MATHEMATICA, 67 (90), N° 1, 2025, pp. 78–90

ON A DYNAMIC BOUNDARY CONDITION FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS

ANASS LAMAIZI, MAHMOUD EL AHMADI, OMAR DARHOUCHE, and MOSTAFA ALLAOUI

Abstract. The main objective of this paper is to study a weak solutions for the following parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \ \Omega, \ t > 0, \\ \sigma u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \ \partial\Omega, \ t > 0, \\ u(x;0) = u_0(x) & \text{in } \ \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$. By using the Galerkin approximation and a family of potential wells, we obtain the existence of global solution and finite time blow-up under some suitable conditions. On the other hand, the results for asymptotic behavior for certain solution with positive initial energy are also given.

MSC 2020. 35K55, 35A01, 35B44.

Key words. Nonlinear parabolic equations, dynamic boundary conditions.

1. INTRODUCTION AND MAIN RESULTS

In this work, we study the following parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \ \Omega, \ t > 0, \\ \sigma u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \ \partial\Omega, \ t > 0, \\ u(x;0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and $g(u)$ satisfies the conditions as follows:

$$\text{(C)} \quad \begin{cases} g \in C^1 \text{ and } g(0) = g'(0) = 0; \\ g(u) \text{ is monotone, concave for } u < 0 \text{ and convex for } u > 0; \\ (q+1)G(u) \leqslant ug(u), |ug(u)| \leqslant \mu|G(u)|; \end{cases}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s,$$

The authors thank the referee for the helpful comments and suggestions.

Corresponding author: Anass Lamaizi.

DOI: 10.24193/mathcluj.2025.1.07

$$\begin{cases} 2 < q+1 \leqslant \mu < \infty, & \text{if } n=2, \\ 2 < q+1 \leqslant \mu \leqslant \frac{2(n-1)}{n-2}, & \text{if } n \geqslant 3, \end{cases} \quad \text{and} \quad \begin{cases} 1 \leq \mu \leq p^{\partial}, & \text{if } p \neq n, \\ 1 \leq \mu < \infty, & \text{if } p = n, \end{cases}$$

with

$$p^{\partial} := \begin{cases} \frac{p(n-1)}{n-p}, & \text{if } 1 < p < n, \\ \infty, & \text{if } p \geq n. \end{cases}$$

The motivation of this paper contains several aspects, the first one is that in general, parabolic equations appear naturally in the modeling of many physical phenomena, such as the diffusion of heat (heat equation), the diffusion of matter (diffusion equation), the motion of viscous fluids (Navier-Stokes equation), wave propagation in a dissipative medium, etc. (see [14, 2, 4, 6, 8, 5, 24, 20]). The physical modeling of parabolic equations often involves the numerical resolution of these equations using methods such as the finite difference method, the finite element method or the finite volume method. These methods discretize the continuous equations on a spatial grid and solve the problem numerically to obtain an approximate solution that represents the physical behavior of the system under study (see [13, 12, 11, 22, 1, 3]).

Equations of the form

$$u_t - \Delta_p u = u^q,$$

are also called the non-Newtonian filtration equations, which are known as fast diffusive for $1 < p < 2$, and as slow diffusive for $p > 2$.

The second interesting aspect of this paper is the dynamical boundary condition imposed on the time lateral boundary relating the outer normal derivative to the time derivative

$$\sigma u_t + |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = g(u).$$

In general, the value of dynamic boundary conditions lies in their ability to capture realistic, dynamic phenomena that cannot be adequately represented by static conditions. They are particularly useful for modeling complex and varied physical processes, where boundary conditions may change in response to external events or dynamic interactions with the environment (see [9, 16]).

In the literature, there are several works dealing with nonlinear parabolic equations with dynamical boundary conditions (see [15, 18, 17]). For example, In [15] A.Lamaizi et al. considered the problem (1) in the particular case $g(u) = \lambda |u|^q u$. Under the following condition

$$(H) \qquad \frac{2n}{n+1} \leq p < +\infty, \quad p < 2+q \quad \text{and} \quad \begin{cases} 1 \leq q+2 \leq p^{\partial} \text{ if } p \neq n, \\ 1 \leq q+2 < \infty \text{ if } p = n, \end{cases}$$

and by using the Galerkin approximation, they established the existence of global solution and finite time blow-up under some suitable conditions.

