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A MODIFIED INERTIAL EXTRAGRADIENT ALGORITHM FOR SOME CLASS OF SPLIT VARIATIONAL INEQUALITY PROBLEM

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Abstract. A new inertial extragradient algorithm for approximating solutions of some class of split variational inequality problem in real Hilbert space is introduced and discussed. Furthermore, the sequence generated by our algorithm is shown to converge strongly to the solution of the aforementioned problem. Our result is obtained without the assumption of the Lipschitz constant of the underline operator, and also with minimal number of projections per iteration compare to other results on split variational inequality problems in the literature. A numerical example is presented to demonstrate and compare the versatility of our result. Our result extends and improves many recent results of this type in the literature.

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Key words. Armijo-line search, Inertia algorithms, extragradient methods, projection methods, split variational inequality problems, strong convergence.

1. INTRODUCTION

Let $f : C \to C$ be a nonlinear map, where C is a nonempty closed and convex subset of a real Hilbert space H. The map f is said to be

(1) $L-$ Lipschitz continuous, if there exits a constant $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \ \forall \ x, y \in C,$$

if the constant L is in the interval $[0, 1)$, then f is called a contraction map, while f is called nonexpansive map if $L = 1$,

(2) v-strongly monotone, if there exists a constant $v > 0$ such that

$$\left\langle f(x) - f(y), (x - y)\right\rangle \geq v\|x - y\|^2 \ \forall \ x, y \in C,$$

(3) v-inverse strongly monotone (v-ism), if there exists a constant $v > 0$ such that

$$\left\langle f(x) - f(y), (x - y)\right\rangle \geq v\|f(x) - f(y)\|^2 \ \forall \ x, y \in C,$$

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if $v = 1$, then f is called firmly nonexpansive,

(4) monotone, if

$$\langle f(x) - f(y), (x - y)\rangle \geq 0 \; \forall \, x, y \in C,$$

(5) α-averaged, if $f = (1-\alpha)I + \alpha T$, for $0 < \alpha < 1$ and $T : C \to C$ is nonexpansive.

It can easily be seen that both v-strongly monotone and v-inverse strongly monotone maps are monotone. Also, firmly nonexpansive maps are $\frac{1}{2}$-averaged while averaged mappings are nonexpansive. It is also known that every v-ism map is $\frac{1}{v}$-Lipschitz continuous.

Recall that, in the sense of Browder and Petryshyn [7] a mapping $T : C \to C$ is called κ-strictly pseudocontractive if for $0 \leq \kappa < 1$,

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \; \forall \, x, y \in C.$$

A point $x \in C$ is called a fixed point of T, if $T(x) = x$. Let $F(T)$ denotes the set of fixed point of T, and it is known generally that if $F(T) \neq \emptyset$, then $F(T)$ is closed and convex. Clearly, nonexpansive mappings are 0-strictly pseudocontractive.

The Variational Inequality Problem (VIP) is defined as: find $x \in C$ such that

$$\langle f(x), (y - x)\rangle \geq 0, \; \forall \, y \in C. \tag{1}$$

The set of solutions of VIP (1) is denoted by $VI(C, f)$.

VIP was first introduced by Stampacchia [42] for modelling problems arising from mechanics. To study the regularity of the problem for partial differential equations, Stampacchia [42] studied a generalization of the Lax-Milgram theorem and called all problems involving inequalities of such kind, the VIPs. The VIP is also known to have numerous applications in diverse fields such as, physics, engineering, economics, mathematical programming, among others. It can also be considered as a central problem in optimization and nonlinear analysis since the theory of variational inequalities provides a simple, natural and unified frame work for a general treatment of many important mathematical problems such as, minimization problems, network equilibrium problems, complementary problems, systems of nonlinear equations and others (see [5, 6, 11, 21, 27, 28, 37, 42, 47] and the references therein). Thus, the theory has become an area of great research interest to numerous researchers. As a result of this, there has been an increasing research interest in developing efficient and implementable methods for solving VIPs.

A simple iterative method for solving VIP (1) is the gradient projection method

$$x_{n+1} = P_C(x_n - \lambda f(x_n)), \; n \geq 1, \tag{2}$$

where $\lambda > 0$ and P_C is the metric projection of H onto C.

Algorithm (2) converges strongly to a unique solution of problem (1) provided that f is v-strongly monotone and L-Lipschitz continuous with $\lambda \in \left(0, \frac{2v}{L^2}\right)$. However, if f is v-inverse strongly monotone, $VI(C,f)$ may not exists. In this case, it is assumed that $VI(C,f) \neq \emptyset$ and $\lambda \in (0, 2v)$, then $VI(C,f)$ is closed and convex, and (2) converges weakly to a solution of (1). An attempt to relax the strong monotonicity assumption (i.e., ν-strongly monotonicity and ν-inverse strongly monotonicity assumptions) to monotonicity would complicate the situation. In fact, Algorithm (2) may not converge if f is monotone and Lipschitz continuous. Therefore, the gradient projection method is only efficient for solving VIP (1) when f is either strongly monotone or inverse strongly monotone (see for example [47]). To overcome this setback, Korpelevich [29] introduced the following extragradient method for solving VIP (1) in the finite dimensional Euclidean space when f is monotone and L-Lipschitz continuous:

$$\left\{ \begin{array}{l} y_n = P_C(x_n - \lambda f(x_n)), \\ x_{n+1} = P_C(x_n - \lambda f(y_n)), \ n \geq 1, \end{array} \right. \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$. Korpelevich [29] proved that the sequence $\{x_n\}$ generated by (3) converges weakly to a solution of $VI(C,f)$ provided that $VI(C,f) \neq \emptyset$. Since then, many authors have studied the extragradient method in the infinite dimensional spaces (see [1, 2, 3, 10, 22, 24, 33] and the references therein).

