$

and therefore

$$\|T_2(x, y_1) - T_2(x, y_2)\|_\infty \leq l_2 \cdot \|y_1 - y_2\|_\infty$$

for all $x \in X_1$ and $y_1, y_2 \in X_2$ which shows that $T_2(x, \cdot) : X_2 \to X_2$ is an l_2-contraction for fixed $x \in X_1$. Also, we have

$$\|T_2(x_1, y) - T_2(x_2, y)\|_\infty \leq l \cdot \|x_1 - x_2\|_\tau,$$

for all $x_1, x_2 \in X_1$ and $y \in X_2$, where $l = \frac{l_H}{\tau}\left(e^{\tau(b-a)} - 1\right)$, which shows that $T_2(\cdot, y) : X_1 \to X_2$ is an l-Lipschitz operator for fixed $y \in X_2$, so, from Theorem 3.1 we get all the conclusions. □

Remark 7.2. For other applications of Fibre Contraction Principle and its variants to the existence of the solutions see [9, 14, 19, 20, 29, 32, 33, 41, 45, 48, 50, 52].

7.2 SMOOTHNESS OF SOLUTIONS

We consider the following system of integral equations:

$$(2)\qquad \begin{cases} x_1(t,\lambda) = \int_a^t K(t,s,x_1(s,\lambda),\lambda)\,\mathrm{d}s + k(t,\lambda), \\ x_2(t,\lambda) = \int_a^b H(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\,\mathrm{d}s + h(t,\lambda), \end{cases}$$

$t \in [a,b]$, $\lambda \in J$, where $J \subseteq \mathbb{R}$ is a closed interval.

We consider $X_1 = (C([a,b]\times J), \|\cdot\|_\tau)$ and $X_2 = (C([a,b]\times J), \|\cdot\|_\infty)$ where

$$\|x\|_\tau = \max_{(t,\lambda)\in[a,b]\times J}\left(\left|x(t,\lambda)\cdot \mathrm{e}^{-\tau(t-a)}\right|\right),\ \tau > 0, \|y\|_\infty = \max_{(t,\lambda)\in[a,b]\times J}(|y(t,\lambda)|).$$

From (1) we define the operators $T_1 : X_1 \to X_1$, $T_1(x_1)(t,\lambda) = \int_a^t K(t,s,x_1(s,\lambda),\lambda)\mathrm{d}s + k(t,\lambda)$, $T_2 : X_1 \times X_2 \to X_2$, $T_2(x_1,x_2)(t,\lambda) = \int_a^b H(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\mathrm{d}s + h(t,\lambda)$.

We have:

Theorem 7.3. *We suppose that:*

(i) $J \subseteq \mathbb{R}$ *is a closed interval;*

(ii) $K \in C([a,b]\times[a,b]\times\mathbb{R}\times J)$, $H \in C\left([a,b]\times[a,b]\times\mathbb{R}^2\times J\right)$ *and* $k,h \in C([a,b]\times J)$;

(iii) *there exists* $L_K > 0$ *such that* $|K(t,s,u_1,\lambda) - K(t,s,u_2,\lambda)| \le L_K \cdot |u_1 - u_2|$ *for all* $t,s \in [a,b]$, $u_1,u_2 \in \mathbb{R}$, $\lambda \in J$;

(iv) *there exists* $l_H, L_H > 0$ *such that*

$$|H(t,s,u_1,v_1,\lambda) - H(t,s,u_2,v_2,\lambda)| \le l_H \cdot |u_1 - u_2| + L_H \cdot |v_1 - v_2|,$$

for all $t,s \in [a,b]$, $u_1,u_2,v_1,v_2 \in \mathbb{R}$, $\lambda \in J$;

(v) $L_H(b-a) < 1$.

Then:

(a) *the system* (2) *has an unique solution* $(x_1^*, x_2^*) \in X_1 \times X_2$.