In [2] J. V. Below et al. considered the nonlinear degenerate parabolic problem

$$\begin{cases} u_t - \Delta_p u = f(t,x,u) & \text{in } \Omega,\ t>0, \\ \sigma u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ t>0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

and they demonstrated the principles of weak comparison under a generalized one-sided Lipschitz condition imposed on a given f. They also compared solutions under different boundary conditions, namely Dirichlet versus dynamic boundary conditions.

In [17] Kun Li and Bo You studied a parabolic problem with dynamic flux boundary conditions of the following form

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u + f(u) = g(x,t) & \text{in } \Omega,\ t>0, \\ u_t + |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + f(u) = 0 & \text{on } \partial\Omega,\ t>0, \\ u(x;\tau) = u_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

They proved the existence of the uniform attractor in $L^2(\overline{\Omega}, \mathrm{d}\rho)$ for non-autonomous p-Laplacian evolution equations subject to nonlinear dynamical boundary conditions using Sobolev's compactness embedding theory, and the existence of the uniform attractor in $W^{1,p}(\Omega) \cap L^q(\Omega) \times L^q(\partial\Omega)$ by a priori asymptotic estimation.

Throughout this work, we designate the Lebesgue space $L^p(\Omega)$ by :

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^p \mathrm{d}x < +\infty \right\},$$

equipped with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

For $p = \infty$, we denote

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that ess-}\sup_\Omega |u| < +\infty \right\},$$

endowed with the norm

$$\text{ess-}\sup_\Omega |u| = \inf\{C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. } \Omega\}.$$

The scalar product of $L^2(\Omega)$ will be denoted by $\langle , \rangle$ and the scalar product of $L^2(\partial\Omega, \rho)$ will be denoted by $\langle , \rangle_0$:

$$\langle u, v \rangle = \int_\Omega uv \ \mathrm{d}x, \quad \langle u, v \rangle_0 = \oint_{\partial\Omega} uv \ \mathrm{d}\rho,$$

where $d\rho$ denotes the restriction to $\partial\Omega$.

Moreover, we denote the usual Sobolev space on Ω

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega) \right\},$$

equipped by the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

or to the equivalent norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty.$$

LEMMA 1.1 (See [2]). *The trace operator* $u : W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ *is continuous if and only if*

$$\begin{cases} 1 \leq q \leq p^\partial, & \textit{if } p \neq n, \\ 1 \leq q < \infty, & \textit{if } p = n. \end{cases}$$

Define the space

$$\mathcal{X}^p = L^p(\Omega) \times L^p(\partial\Omega, \rho), \text{ for } 1 \leq p \leq \infty,$$

endowed with the norm

$$\|U\|_{\mathcal{X}^p} := \left(\|u\|_p^p + \sigma\|\varphi\|_{p,\partial\Omega}^p\right)^{1/p},$$

for $U = (u, \varphi) \in \mathcal{X}^p$ and $\sigma > 0$.

In particular for $p = 2$, we denote

$$\langle u_t, \varphi\rangle_{\mathcal{X}^2} := \langle u_t, \varphi\rangle + \sigma\langle u_{t|\partial\Omega}, \varphi\rangle_0,$$

for any $\varphi \in W^{1,p}(\Omega)$, and

$$\|u_t\|_{\mathcal{X}^2}^2 := \|u_t\|_2^2 + \sigma\left\|u_{t|\partial\Omega}\right\|_{2,\partial\Omega}^2.$$

Let X be a Banach space and $T > 0$. Denote the following spaces:

$$C([0,T]; X) = \{u : [0,T] \longrightarrow X \text{ continue }\},$$

$L^p(0,T;X) = \{u : [0,T] \longrightarrow X$ is a measurable such that $\int_0^T \|u(t)\|_X^p \mathrm{d}t < \infty\}$, equipped with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \mathrm{d}t\right)^{\frac{1}{p}},$$

and $L^\infty(0,T;X) = \{u : [0,T] \longrightarrow X$ is a measurable such that $: \exists C > 0;$ $\|u(t)\|_X < C$ a.e.t$\}$, endowed with the norm

$$\|u\|_{L^\infty(0,T;X)} = \inf\{C > 0; \|u(t)\|_X < C \text{ a.e.t}\}.$$

Further, we define the energy functional $B(u)$ of problem (1) as follows:

$$B(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_{\partial\Omega} G(u)\mathrm{d}\rho.$$