The VIP has also been studied as a split type problem, namely the Split Variational Inequality Problem (SVIP) which was introduced by Censor et al. [11] and defined as: find $x \in C$ such that

$$\big\langle f(x), (y - x) \big\rangle \geq 0 \ \forall \ y \in C \tag{4}$$

and

$$\big\langle g(Ax), z - Ax \big\rangle \geq 0 \ \forall \ z \in Q, \tag{5}$$

where C and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_2$ is a bounded linear operator and f, g are nonlinear mappings on C and Q respectively. As observed in [11], the SVIP can be seen as a pair of VIPs in which a solution of one VIP occur in the first space H_1 whose image under a given bounded linear operator A is a solution of the second VIP in the second space H_2. Furthermore, SVIPs are very important in optimization, nonlinear and convex analysis. They can be viewed as an important generalization of the spilt feasibility problems introduced by Censor and Elfving [14] which are known to have applications in many fields such as phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning among others (for example, see [9, 12, 14, 13, 20, 25, 26] and the references therein). To solve SVIP (4)-(5), Censor [11] proposed the following algorithm:

$$x_{n+1} = P_C(I - \lambda f)(x_n + \tau A^*(P_Q(I - \lambda g) - I)Ax_n) \ n \geq 1, \tag{6}$$

where $\tau \in \left(0, \frac{1}{L}\right)$, L being the spectral radius of A^*A. They proved that the sequence $\{x_n\}$ generated by (6) converges weakly to a solution of (4)-(5) provided that the solution set of problem (4)-(5) is nonempty, f, g are α_1, α_2-inverse strongly monotone mappings, $\lambda \in (0, 2\alpha)$, where $\alpha := \min\{\alpha_1, \alpha_2\}$, and for all x solution of (4),

$$\langle f(y), P_C(I - \lambda f)(y) - x\rangle \geq 0, \ \forall \ y \ \in H. \tag{7}$$

Indeed, the weak convergence of Algorithm (6) requires some slightly strong assumptions (assumption (7) and the fact that both mappings are inverse strongly monotone). To overcome these assumptions, Tian and Jiang [47] proposed the following algorithm by combining Algorithm (3) and (6):

$$\begin{cases} y_n = P_C(x_n - \tau_n A^*(I - T)Ax_n), \\ x_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(x_n)), \quad n \geq 1. \end{cases} \tag{8}$$

They obtained the following results without assumption (7), and under the condition that f is monotone and L-Lipschitz continuous.

Theorem 1.1. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$, and $T : H_2 \to H_2$ be a nonexpansive mapping. Let $f : C \to H_1$ be a monotone and L-Lipschitz continuous mapping. Suppose that $\Gamma := \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$ and the sequence $\{x_n\}$ is defined for arbitrary $x_1 \in C$ by (8), where $\{\tau_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in \left(0, \frac{1}{L}\right)$. Then $\{x_n\}$ converges weakly to $z \in \Gamma$.*

We also note that the class of SVIP considered by Tian and Jian [47], that is, find $x \in C$ satisfying

$$\langle f(x), (y - x)\rangle \geq 0 \ \forall \ y \in C \text{ and } Ax \in F(T), \tag{9}$$

generalizes the class of SVIP considered by Censor et al. [11] (see [47, Theorem 3.3]). We now make the following observations about the results of Tian and Jiang [47].

Remark 1.2. (1) The sequence generated by Algorithm (8) converges weakly to a solution of problem (9). However, we know that strong convergence results are more desirable than weak convergence results in infinite dimensional spaces.

(2) For the weak convergence of Algorithm (8) to the solution of problem (9), one needs to compute three projections onto the closed convex set C in each iteration which seems very difficult to do in practice when C does not posses a simple structure, and this could seriously affect the efficiency of the algorithm. Thus, for the sake of computation, it is more desirable to develop algorithms with minimized number of evaluations of P_C per iteration.

(3) To implement Theorem 1.1, one needs to compute the Lipschitz constant L before the control sequence $\{\lambda_n\}$ can be computed. Thus, Theorem 1.1 is dependent on the knowledge of the Lipschitz constant L.
(4) Problem (9) can be viewed as a class of SVIP for which a solution of a VIP occur in the first space H_1 whose image under a given bounded linear operator A is a fixed point of a nonexpansive mapping in the second space H_2.

We note here that items (1)-(3) of Remark 1.2 can also be attributed to the work of Korpelevich [29]. In this case, two projections onto C needs to be computed for solving VIP (1). To reduce the number of projections onto C from two to one, Thong and Hieu [37] proposed the following two iterative methods for approximating solutions of (1).

Algorithm 1
Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0 \in H$ be arbitrary.
Iterative Steps: Assume that $x_n \in H$ is known, calculate x_{n+1} as follows:
Step 1. Compute

$$y_n = P_C(x_n - \lambda_n f(x_n)),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda \|f(x_n) - f(y_n)\| \leq \mu \|x_n - y_n\|. \tag{10}$$

If $x_n = y_n$, then stop and x_n is the solution of VIP. Otherwise,
Step 2. Compute

$$x_{n+1} = y_n - \lambda_n (f(y_n) - f(x_n)),$$

where $f : H \to H$ is monotone and Lipschitz continuous.
Set $n := n + 1$ and return to **Step 1**.

Algorithm 2
Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step1. Compute

$$y_n = P_C(x_n - \lambda_n f(x_n)),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda \|f(x_n) - f(y_n)\| \leq \mu \|x_n - y_n\|. \tag{11}$$

If $x_n = y_n$, then stop and x_n is the solution of VIP. Otherwise,
Step 2. Compute

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) z_n,$$

where $z_n = y_n - \lambda_n (f(y_n) - f(x_n))$, where $f : H \to H$ is monotone and Lipschitz continuous, and $g : H \to H$ is a contraction with constant $\rho \in [0, 1)$.
Set $n := n + 1$ and return to **Step 1**.

They obtained weak and strong convergence of Algorithm 1 and Algorithm 2 respectively, to a solution of (1) in a real Hilbert space. The main features of Algorithm 1 and Algorithm 2 are that:

(1) Only one projection onto C is required to be computed in each iteration,
(2) the Armijo-like search rule (10) (see also (11)) which has been established in [37, Lemma 3.1] to be well-defined, can be seen as a local approximation of the Lipschitz constant of the mapping f. Thus, the Lipschitz constant need not to be known. Hence, the control sequence $\{\lambda_n\}$ is given self-adaptively (see [21]) unlike Algorithm (2), (3), (6) and (8) where the knowledge of $\{\lambda_n\}$ (or λ) depends on the knowledge of L (see also (3) of Remark 1.2).

Therefore, Algorithm 1 and Algorithm 2 are very efficient for solving problem (1). However, our interest in this paper is in the development of efficient and better implementable algorithm for solving problem (9) which is more general and known to have more applications than problem (1).