(b) *the sequence* $(x_{1,n}, x_{2,n})$ *given by* $x_{1,n+1} = T_1(x_{1,n})$, $x_{2,n+1} = T_2(x_{1,n}, x_{2,n})$ *with* $(x_{1,0}, x_{2,0}) \in X_1 \times X_2$, *converges uniformly to* (x_1^*, x_2^*) *for all* $(x_{1,0}, x_{2,0}) \in X_1 \times X_2$;

(c) *If* $K(t,s,\cdot,\cdot) \in C^1(\mathbb{R}\times J)$, $H(t,s,\cdot,\cdot\cdot) \in C^1\left(\mathbb{R}^2\times J\right)$, $k(t,\cdot), h(t,\cdot) \in C^1(J)$, *for every* $t,s \in [a,b]$ *then* $x_1^*(t,\cdot)$, $x_2^*(t,\cdot) \in C^1(J)$, *for every* $t,s \in [a,b]$.

Proof. (a)+(b) We consider $X_1 = (C([a,b]\times J), \|\cdot\|_\tau)$ and $X_2 = (C([a,b]\times J), \|\cdot\|_\infty)$ where

$$\|x\|_\tau = \max_{(t,\lambda)\in[a,b]\times J}\left(\left|x(t,\lambda)\cdot \mathrm{e}^{-\tau(t-a)}\right|\right),\ \tau > 0, \|y\|_\infty = \max_{(t,\lambda)\in[a,b]\times J}(|y(t,\lambda)|).$$

From (2) we define the operators $T_1 : X_1 \to X_1$,

$$T_1(x_1)(t,\lambda) = \int_a^t K(t,s,x_1(s,\lambda),\lambda)\,\mathrm{d}s + k(t,\lambda),$$

$T_2 : X_1 \times X_2 \to X_2$,

$$T_2(x_1,x_2)(t,\lambda) = \int_a^b H(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\,\mathrm{d}s + h(t,\lambda)$$

and the triangular operator $T = (T_1,T_2) : X_1 \times X_2 \to X_1 \times X_2$. As in the proof of Theorem 7.1 we have that $T_1 : X_1 \to X_1$ is a l_1-contraction with $l_1 = \frac{L_K}{\tau}$ for suitable choice of $\tau > 0$, $(\tau > L_K)$, and $T_2(x_1,\cdot) : X_2 \to X_2$ is an l_2-contraction for fixed $x_1 \in X_1$, $l_2 = L_H(b-a)$, so, from Theorem 3.1 we have that $T = (T_1,T_2)$ is a Picard operator, therefore we get (i) and (ii).

(c) We formally differentiate the equations of the system (2) with respect to λ:

$$\begin{aligned}\frac{\partial x_1}{\partial\lambda}(t,\lambda) &= \int_a^t \frac{\partial K}{\partial u}(t,s,x_1(s,\lambda),\lambda)\cdot\frac{\partial x_1}{\partial\lambda}(s,\lambda)\,\mathrm{d}s\\ &\quad+\int_a^t \frac{\partial K}{\partial\lambda}(t,s,x_1(s,\lambda),\lambda)\,\mathrm{d}s + \frac{\partial k}{\partial\lambda}(t,\lambda),\end{aligned}$$

$$\begin{aligned}\frac{\partial x_2}{\partial\lambda}(t,\lambda) &= \int_a^b \frac{\partial H}{\partial u}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\cdot\frac{\partial x_1}{\partial\lambda}(s,\lambda)\,\mathrm{d}s\\ &\quad+\int_a^b \frac{\partial H}{\partial v}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\cdot\frac{\partial x_2}{\partial\lambda}(s,\lambda)\,\mathrm{d}s\\ &\quad+\int_a^b \frac{\partial H}{\partial\lambda}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\,\mathrm{d}s + \frac{\partial h}{\partial\lambda}(t,\lambda)\end{aligned}$$