In addition, we define the auxiliary functional

$$A(u) = \|u\|_{1,p}^p - \int_{\partial\Omega} ug(u)\mathrm{d}\rho.$$

Let

$$S = \left\{u \in W^{1,p}(\Omega) \mid A(u) > 0, B(u) < h\right\} \cup \{0\},$$

where $h = \inf_{u \in Y} B(u)$,

$$Y = \left\{u \in W^{1,p}(\Omega) \mid A(u) = 0, \|u\|_{1,p} \neq 0\right\},$$

and

$$U = \left\{u \in W^{1,p}(\Omega) \mid A(u) < 0, B(u) < h\right\}.$$

For $\theta > 0$ we define

$$A_\theta(u) = \theta \|u\|_{1,p}^p - \int_{\partial\Omega} u g(u) \mathrm{d}\rho,$$

$$h(\theta) = \inf_{u \in Y_\theta} B(u),$$

$$Y_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) = 0, \|u\|_{1,p} \neq 0\right\},$$

$$S_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) > 0, B(u) < h(\theta)\right\} \cup \{0\},$$

and

$$U_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) < 0, B(u) < h(\theta)\right\}.$$

Now, we present the main results of this paper.

Theorem 1.2 (**Global Existence**). *Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy* (C). *Suppose that $0 < B(u_0) < h$ and $A(u_0) > 0$. Then, problem* (1) *admits a global weak solution $u(t) \in L^\infty\left(0, \infty; W^{1,p}(\Omega)\right) \cap C\left([0,T]; \mathcal{X}^2\right)$ with $u_t(t) \in L^2\left(0, \infty; L^2(\Omega)\right)$ and $u(t) \in S$ for $0 \leqslant t < \infty$.*

Theorem 1.3 (**Finite Time Blow-up**). *Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy* (C). *Suppose that $B(u_0) < h$ and $A(u_0) < 0$. Then, the weak solution of problem* (1) *must blow up in finite time.*

Theorem 1.4 (**Asymptotic Behavior**). *Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy* (C). *Suppose also that $B(u_0) < h$ and $A(u_0) > 0$. Then, for all weak global solution $u(t)$ of problem* (1), *there exists a constant $\omega > 0$ such as:*

$$\|u(t)\|_{\mathcal{X}^2}^2 \leq \|u_0\|_{\mathcal{X}^2}^2 \mathrm{e}^{-\omega t}, \quad 0 \leq t < \infty. \tag{2}$$

2. PROOF OF MAIN RESULTS

2.1 PROOF OF THEOREM 1.2

Before giving the proof of first result, we give the definition of weak solution and state some lemmas which will be used later.

Definition 2.1. Let $T>0$. A function $u=u(x,t)\in L^{\infty}\left(0,\infty;W^{1,p}(\Omega)\right)\cap C\left([0,T];\mathcal{X}^2\right)$ with $u_t(t)\in L^2\left(0,\infty;L^2(\Omega)\right)$ is said to be a **weak solution** to the problem (1) in $\Omega\times[0,T)$, if $u(x,0)=u_0\in W^{1,p}(\Omega)$, and satisfies

$$\langle u_t,v\rangle_{\mathcal{X}^2}+\langle|u|^{p-2}u,v\rangle+\langle|\nabla u|^{p-2}\nabla u,\nabla v\rangle=\langle g(u),v\rangle_0,$$

for all $v\in W^{1,p}(\Omega)$ and a.e. $t\in(0,T]$. Moreover,

$$\int_0^t\|u_t\|^2_{\mathcal{X}^2}\ \mathrm{d}\tau+B(u)\leqslant B\left(u_0\right),\quad\forall t\in[0,T).\tag{3}$$

Lemma 2.2 ([21]). *Let $g(u)$ satisfy* (C). *Then,*

(1) $|G(u)|\leqslant M|u|^{\mu}$ *for some* $M>0$ *and all* $u\in\mathbb{R}$.
(2) $G(u)\geqslant N|u|^{q+1}$ *for some* $N>0$ *and* $|u|\geqslant1$.
(3) *The equality* $u\left(ug'(u)-g(u)\right)\geqslant0$ *holds only for* $u=0$.

Corollary 2.3 ([21]). *Let $g(u)$ satisfy* (C). *Then,*

(1) $|ug(u)|\leqslant\mu M|u|^{\mu},|g(u)|\leqslant\mu M|u|^{\mu-1}$ *for all* $u\in\mathbb{R}$.
(2) $ug(u)\geqslant(q+1)N|u|^{q+1}$ *for* $|u|\geqslant1$.