We would also like to mention here that, the construction of inertial-type algorithms have been of great interest ever since it was first introduced in [4]. It has been successfully applied for solving various optimization problems emanating from the area of applied sciences [8, 30]. For instance, the method was applied for solving the split feasibility problem [17, 34]. In recent time, it has also proved successful in speeding up convergence rate of iterative algorithms. Some more contributions in this direction are Hybrid inertial proximal algorithms, Chuang [16], inertial method for split common fixed point problems, Thong and Hieu [39], inertial subgradient extragradient algorithms, Thong and Hieu [38], modified inertial Mann algorithm and inertial CQ algorithm, Dong *et al.* [18], inertial projection and contraction methods for split feasibility problem applied to compressed sensing and image restoration, Suanti *et al.* [43], inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space, Reich *et al.* [35], Iterative method with inertial for variational inequalities in Hilbert spaces, Shehu and Cholamjiak [40], among others in the literature. As a passing remark, we point out here that most of the results involving inertial-type algorithms only yielded weak convergence results. In very few cases where strong convergence results were obtained, the authors employed the inertial CQ algorithms which requires that, at each step of the iteration process, the computation of the two subsets C_n and Q_n, the computation of their intersection $C_n \cap Q_n$ and the computation of the projection of the initial vector onto this intersection (see [32, 18, 19, 45, 44] and the references therein). Thus, it will be of great interest to study the strong convergence of an inertial-type algorithm which does not involve any of the above mentioned computations at each step of the iteration process.

Motivated by the works of Tain and Jaing [47], Thong and Hieu [37, 39, 38], Dong et al. [18] and Chuang [16], we propose an iteration method which does not require prior knowledge of the Lipschitz constant L, and which has a minimized number of evaluations of P_C (unlike algorithm (8)) for solving SVIP (9) (which clearly extends the VIP (1) studied in [37]. Furthermore, we prove that the sequence generated by our iteration converges strongly to a solution of SVIP (9) for which T is strictly pseudocontractive (unlike the nonexpansive mapping considered in [47]). Our strong convergent algorithm (interia-type algorithm) does not require the construction of any of the subsets used in [18, 19, 45, 44]. Also, a numerical example of our algorithm in comparison with Algorithm (8) of Tian and Jiang [47] is given to show the applicability of our result. The numerical experiment shows that our algorithm converges faster than that proposed by Tian and Jiang [47]. Our result extends and improves the results of Tian and Jiang [47], Thong and Hieu [37, 39, 38], Dong et al. [18] and Chuang [16], and many important results in this direction.

2. PRELIMINARIES

We state some useful results which will be needed in the proof of our main theorem.

LEMMA 2.1 ([15]). *Let H be a real Hilbert space, then for all $x, y \in H$ and $\alpha \in (0,1)$, the following hold:*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2$,

(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$,

(iii) $||x-y||^2 \leq ||x||^2 + 2\langle y, x-y\rangle$.

LEMMA 2.2 ([48]). *Let H be a Hilbert space and $f : H \to H$ be a nonlinear mapping, then the following hold.*

(i) *f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism.*

(ii) *f is ν-ism and $\gamma > 0$, then γf is $\frac{\nu}{\gamma}$-ism.*

(iii) *f is averaged if and only if the complement $I - f$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\beta \in (0,1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism.*

(iv) *If f_1 is β_1-averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0,1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.*

(v) *If f_1 and f_2 are averaged and have a common fixed point, then $F(f_1 f_2) = F(f_1) \cap F(f_2)$.*

LEMMA 2.3 ([46]). *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$, and $T : H_2 \to H_2$ be a nonexpansive mapping. Then $A^*(I - T)A$ is $\frac{1}{2\|A\|^2}$-ism.*

LEMMA 2.4 ([47]). *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent.*

(i) $x^* = P_C(I - \gamma A^*(I-T)A)x^*$;
(ii) $0 \in A^*(I-T)Ax^* + N_C x^*$;
(iii) $x^* \in C \cap A^{-1}F(T)$.

LEMMA 2.5 ([49]). *Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H converging weakly to x^* and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x^* = y$.*

LEMMA 2.6 ([50]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n,\ n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\alpha_n\} \subset [0,1]$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0\ (n \geq 0)$, $\sum\limits_{n=0}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.7 ([51]). *Let H be a real Hilbert space and $T : H \to H$ be a κ-strictly pseudocontractive mapping with $\kappa \in [0,1)$. Let $T_\beta := \beta I + (1-\beta)T$, where $\beta \in [\kappa, 1)$, then*
(i) $F(T) = F(T_\beta)$,
(ii) T_β *is a nonexpansive mapping.*

LEMMA 2.8 ([31]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1}\ \forall j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\Gamma_n\}_{n\geq n_0}$ is a nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to 0$, and for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1},\ \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

LEMMA 2.9 ([37]). *The Armijo-like search rule* (10) *is well defined and* $\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma$.

LEMMA 2.10 ([36]). *Assume that $f : H \to H$ is a continuous and monotone operator. Then x^* is a solution of* (1) *if and only if x^* is a solution of following problem: find $x \in C$ such that*

$$\langle fx, x-y \rangle \geq 0,\ \forall\ y \in C.$$

3. MAIN RESULT

We now present and study our inertial extragradient algorithm in this section, for solving the SVIP (9). Throughout this section, we assume that H_1 and H_2 are two real Hilbert spaces, C is a nonempty closed and convex subset of H_1 and $f : H_1 \to H_1$ is a monotone and Lipschitz continuous operator, but the Lipschitz constant need not to be known. We also assume that $g : H_1 \to H_1$ is a contraction mapping with constant $\rho \in [0, 1)$, $A : H_1 \to H_2$ is a bounded linear operator and $T : H_2 \to H_2$ is a κ-strictly pseudocontractive mapping with $\kappa \in [0, 1)$. Finally, we assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\theta_n\} \subset [0, \theta)$, $\theta \in [0, 1)$ and the solution set $\Gamma := \{z \in VI(C, f) : Az \in F(T)\} \neq \emptyset$.