These relations suggest us to consider the spaces $X_3 = X_1$, $X_4 = X_2$ and the following operators $T_3 : X_1 \times X_2 \times X_3 \to X_3$,

$$\begin{aligned}T_3(x_1,x_2,x_3)(t,\lambda) &= \int_a^t \frac{\partial K}{\partial u}(t,s,x_1(s,\lambda),\lambda)\cdot x_3(s,\lambda)\,\mathrm{d}s +\\ &\quad+\int_a^t \frac{\partial K}{\partial\lambda}(t,s,x_1(s,\lambda),\lambda)\,\mathrm{d}s + \frac{\partial k}{\partial\lambda}(t,\lambda),\end{aligned}$$

$T_4 : X_1 \times X_2 \times X_3 \times X_4 \to X_4$

$$\begin{aligned} T_4(x_1,x_2,x_3,x_4)(t,\lambda) &= \int_a^b \frac{\partial H}{\partial u}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\cdot x_3(t,\lambda)\,\mathrm{d}s \\ &+ \int_a^b \frac{\partial H}{\partial v}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\cdot x_4(t,\lambda)\,\mathrm{d}s \\ &+ \int_a^b \frac{\partial H}{\partial \lambda}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\,\mathrm{d}s + \frac{\partial h}{\partial \lambda}(t,\lambda) \end{aligned}$$

and the triangular operator $T = (T_1, T_2, T_3, T_4) : \prod_{i=1}^{4} X_i \to \prod_{i=1}^{4} X_i$.

We already proved that $T_1 : X_1 \to X_1$ is a l_1-contraction, so it is Picard operator, and $T_2(x_1,\cdot) : X_2 \to X_2$ is an l_2-contraction for fixed $x_1 \in X_1$. We have:

$$\begin{aligned} &\left|T_3(x_1,x_2,x_3)(t,\lambda) - T_3(x_1,x_2,\tilde{x}_3)(t,\lambda)\right| \\ \leq\ & \int_a^t \left|\frac{\partial K}{\partial u}(t,s,x_1(s,\lambda),\lambda)\right|\cdot\left|x_3(s,\lambda)-\tilde{x}_3(s,\lambda)\right|\mathrm{d}s \\ \leq\ & \frac{L_K}{\tau}\cdot\left\|x_3-\tilde{x}_3\right\|_\tau\cdot \mathrm{e}^{\tau(t-a)} \end{aligned}$$

for all $(t,\lambda) \in [a,b]\times J$, therefore

$$\left\|T_3(x_1,x_2,x_3) - T_3(x_1,x_2,\tilde{x}_3)\right\|_\tau \leq \frac{L_K}{\tau}\cdot\left\|x_3-\tilde{x}_3\right\|_\tau$$

for all $(x_1,x_2) \in X_1 \times X_2$ and x_3, $\tilde{x}_3 \in X_3$, so $T_3(x_1,x_2,\cdot) : X_3 \to X_3$ is an l_3-contraction, where $l_3 = \frac{L_K}{\tau}$ for suitable choice of $\tau > 0$, $(\tau > L_K)$.

Also, we have:

$$\begin{aligned} &\left|T_4(x_1,x_2,x_3,x_4)(t,\lambda) - T_4(x_1,x_2,x_3,\tilde{x}_4)(t,\lambda)\right| \leq \\ \leq\ & \int_a^b \left|\tfrac{\partial H}{\partial v}(t,s,x_1(s,\lambda),x_2(s,\lambda),\lambda)\right|\cdot\left|x_4(s,\lambda)-\tilde{x}_4(s,\lambda)\right|\mathrm{d}s \leq \\ \leq\ & L_H(b-a)\cdot\left\|x_4-\tilde{x}_4\right\|_\infty, \end{aligned}$$

for all $(t,\lambda) \in [a,b]\times J$, hence

$$\left\|T_4(x_1,x_2,x_3,x_4) - T_4(x_1,x_2,x_3,\tilde{x}_4)\right\|_\infty \leq L_H(b-a)\cdot\left\|x_4-\tilde{x}_4\right\|_\infty,$$

for all $(x_1,x_2,x_3) \in X_1 \times X_2 \times X_3$ and x_4, $\tilde{x}_4 \in X_3$, so $T_4(x_1,x_2,x_3,\cdot) : X_4 \to X_4$ is an l_4-contraction, where $l_4 = L_H(b-a)$.