Lemma 2.4. *Let $\theta_1<\theta_2$ are the two roots of equation $h(\theta)=B(u)$. Then, the sign of $A_\theta(u)$ does not change for $\theta_1<\theta<\theta_2$, provided $0<B(u)<h$ for some $u\in W^{1,p}(\Omega)$.*

Proof. If it is false, thus there exist a $\theta_0\in(\theta_1,\theta_2)$ such as $A_{\theta_0}(u)=0$. By $B(u)>0$, we have $\|u\|_{1,p}\neq0$, consequently $u\in Y_{\theta_0}$. Then, $B(u)\geq h(\theta_0)$, which contradicts

$$B(u)=h\left(\theta_1\right)=h\left(\theta_2\right)<h(\theta_0).$$

□

Lemma 2.5. *Let $g(u)$ satisfy* (C), *$u_0(x)\in W^{1,p}(\Omega),0<e<h$ and $\theta_1<\theta_2$ be the two roots of equation $h(\theta)=e$. Suppose that $A\left(u_0\right)>0$, thus all weak solutions $u(t)$ of problem* (1) *with $B\left(u_0\right)=e$ belong to S_θ for $\theta_1<\theta<\theta_2$ and $0\leq t<T$.*

Proof. By $B\left(u_0\right)=e,A\left(u_0\right)>0$ and Lemma 2.4 we obtain $A_\theta\left(u_0\right)>0$ and $B\left(u_0\right)<h(\theta)$ i.e. $u_0(x)\in S_\theta$ for $\theta_1<\theta<\theta_2$.

Let $u(t)$ be any weak solution of problem (1) with $A\left(u_0\right)>0$ and $B\left(u_0\right)=e$, and T be the maximal existence time of $u(t)$. Next, we show that $u(t)\in S_\theta$ for $\theta_1<\theta<\theta_2$ and $0<t<T$. If it is false, so it must exist a $\theta_0\in(\theta_1,\theta_2)$ and $t_0\in(0,T)$ such as

$$A_{\theta_0}\left(u\left(t_0\right)\right)=0,\left\|u\left(t_0\right)\right\|_{1,p}\neq0\text{ or }B\left(u\left(t_0\right)\right)=h\left(\theta_0\right).$$

From (3), it follows that

$$\int_0^t\|u_\tau\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau+B(u)\leq B\left(u_0\right)<h(\theta),\quad\theta_1<\theta<\theta_2,\ 0\leq t<T.\tag{4}$$

As a result $B\left(u\left(t_0\right)\right)\neq h\left(\theta_0\right)$. If $A_{\theta_0}\left(u\left(t_0\right)\right)=0,\left\|u\left(t_0\right)\right\|_{1,p}\neq0$, thus the definition of $h(\theta)$ means that $B\left(u\left(t_0\right)\right)\geq h\left(\theta_0\right)$, which contradicts (4). □

Proof of Theorem 1.2. The idea of proof is classical, for more information see [7, 14, 23]. Let $w_j(x)$ be a system of base functions in $W^{1,p}(\Omega)$. Define the approximate solutions $u_m(x,t)$ of problem (1)

$$u_m(x,t)=\sum_{j=1}^{m}\Phi_{jm}(t)w_j(x), \quad m=1,2,\dots$$

verifying

$$\begin{aligned}&\langle u_{mt},w_s\rangle_{\mathcal{X}^2}+\langle|u_m|^{p-2}u_m,w_s\rangle+\langle|\nabla u_m|^{p-2}\nabla u_m,\nabla w_s\rangle=\langle g\left(u_m\right),w_s\rangle_0,\\ &s=1,2,\dots,m\end{aligned}\tag{5}$$

$$u_m(x,0)=\sum_{j=1}^{m}a_{jm}w_j(x)\to u_0(x) \quad \text{in } W^{1,p}(\Omega)\tag{6}$$

and

$$\int_0^t\|u_{mt}\|_{\mathcal{X}^2}^2\ \mathrm{d}\tau+B(u_m)\leqslant B\left(u_0\right) \quad \forall t\in[0,T).$$