Algorithm 3.1
Initialization: Let $\gamma > 0$, $l, \mu \in (0, 1)$ and $x_0, x_1 \in H$ be given arbitrarily.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Set $u_n = x_n + \theta_n(x_n - x_{n-1})$ and compute

(12) $w_n = P_C(u_n - \tau_n A^*(I - T_\beta)Au_n)$ and $y_n = P_C(w_n - \lambda_n f(w_n))$,

where T_β is as defined in Lemma 2.7 and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda_n\|f(w_n) - f(y_n)\| \leq \mu\|w_n - y_n\|. \tag{13}$$

Step 2. Compute

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)z_n, \tag{14}$$

where $z_n = y_n - \lambda_n(f(y_n) - f(w_n))$. Set $n := n + 1$ and go back to **Step 1.**

We highlight below some of the features or advantages of Algorithm 3.1 stated above;

(1) The iterative method used does not require prior knowledge of lipschitz constant L.
(2) Minimized number of evaluations of P_C per iteration unlike algorithm (1.8) for solving SVIP (1.9).
(3) The sequence generated converges strongly to a solution of SVIP (1.9) for which T is strictly pseudocontractive unlike the nonexpansive mapping considered in [47].
(4) The strong convergence results obtained does not require construction of the subsets used in [18, 19, 44, 45].

Lemma 3.1. *Let $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.1, then*

(1) $\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \mu^2)\|y_n - w_n\|^2 \ \forall\, p \in \Gamma$.
(2) $\|x_{n+1}-p\|^2 \leq \alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)+2\theta_n(1-\alpha_n)\|x_n-x_{n-1}\|^2-(1-\alpha_n)(1-\mu^2)\|y_n-w_n\|^2 \ \forall\, p \in \Gamma$.

Proof. (1) Let $p \in \Gamma$, then by the monotonicity of f, we obtain from (13) and (14) that

$$
\begin{aligned}
\|z_n - p\|^2 &= \|y_n - p\|^2 + \lambda_n^2 \|f(y_n) - f(w_n)\|^2 \\
&\quad - 2\lambda_n \langle y_n - p, f(y_n) - f(w_n)\rangle \\
&= \|y_n - w_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, w_n - p\rangle \\
&\quad + \lambda_n^2 \|f(y_n) - f(w_n)\|^2 \\
&\quad - 2\lambda_n \langle y_n - p, f(y_n) - f(w_n)\rangle \\
&\leq \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle w_n - p, y_n \\
&\quad - w_n\rangle + \mu^2 \|y_n - w_n\|^2 \\
&\quad - 2\lambda_n \langle y_n - p, f(y_n) - f(w_n)\rangle \\
&= \|w_n - p\|^2 + (1 + \mu^2)\|y_n - w_n\|^2 + 2\langle y_n - p, y_n - w_n\rangle \\
&\quad - 2\langle y_n - w_n, y_n - w_n\rangle - 2\lambda_n \langle y_n - p, f(y_n) - f(w_n)\rangle \\
&= \|w_n - p\|^2 - (1 - \mu^2)\|y_n - w_n\|^2 \\
&\quad + 2\langle y_n - p, y_n - w_n - \lambda_n (f(y_n) - f(w_n))\rangle.
\end{aligned} \tag{15}
$$

Now, since from (12), $y_n = P_C(w_n - \lambda_n f(w_n))$, we obtain from the characteristics property of P_C that

$$\langle y_n - p, y_n - w_n + \lambda_n f(w_n)\rangle \leq 0.$$

Thus, we obtain from the monotonicity of f that

$$
\begin{aligned}
&2\langle (y_n - p), y_n - w_n - \lambda_n (f(y_n) - f(w_n))\rangle \\
&= 2\langle (y_n - p), y_n - w_n + \lambda_n f(w_n)\rangle - 2\lambda_n \langle y_n - p, f(y_n)\rangle \\
&\leq -2\lambda_n \langle y_n - p, f(y_n)\rangle \\
&= -2\lambda_n \langle y_n - p, f(y_n) - f(p)\rangle - 2\lambda_n \langle y_n - p, f(p)\rangle \\
&\leq 0.
\end{aligned} \tag{16}
$$

From (15) and (16), we obtain the desired conclusion.

(2) For $p \in \Gamma$, we obtain

$$
\begin{aligned}
\|u_n - p\|^2 &= \|x_n - p\|^2 + 2\theta_n \big\langle x_n - p, x_n - x_{n-1}\big\rangle + {\theta_n}^2 \|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n (\|x_n - x_{n-1}\|^2 + \|x_n - p\|^2 \\
&\quad - \|x_{n-1} - p\|^2) + {\theta_n}^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n (\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\
&\quad + 2\theta_n \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{17}
$$

LEMMA 3.2. *Let $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.1 such that $\sum\limits_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$, then the sequences $\{x_n\}$,*

$\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded, and $\lim_{n\to\infty}\|x_n-u_n\|=0$. Furthermore, if $\lim_{n\to\infty}\alpha_n=0$, then $\lim_{n\to\infty}\|x_{n+1}-z_n\|=0$.

Proof. From Lemma 2.2 (ii),(iii),(iv), Lemma 2.3 and Lemma 2.7, we obtain that $P_C(I-\tau_nA^*(I-T_\lambda)A)$ is $\frac{1+\tau_n\|A\|^2}{2}$-average. That is $P_C(I-\tau_nA^*(I-T_\lambda)A)=(1-\beta_n)I+\beta_nT_n$, $\forall n\geq 1$, where $\beta_n=\frac{1+\tau_n\|A\|^2}{2}$ and T_n is nonexpansive for all $n\geq 1$. Therefore, we can rewrite w_n from (12) as

$$w_n=(1-\beta_n)u_n+\beta_nT_nu_n,\ n\geq 1. \tag{18}$$

Let $p\in\Gamma$, then from (18), we obtain that

$$\begin{aligned}\|w_n-p\|^2&\leq(1-\beta_n)\|u_n-p\|^2+\beta_n\|T_nu_n-p\|^2\\&-\beta_n(1-\beta_n)\|u_n-T_nu_n\|^2\\&\leq\|u_n-p\|^2-\beta_n(1-\beta_n)\|u_n-T_nu_n\|^2\\&\leq\|u_n-p\|^2,\end{aligned} \tag{19}$$

which implies that

$$\begin{aligned}\|w_n-p\|&\leq\|u_n-p\|\\&\leq\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|.\end{aligned} \tag{20}$$