Applying Theorem 6.1 for triangular operator $T=(T_1,T_2,T_3,T_4):\prod_{i=1}^{4}X_i\to\prod_{i=1}^{4}X_i$ we conclude that T is Picard operator and the sequences $(x_{1,n})_{n\in\mathbb{N}}$, $(x_{2,n})_{n\in\mathbb{N}}$, $(x_{3,n})_{n\in\mathbb{N}}$, $(x_{4,n})_{n\in\mathbb{N}}$ defined by:

$$x_{1,n+1}(t,\lambda)=\int_a^t K(t,s,x_{1,n}(s,\lambda),\lambda)\,\mathrm{d}s+k(t,\lambda),$$

$$x_{2,n+1}(t,\lambda)=\int_a^b H(t,s,x_{1,n}(s,\lambda),x_{2,n}(s,\lambda),\lambda)\,\mathrm{d}s+h(t,\lambda),$$

$$\begin{aligned}x_{3,n+1}(t,\lambda) &= \int_a^t \frac{\partial K}{\partial u}(t,s,x_{1,n}(s,\lambda),\lambda)\cdot x_{3,n}(s,\lambda)\,\mathrm{d}s+\\ &\quad+\int_a^t \frac{\partial K}{\partial \lambda}(t,s,x_{1,n}(s,\lambda),\lambda)\,\mathrm{d}s+\frac{\partial k}{\partial \lambda}(t,\lambda)\end{aligned}$$

$$\begin{aligned}x_{4,n+1}(t,\lambda) &= \int_a^b \frac{\partial H}{\partial u}(t,s,x_{1,n}(s,\lambda),x_{2,n}(s,\lambda),\lambda)\cdot x_{3,n}(t,\lambda)\,\mathrm{d}s+\\ &\quad+\int_a^b \frac{\partial H}{\partial v}(t,s,x_{1,n}(s,\lambda),x_{2,n}(s,\lambda),\lambda)\cdot x_{4,n}(t,\lambda)\,\mathrm{d}s+\end{aligned}$$

$$+\int_a^b \frac{\partial H}{\partial \lambda}(t,s,x_{1,n}(s,\lambda),x_{2,n}(s,\lambda),\lambda)\,\mathrm{d}s+\frac{\partial h}{\partial \lambda}(t,\lambda)$$

converge uniformly to $(x_1^*,x_2^*,x_3^*,x_4^*)\in F_T$ for all $(x_{1,0},x_{2,0},x_{3,0},x_{4,0})\in\prod_{i=1}^{4}X_i$. If for fixed $(x_{1,0},x_{2,0},x_{3,0},x_{4,0})\in\prod_{i=1}^{4}X_i$ we chose $x_{3,0}=\frac{\partial x_{1,0}}{\partial\lambda}$ and $x_{4,0}=\frac{\partial x_{2,0}}{\partial\lambda}$ then $x_{3,1}=\frac{\partial x_{1,1}}{\partial\lambda}$ and $x_{4,1}=\frac{\partial x_{2,1}}{\partial\lambda}$. By induction we have $x_{3,n}=\frac{\partial x_{1,n}}{\partial\lambda}$, $x_{4,n}=\frac{\partial x_{2,n}}{\partial\lambda}$ and $x_{3,n}\rightrightarrows x_3^*$, $x_{4,n}\rightrightarrows x_4^*$, these imply that there exist $\frac{\partial x_1^*}{\partial\lambda}$, respectively $\frac{\partial x_2^*}{\partial\lambda}$ and $\frac{\partial x_1^*}{\partial\lambda}=x_3^*$, respectively $\frac{\partial x_2^*}{\partial\lambda}=x_4^*$ □

Remark 7.4. For other applications of Fibre Contraction Principle and its variants to the derivability of the solutions with respect to a parameter see [37, 52, 54, 55, 57, 58, 59].

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