Consequently

$$\int_0^t\|u_{mt}\|_{\mathcal{X}^2}^2\ \mathrm{d}\tau+B(u_m)\leqslant B\left(u_0\right)<h, \quad \forall t\in[0,T),\tag{7}$$

and $u_m\in S$ for $0\leqslant t<\infty$ (see the proof of Lemma 2.5).
Combining (7) and

$$\begin{aligned}B\left(u_m\right)=\frac{1}{p}\|u_m\|_{1,p}^p-\int_{\partial\Omega}G\left(u_m\right)\mathrm{d}\rho&\geqslant\frac{1}{p}\|u_m\|_{1,p}^p-\frac{1}{q+1}\int_{\partial\Omega}u_m g\left(u_m\right)\mathrm{d}\rho\\&=\left(\frac{1}{p}-\frac{1}{q+1}\right)\|u_m\|_{1,p}^p+\frac{1}{q+1}A\left(u_m\right)\\&\geqslant\frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p,\end{aligned}$$

we obtain

$$\int_0^t\|u_{mt}\|_{\mathcal{X}^2}^2\ \mathrm{d}\tau+\frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p<h, \quad 0\leqslant t<\infty.\tag{8}$$

From (8), we get

$$\|u_m\|_{1,p}^p<\frac{p(q+1)}{q-p+1}h, \quad 0\leqslant t<\infty,\tag{9}$$

$$\||u_m|^{p-2}u_m\|_s^s=\|u_m\|_p^p<\frac{p(q+1)}{q-p+1}h, \quad s=\frac{p}{p-1},\ 0\leqslant t<\infty,$$

$$\|u_m\|_{\mu,\partial\Omega}\leqslant C_*\|u_m\|_{1,p}<C_*\left(\frac{p(q+1)}{q-p+1}h\right)^{\frac{1}{p}}, \quad 0\leqslant t<\infty,\tag{10}$$

$$
\begin{aligned}
\|g(u_m)\|^r_{r,\partial\Omega} &\leqslant \int_{\partial\Omega} \left(\mu M\,|u_m|^{\mu-1}\right)^r \,\mathrm{d}\rho \\
&= (\mu M)^r\, \|u_m\|^{\mu}_{\mu,\partial\Omega} \\
&\leqslant (\mu M)^r C_*^{\mu} \left(\frac{p(q+1)}{q-p+1}h\right)^{\frac{\mu}{p}}, \quad r=\frac{\mu}{\mu-1}, \quad 0\leqslant t<\infty,
\end{aligned} \tag{11}
$$

where C_* is the embedding constant form $W^{1,p}(\Omega)$ into $L^{\mu}(\partial\Omega)$.

Furthermore

$$
\int_0^t \|u_{mt}\|^2_{\mathcal{X}^2}\,\mathrm{d}\tau < h, \quad 0\leqslant t<\infty. \tag{12}
$$

Therefore, there exist u, ϕ and a subsequence $\{u_v\}$ of $\{u_m\}$ such as:

$u_v \to u$ in $L^{\infty}\left(0,\infty; W^{1,p}(\Omega)\right)$ weakly star,

$u_{vt} \to u_t$ in $L^{2}\left(0,\infty; L^{2}(\Omega)\right)$ weakly ,

$|u_v|^{p-2}u_v \to |u|^{p-2}u$ in $L^{\infty}\left(0,\infty; L^{s}(\Omega)\right)$ weakly star,

$g(u_v) \to \phi$ in $L^{\infty}\left(0,\infty; L^{r}(\partial\Omega)\right)$ weakly star, and a.e. in $\partial\Omega\times[0,\infty)$.

Consequently, from Lemma 1.3 in [19], we deduce $\phi = g(u)$. In (5) for fixed s letting $m=v\to\infty$, we have

$$
\langle u_t, w_s\rangle_{\mathcal{X}^2} + \langle |u|^{p-2}u, w_s\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla w_s\rangle = \langle g(u), w_s\rangle_0, \quad \forall s,
$$

and

$$
\langle u_t, v\rangle_{\mathcal{X}^2} + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g(u), v\rangle_0, \ \forall v\in W^{1,p}(\Omega).
$$

By (6), we obtain $u(x,0)=u_0(x)$ in $W^{1,p}(\Omega)$. Then $u(t)$ is a global weak solution of problem (1). Finally, by applying Lemma 2.5 we deduce that the solution $u(t)\in S$.

2.2 PROOF OF THEOREM 1.3

To prove Theorem 1.3, we need the following auxiliary lemmas.