Again, from (14), (15) and (17), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&=\alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\&-\alpha_n(1-\alpha_n)\|g(x_n)-z_n\|^2\\&\leq\alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|w_n-p\|^2\\&-(1-\alpha_n)(1-\mu^2)\|y_n-w_n\|^2\\&\leq\alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|u_n-p\|^2\\&-(1-\alpha_n)(1-\mu^2)\|y_n-w_n\|^2\\&\leq\alpha_n\|g(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\&+(1-\alpha_n)\theta_n(\|x_n-p\|^2-\|x_{n-1}-p\|^2)\\&+2\theta_n(1-\theta_n)\|x_n-x_{n-1}\|^2\\&-(1-\alpha_n)(1-\mu^2)\|y_n-w_n\|^2.\end{aligned} \tag{21}$$

□

Thus, from (14) and Lemma 3.1, we obtain that

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_n\|g(x_n)-p\| + (1-\alpha_n)\|z_n-p\| \\
&\le \alpha_n\rho\|x_n-p\| + \alpha_n\|g(p)-p\| + (1-\alpha_n)\|z_n-p\| \\
&\le \alpha_n\rho\|x_n-p\| + (1-\alpha_n)\|w_n-p\| + \alpha_n\|g(p)-p\| \\
&\le \alpha_n\rho\|x_n-p\| + (1-\alpha_n)\|x_n-p\| \\
&+ (1-\alpha_n)\theta_n\|x_n-x_{n-1}\| + \alpha_n\|g(p)-p\| \\
&= (1-\alpha_n(1-\rho))\|x_n-p\| + (1-\alpha_n)\theta_n\|x_n-x_{n-1}\| \\
&+ \alpha_n\|g(p)-p\| \\
&\le \max\left\{\|x_n-p\|, \frac{\|g(p)-p\|}{1-\rho}\right\} + \theta_n\|x_n-x_{n-1}\| \\
&\le \max\Big\{\max\Big\{\|x_{n-1}-p\|, \frac{\|g(p)-p\|}{1-\rho}\Big\} \\
&+ \theta_{n-1}\|x_{n-1}-x_{n-2}\|, \frac{\|g(p)-p\|}{1-\rho}\Big\} \\
&+ \theta_n\|x_n-x_{n-1}\| \\
&= \max\left\{\|x_{n-1}-p\|, \frac{\|g(p)-p\|}{1-\rho}\right\} \\
&+ \theta_{n-1}\|x_{n-1}-x_{n-2}\| + \theta_n\|x_n-x_{n-1}\|.
\end{aligned}$$

Let $M := \sum_{i=1}^{n}\theta_i\|x_i-x_{i-1}\|$, since $\sum_{i=1}^{n}\theta_i\|x_i-x_{i-1}\| < \infty$, we obtain that

$$\|x_{n+1}-p\| \le \max\left\{\|x_1-p\|, \frac{\|g(p)-p\|}{1-\rho}\right\} + M.$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded. More so, we obtain from Algorithm 3.1 that

$$\|u_n-x_n\| = \theta_n\|x_n-x_{n-1}\| \to 0, \text{ as } n\to\infty. \tag{22}$$

Furthermore, from (14), we obtain that

$$\|x_{n+1}-z_n\| = \alpha_n\|g(x_n)-z_n\| \to 0, \text{ as } n\to\infty. \tag{23}$$

□

LEMMA 3.3. *Let $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 3.1 such that $\sum\limits_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\| < \infty$ and $\lim\limits_{n\to\infty}\|w_n-y_n\| = 0 = \lim\limits_{n\to\infty}\|w_n-u_n\|$. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ that converges weakly to some $v\in H$, then $v\in\Gamma$.*

Proof. Since $\lim_{n\to\infty}\|w_n-u_n\| = 0$, we obtain that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to $v\in H$. Without loss of generality, we may also assume that the subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ converges to a point say $\overline{\tau} \in \left(0, \frac{1}{\|A\|^2}\right)$. Also, by Lemma 2.3, $A^*(I - T_\beta)A$ is an inverse strongly monotone operator. Therefore, $\{A^*(I-T_\beta)Au_{n_k}\}$ is bounded. Hence, by the firmly nonexpansivity of P_C, we obtain that

$$\begin{aligned}&\|P_C(I - \tau_{n_k}A^*(I - T_\beta)A)u_{n_k} - P_C(I - \overline{\tau}A^*(I - T_\beta)A)u_{n_k}\|\\ &\leq |\tau_{n_k} - \overline{\tau}|\|A^*(I - T_\beta)Au_{n_k}\| \to 0, \text{ as } k \to \infty.\end{aligned}$$

That is,

$$\lim_{k\to\infty} \|w_{n_k} - P_C(I - \overline{\tau}A^*(I - T_\beta)A)u_{n_k}\| = 0,$$

which implies that

$$\lim_{k\to\infty} \|u_{n_k} - P_C(I - \overline{\tau}A^*(I - T_\beta)A)u_{n_k}\| = 0. \tag{24}$$

Thus, by Lemma 2.5, we obtain that $v \in F(P_C(I - \overline{\tau}A^*(I - T_\beta)A)$. It then follows from Lemma 2.4 that $v \in C \cap A^{-1}F(T_\beta)$, which together with Lemma 2.7 implies that

$$Av \in F(T_\beta) = F(T). \tag{25}$$

Now, by the monotonicity of f and the characteristic property of P_C, we obtain for all $x \in C$ that

$$\begin{aligned}0 &\leq \langle y_{n_k} - w_{n_k} + \lambda_{n_k} f w_{n_k}, x - y_{n_k}\rangle\\ &= \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda_{n_k}\langle f w_{n_k}, w_{n_k} - y_{n_k}\rangle\\ &+ \lambda_{n_k}\langle f w_{n_k}, x - w_{n_k}\rangle\\ &\leq \|y_{n_k} - w_{n_k}\|\|x - y_{n_k}\| + \lambda_{n_k}\|f w_{n_k}\|\|w_{n_k} - y_{n_k}\|\\ &+ \lambda_{n_k}\langle f w_{n_k}, x - w_{n_k}\rangle \ \forall x \in C\\ &= \|y_{n_k} - w_{n_k}\|\|x - y_{n_k}\| + \lambda_{n_k}\|f w_{n_k}\|\|w_{n_k} - y_{n_k}\|\\ &+ \lambda_{n_k}\left(\langle f w_{n_k} - fx, x - w_{n_k}\rangle + \langle fx, x - w_{n_k}\rangle\right)\\ &\leq \|y_{n_k} - w_{n_k}\|\|x - y_{n_k}\| + \lambda_{n_k}\|f w_{n_k}\|\|w_{n_k} - y_{n_k}\|\\ &+ \lambda_{n_k}\langle fx, x - w_{n_k}\rangle \ \forall x \in C.\end{aligned} \tag{26}$$