LEMMA 2.6. *Let $g(u)$ satisfy* (C) *and $A_\theta(u)<0$. Then, $\|u\|_{1,p}>z(\theta)$. In particular, if $A(u)<0$, then $\|u\|_{1,p}>z(1)$. Where*

$$
z(\theta) = \left(\frac{\theta}{aC_*^{\mu}}\right)^{1/(\mu-2)}
$$

and

$$
a = \sup \frac{ug(u)}{|u|^{\mu}}.
$$

Proof. $A_\delta(u)<0$ gives

$$
\theta\|u\|^2_{1,p} < \int_{\partial\Omega} ug(u)\mathrm{d}\rho \leqslant a\|u\|^{\mu}_{\mu,\partial\Omega} \leqslant aC_*^{\mu}\|u\|^{\mu-2}_{1,p}\|u\|^2_{1,p}, \tag{13}
$$

then $\|u\|_{1,p}>z(\theta)$. □

LEMMA 2.7. *Let $g(u)$ satisfy $(C), u_0(x) \in W^{1,p}(\Omega)$. Suppose that $0 < e < h$, $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = e$. Then, all weak solutions of problem (1) with $B(u_0) = e$ belong to U_θ for $\theta_1 < \theta < \theta_2$, provided $A(u_0) < 0$.*

Proof. Let $u(t)$ be any solution of problem (1) with $B(u_0) = e$, $A(u_0) < 0$ and T be the existence time of $u(t)$. First from $B(u_0) = e$, $A(u_0) < 0$ and Lemma 2.4 we can deduce $A_\theta(u_0) < 0$ and $B(u_0) < h(\theta)$, i.e. $u_0(x) \in U_\theta$ for $\theta_1 < \theta < \theta_2$.

Next, we prove $u(t) \in U_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 < t < T$. If it is false, let $t_0 \in (0, T)$ be the first time such that $u(t) \in U_\theta$ for $0 \leqslant t < t_0$ and $u(t_0) \in \partial U_\theta$, i.e. $A_\theta(u(t_0)) = 0$ or $B(u(t_0)) = h(\theta)$ for some $\theta \in (\theta_1, \theta_2)$. So (4) implies $B(u(t_0)) = h(\theta)$ is impossible. If $A_\theta(u(t_0)) = 0$, thus $A_\theta(u(t)) < 0$ for $0 < t < t_0$ and Lemma 2.6 yield $\|u(t)\|_{1,p} > z(\theta)$ and $\|u(t_0)\|_{1,p} \geqslant z(\theta)$. Therefore by the definition of $h(\theta)$ we have $B(u(t_0)) \geqslant h(\theta)$ which contradicts (4). □

Proof of Theorem 1.3. Let $u(t)$ be any solution of problem (1) with $B(u_0) < h$ and $A(u_0) < 0$.

We consider the auxiliary function

$$\varphi_1(t) = \int_0^t \|u\|_{\mathcal{X}^2}^2 \, \mathrm{d}\tau.$$

A direct calculation gives

$$\dot{\varphi}_1(t) = \|u\|_{\mathcal{X}^2}^2,$$

and

$$\ddot{\varphi}_1(t) = 2\langle u_t, u\rangle_{\mathcal{X}^2} = 2\left(\langle g(u), u\rangle_0 - \|u\|_{1,p}^p\right) = -2A(u). \tag{14}$$

By (14), (3) and

$$\int_{\partial\Omega} ug(u)\mathrm{d}\rho \geqslant (q+1)\int_{\partial\Omega} G(u)\mathrm{d}\rho$$

we can deduce

$$\begin{aligned}\ddot{\varphi}_1(t) &\geqslant 2(q+1)\int_0^t \|u_t\|_{\mathcal{X}^2}^2 \, \mathrm{d}\tau + (q-1)\|u\|_{1,p}^2 - 2(q+1)B(u_0)\\ &\geqslant 2(q+1)\int_0^t \|u_t\|_{\mathcal{X}^2}^2 \, \mathrm{d}\tau + (q-1)\dot{\varphi}_1(t) - 2(q+1)B(u_0),\end{aligned}$$

and

$$\begin{aligned}\varphi_1\ddot{\varphi}_1 - \frac{q+1}{2}(\dot{\varphi}_1)^2 \geqslant& 2(q+1)\Big[\int_0^t \|u\|_{\mathcal{X}^2}^2 \, \mathrm{d}\tau \int_0^t \|u_t\|_{\mathcal{X}^2}^2 \, \mathrm{d}\tau\\ &- \left(\int_0^t \langle u, u_t\rangle_{\mathcal{X}^2}\mathrm{d}\tau\right)^2\Big]\\ &+ (q-1)\varphi_1\dot{\varphi}_1 - (q+1)\|u_0\|_{\mathcal{X}^2}^2\dot{\varphi}_1\\ &- 2(q+1)B(u_0)\varphi_1 + \frac{q+1}{2}\|u_0\|_{\mathcal{X}^2}^2.\end{aligned}$$