Since by Lemma 2.9, $\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma$, we obtain by passing limit as $n \to \infty$ in (26) that

$$\langle fx, x - v\rangle \geq 0 \ \forall x \in C.$$

Thus, by Lemma 2.10, we have that $v \in VI(C, f)$. This together with (25) gives that $v \in \Gamma$. □

Theorem 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that $\sum_{n=1}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty$. Assume that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, $\lim_{n\to\infty} \theta_n||x_n - x_{n-1}|| = 0$ and hence bounded. Then, $\{x_n\}$ converges strongly to $z = P_\Gamma g(z)$.*

Proof. We consider two cases for our proof.

Case 1. Let $z = P_\Gamma g(z)$. Suppose that $\{\|x_n - z\|^2\}$ is monotone decreasing, then $\{\|x_n - z\|^2\}$ is convergent. Thus,

$$\lim_{n\to\infty} \|x_n - z\|^2 = \lim_{n\to\infty} \|x_{n+1} - z\|^2. \tag{27}$$

Since $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$ and $\lim_{n\to\infty} \alpha_n = 0$, we obtain from (21) and (27) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{28}$$

Again, from Algorithm 3.1, we obtain that

$$\|z_n - y_n\| = \lambda_n \|fy_n - fw_n\| \leq \mu \|w_n - y_n\| \to 0, \text{ as } n \to \infty. \tag{29}$$

From (23), (28) and (29), we obtain that

$$\|z_n - w_n\| \to 0 \text{ and } \|x_{n+1} - w_n\| \to 0, \text{ as } n \to \infty. \tag{30}$$

From (17) ,(19), (21) and (27), we obtain that

$$\begin{aligned}
&\beta_n(1-\beta_n)\|u_n - T_n u_n\|^2 \\
&\leq \|u_n - z\|^2 - \|w_n - z\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
&+ 2\theta_n \|x_n - x_{n-1}\|^2 - \|w_n - z\|^2 \\
&\leq \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
&+ 2\theta_n \|x_n - x_{n-1}\|^2 + \alpha_n \|g(x_n) - z\|^2 - \|x_{n+1} - z\|^2 \\
&= \|x_n - z\|^2 - \|x_{n-1} - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
&+ 2\theta_n \|x_n - x_{n-1}\|^2 + \alpha_n \|g(x_n) - z\|^2 \to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies from the definition of β_n that

$$\lim_{n\to\infty} \|u_n - T_n u_n\| = 0. \tag{31}$$

Again, from (18) and (31), we obtain that

$$\|w_n - u_n\| = \beta_n \|u_n - T_n u_n\| \to 0, \text{ as } n \to \infty. \tag{32}$$

Thus, we obtain from (22) that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{33}$$

From (30) and (33), we obtain that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{34}$$

By Lemma 3.2, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $v \in H$ such that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle g(z)-z, x_n - z\big\rangle &= \lim_{k\to\infty}\big\langle g(z)-z, x_{n_k}-z\big\rangle \\ &= \big\langle g(z)-z, v-z\big\rangle.\end{aligned} \tag{35}$$

By (33), there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ that converges weakly to $v \in H$. It the follows from (28), (32) and Lemma 3.3 that $v \in \Gamma$. Furthermore, since $z = P_\Gamma g(z)$, we obtain from (35) that

$$\limsup_{n\to\infty}\big\langle g(z)-z, x_n - z\big\rangle \le 0,$$

which implies from (34) that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle g(z)-z, x_{n+1} - z\big\rangle &= \limsup_{n\to\infty}\Big(\big\langle g(z)-z, x_{n+1}-x_n\big\rangle \\ &\quad + \big\langle g(z)-z, x_n - z\big\rangle\Big) \le 0.\end{aligned} \tag{36}$$

Thus, from (14) and Lemma 2.1 (iii), we obtain that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \le (1-\alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\big\langle g(x_n)-z, x_{n+1}-z\big\rangle \\
&= (1-\alpha_n)^2\|z_n-z\|^2 + 2\alpha_n\Big(\big\langle g(x_n)-g(z), x_{n+1}-z\big\rangle \\
&+ \big\langle g(z)-z, x_{n+1}-z\big\rangle\Big) \\
&\le (1-\alpha_n)^2\|u_n-z\|^2 + 2\alpha_n\gamma\|x_n-z\|\|x_{n+1}-z\| \\
&+ 2\alpha_n\big\langle g(z)-z, x_{n+1}-z\big\rangle \\
&\le (1-\alpha_n)^2\|x_n-z\|^2 + (1-\alpha_n)^2\theta_n\left(\|x_n-z\|^2 - \|x_{n-1}-z\|^2\right) \\
&+ 2(1-\alpha_n)^2\theta_n\|x_n-x_{n-1}\|^2 + 2\alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| \\
&+ 2\alpha_n\big\langle g(z)-z, x_{n+1}-z\big\rangle \\
&\le (1-2\alpha_n(1-\rho))\|x_n-z\|^2 + \alpha_n^2\|x_n-z\|^2 + 2\alpha_n\big\langle g(z)-z, x_{n+1}-z\big\rangle \\
&+ \theta_n\left(\|x_n-z\|^2 - \|x_{n-1}-z\|^2 + 2\|x_n-x_{n-1}\|^2\right) \\
&\le (1-2\alpha_n(1-\rho))\|x_n-z\|^2 + 2\alpha_n(1-\rho) \\
&\times\Big[\frac{\alpha_n\|x_n-z\|^2}{2(1-\rho)} + \frac{\big\langle g(z)-z, x_{n+1}-z\big\rangle}{1-\rho}\Big] \\
&+ 2\theta_n\|x_n-x_{n-1}\|^2.
\end{aligned} \tag{37}$$

Thus, by Lemma 2.6, we obtain that $\lim_{n\to\infty}\|x_n - z\|^2 = 0$. Hence, $\{x_n\}$ converges strongly to $z = P_\Gamma g(z)$.

Case 2. Suppose that $\{\|x_n - z\|^2\}$ is not monotone decreasing, then there exists a subsequence $\{\|x_{n_i} - z\|^2\}$ of $\{\|x_n - z\|^2\}$ such that $\|x_{n_i} - z\|^2 < \|x_{n_{i+1}} - z\|^2 \; \forall i \in \mathbb{N}$. Thus, by Lemma 2.8 there exists a nondecreasing sequence

$\{m_k\}$ of $\mathbb{N}$ such that $k \to \infty$ and the following holds

$$\|x_{m_k} - z\|^2 \le \|x_{m_{k+1}} - z\|^2 \text{ and } \|x_k - z\|^2 \le \|x_{m_k} - z\|^2. \tag{38}$$

Thus, we obtain from (21) that

$$\begin{aligned}
&(1-\alpha_{m_k})(1-\mu^2)\|y_{m_k} - w_{m_k}\|^2 \\
&\le \|x_{m_k} - z\|^2 - \|x_{m_{k+1}} - z\|^2 + \alpha_{m_k}\|g(x_{m_k}) - z\|^2 \\
&+ \theta_{m_k}\left(\|x_{m_k} - z\|^2 - \|x_{m_{k-1}} - z\|^2\right) + 2\theta_{m_k}\|x_{m_k} - x_{m_{k-1}}\|^2 \\
&\le \alpha_{m_k}\|g(x_{m_k}) - z\|^2 + 2\theta_{m_k}\|x_{m_k} - x_{m_{k-1}}\|^2 \\
&+ \theta_{m_k}\left(\|x_{m_k} - z\|^2 - \|x_{m_{k-1}} - z\|^2\right) \to 0,\ k \to \infty,
\end{aligned}$$

which implies that $\lim_{k\to\infty}\|y_{m_k} - w_{m_k}\| = 0$. Similarly, we obtain that $\lim_{k\to\infty}\|w_{n_k} - u_{m_k}\| = 0$. By similar arguments as in **Case 1**, we obtain $\lim_{k\to\infty}\|x_{m_k} - w_{m_k}\| = 0 = \lim_{k\to\infty}\|x_{m_k} - x_{m_{k+1}}\| = 0$ and

$$\limsup_{k\to\infty}\left\langle g(z) - z, x_{m_{k+1}} - z\right\rangle \le 0.$$

Now, for all $k \ge k_0$, we obtain from (37) that

$$\begin{aligned}
\|x_{m_{k+1}} - z\|^2 &\le (1 - 2\alpha_{m_k}(1-\rho))\|x_{m_k} - z\|^2 \\
&+ 2\alpha_{m_k}(1-\rho)\left[\frac{\alpha_{m_k}\|x_{m_k} - z\|^2}{2(1-\rho)} + \frac{\left\langle g(z) - z, x_{m_{k+1}} - z\right\rangle}{1-\rho}\right] \\
&+ 2\alpha_{m_k}(1-\rho)\frac{\theta_{mk}}{2\alpha_{mk}(1-\rho)}\left[\left(\|x_{m_k} - z\|^2 + 2\|x_{m_k} - x_{m_{k-1}}\|^2\right)\right] \\
&\le (1 - 2\alpha_{m_k}(1-\rho))\|x_{m_k} - z\|^2 \\
&+ 2\alpha_{m_k}(1-\rho)\Big[\frac{\alpha_{m_n}M_1}{2(1-\rho)} + \frac{\left\langle g(z) - z, x_{m_{k+1}} - z\right\rangle}{1-\rho} \\
&+ \frac{\theta_{m_k}}{2\alpha_{m_k}(1-\rho)}(M_1 + M_2)\Big],
\end{aligned}$$

for some $M_1, M_2 > 0$. Thus, we obtain from (38) that

$$\begin{aligned}
\|x_k - z\|^2 \le \|x_{m_{k+1}} - z\|^2 \le \frac{\alpha_{m_n}M_1}{2(1-\rho)} &+ \frac{\left\langle g(z) - z, x_{m_{k+1}} - z\right\rangle}{1-\rho} \\
&+ \frac{\theta_{m_k}}{2\alpha_{m_k}(1-\rho)}(M_1 + M_2),
\end{aligned}$$

which implies that $\limsup_{k\to\infty}\|x_k - z\| \le 0$. Hence, $\{x_k\}$ converges strongly to z, where $z = P_\Gamma g(z)$. □

By setting $H_1 = H_2$ and $T = I = A$ in Algorithm 3.1, we obtain the following result as a corollary of Theorem 3.4.

COROLLARY 3.5. *Let* $\gamma > 0$, $l, \mu \in (0,1)$ *and* $x_0, x_1 \in H$ *be given arbitrary. Then calculate* x_{n+1} *as follows:*

Step 1. *Set* $w_n = x_n + \theta_n(x_n - x_{n-1})$ *and compute* $y_n = P_C(w_n - \lambda_n f(w_n))$, *where* λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying*

$$\lambda_n\|f(w_n) - f(y_n)\| \leq \mu\|w_n - y_n\|.$$

Step 2. *Compute* $x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)z_n$, *where* $z_n = y_n - \lambda_n(f(y_n) - f(w_n))$. *Assume that* $\sum\limits_{n=1}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $z = P_\Gamma g(z)$.

REMARK 3.6. Notice that our algorithm (Algorithm 3.1) is also of viscosity-type. The motivation for using the viscosity-type algorithm over the Halpern-type (which also converges strongly) stems from the fact that viscosity-type algorithms have higher rate of convergence than the Halpern-type. More so, it was established in [41] (see also [25, Remark 3.7]) that viscosity convergence theorems imply Halpern-type convergence theorems for weak contractions. In fact, setting $g(x) = u$ for all $x \in H$ in Algorithm 3.1, we obtain the following result as a corollary of Theorem 3.4.

COROLLARY 3.7. *Let* $\gamma > 0$, $l, \mu \in (0,1)$ *and* $x_0, x_1 \in H$ *be given arbitrary. Then calculate* x_{n+1} *as follows:*

Step 1. *Set* $u_n = x_n + \theta_n(x_n - x_{n-1})$ *and compute* $w_n = P_C(u_n - \tau_n A^*(I - T_\beta)Au_n)$ *and* $y_n = P_C(w_n - \lambda_n f(w_n))$, *where* T_β *is as defined in* Lemma *2.7 and* λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying* $\lambda_n\|f(w_n) - f(y_n)\| \leq \mu\|w_n - y_n\|$.

Step 2. *Compute* $x_{n+1} = \alpha_n u + (1-\alpha_n)z_n$, *where* $z_n = y_n - \lambda_n(f(y_n) - f(w_n))$.

Assume that $\sum\limits_{n=1}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $z = P_\Gamma u$.

4. NUMERICAL EXAMPLE

In this section, we give a numerical example of our algorithm in comparison with Algorithm (8) of Tian and Jiang [47] in an infinite dimensional Hilbert space. Let $H_1 = H_2 = L_2([0, 2\pi])$ be endowed with inner product $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)\mathrm{d}t \ \forall\, x, y \in L_2([0, 2\pi])$ and norm $||x|| := \left(\int_0^{2\pi} |x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \ \forall\, x, y \in L_2([0, 2\pi])$. Let $C = \{x \in L_2([0, 2\pi]) : \langle y, x\rangle \leq a\}$, where $y = \mathrm{e}^{2t}$ and $a = 3$. Then, $P_C(x) = \begin{cases} \frac{a-\langle y,x\rangle}{||y||^2_{L_2}} y + x, & \text{if } \langle y, x\rangle > a, \\ x, & \text{if } \langle y, x\rangle \leq a. \end{cases}$ Now, define the operator f :

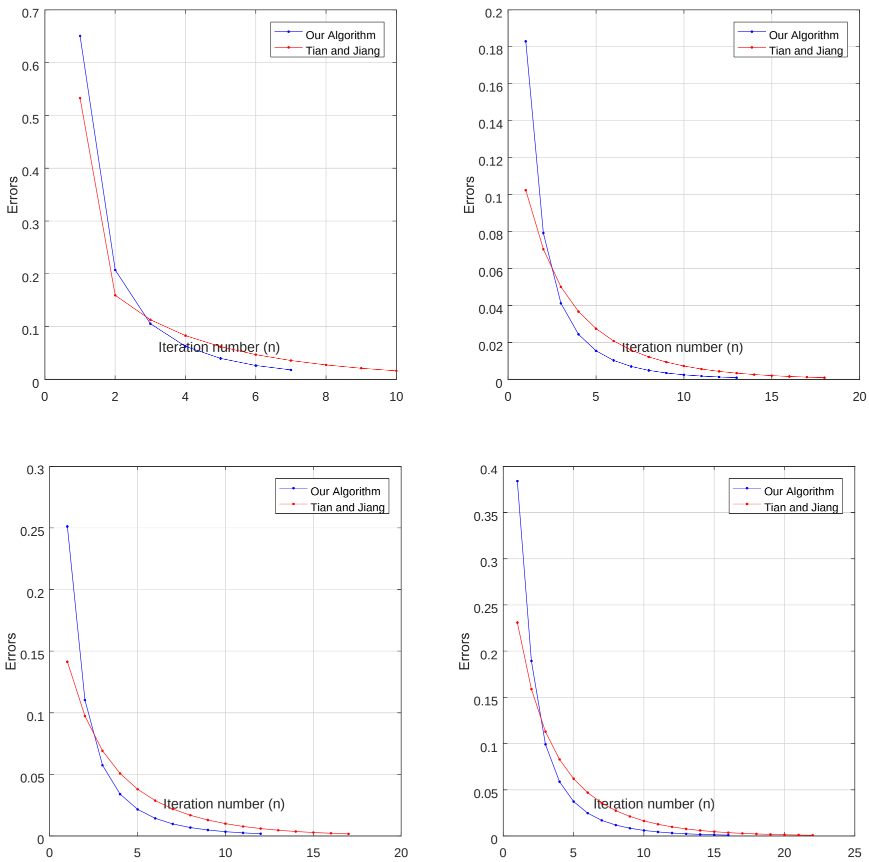

Fig. 4.1 – Errors *vs* Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

$L_2([0,2\pi]) \to L_2([0,2\pi])$ by

$$fx(t) = \int_0^{2\pi} \Big(x(t) - \Big(\frac{2ts\mathrm{e}^{t+s}}{\mathrm{e}\sqrt{\mathrm{e}^2-1}} \Big) \cos x(s) \Big) \mathrm{d}s + \frac{2t\mathrm{e}^{t}}{\mathrm{e}\sqrt{\mathrm{e}^2-1}},$$

$x \in L_2([0,2\pi])$, $t \in [0,1]$. Then f is 2-Lipschitz continuous and monotone on $L_2([0,2\pi])$ (see [23]). Let $A, g, T : L_2([0,2\pi]) \to L_2([0,2\pi])$ be defined by $Ax(t) = \frac{x(t)}{2}$, $gx(t) = \frac{x(t)}{3}$ and $Tx(t) = -4x(t)$. Then, A is a bounded linear operator with adjoint $A^*x(t) = \frac{x(t)}{2}$, g is a contraction with coefficient $\rho = \frac{1}{3}$ and T is $\frac{3}{5}$-strictly pseudocontractive. Thus, we can choose $\beta = \frac{3}{5}$, so that $T_\beta x(t) = -x(t)$. Take $\mu = \frac{1}{2} = l$, $\gamma = 1$, $\alpha_n = \frac{1}{n+1}$ and $\theta_n = \frac{n}{9n^2+1}$ for all $n \geq 1$, then the conditions in Theorem 3.4 are satisfied. Now, consider the following cases.

Case 1: Take $x_0(t) = t^3$ and $x_1(t) = 2t$.

Case 2: Take $x_0(t) = 2t$ and $x_1(t) = t^3$.
Case 3: Take $x_0(t) = \cos t$ and $x_1(t) = \sin t$.
Case 4: Take $x_0(t) = \sin t$ and $x_1(t) = \cos t$.
By using these cases (**Case 1-Case 4** above), we compared Algorithms 3.1 (studied in this paper) with Algorithm (8) of Tian and Jiang [47] as shown in the graphs above. The graphs show that our algorithm converges faster than Algorithms (8) of Tian and Jiang [47].

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