Making use of the Hölder inequality, we get

$$\begin{aligned}\varphi_1\ddot{\varphi}_1-\frac{q+1}{2}\left(\dot{\varphi}_1\right)^2 \geqslant&(q-1)\varphi_1\dot{\varphi}_1-(q+1)\left\|u_0\right\|_{\mathcal{X}^2}^2\dot{\varphi}_1\\ &-2(q+1)B\left(u_0\right)\varphi_1+\frac{q+1}{2}\left\|u_0\right\|_{\mathcal{X}^2}^2 .\end{aligned} \tag{15}$$

(1) If $B\left(u_0\right)\leqslant 0$, then

$$\varphi_1\ddot{\varphi}_1-\frac{q+1}{2}\left(\dot{\varphi}_1\right)^2 \geqslant (q-1)\varphi_1\dot{\varphi}_1-(q+1)\left\|u_0\right\|_{\mathcal{X}^2}^2\dot{\varphi}_1.$$

The following task is to claim that $A(u)<0$ for $t>0$. Arguing by contradiction, we assume the existence of a $t_0>0$ so that $A\left(u\left(t_0\right)\right)=0$.

Next, let $t_0>0$ be the first time such as $A(u(t))=0$, thus $A(u(t))<0$ for $0\leqslant t<t_0$. From Lemma 2.6 we obtain $\|u\|_{1,p}>z(1)$ for $0<t<t_0$. Consequently, we obtain $\left\|u\left(t_0\right)\right\|_{1,p}\geqslant z(1)$ and $B\left(u\left(t_0\right)\right)\geqslant h$ which contradicts (3). Then, from (14) we have $\ddot{\varphi}_1(t)>0$ for $t>0$. By this and $\dot{\varphi}_1(0)=\left\|u_0\right\|_{\mathcal{X}^2}^2\geqslant 0$, then there exists a $t_0\geqslant 0$ such as $\dot{\varphi}_1\left(t_0\right)>0$ and

$$\varphi_1(t)\geqslant\dot{\varphi}_1\left(t_0\right)\left(t-t_0\right)+\varphi_1\left(t_0\right)\geqslant\dot{\varphi}_1\left(t_0\right)\left(t-t_0\right),\quad t\geqslant t_0.$$

Therefore for sufficiently large t we can deduce

$$(q-1)\varphi_1>(q+1)\left\|u_0\right\|_{\mathcal{X}^2}^2,$$

and

$$\varphi_1\left(t\right)\ddot{\varphi}_1\left(t\right)-\frac{q+1}{2}\left(\dot{\varphi}_1\left(t\right)\right)^2>0. \tag{16}$$

Since, for $t>0$

$$\left(\varphi_1^{-\beta}\left(t\right)\right)''=-\frac{\beta}{\varphi_1^{\beta+2}\left(t\right)}\left(\varphi_1\left(t\right)\ddot{\varphi}_1\left(t\right)-(\beta+1)\dot{\varphi}_1\left(t\right)^2\right),$$

we see that for $\beta=\frac{q-1}{2}$ we have $\left(\varphi_1^{-\beta}\left(t\right)\right)''<0$. Therefore $\varphi_1^{-\beta}\left(t\right)$ is concave for sufficiently large t, and there exists a finite time T for which $\varphi_1^{-\beta}\left(t\right)\to 0$. In other words,

$$\lim_{t\to T^-}\varphi_1(t)=+\infty.$$

(2) If $0<B\left(u_0\right)<h$, thus by Lemma 2.7, we have $u(t)\in U_\theta$ for $1<\theta<\theta_2$ and $t>0$, where θ_2 is the larger root of equation $h(\theta)=B(u_0)$. Therefore $A_\theta(u)<0$ and from Lemma 2.6 we deduce $\|u\|_{1,p}>z(\theta)$ for $1<\theta<\theta_2$ and $t>0$. Then, we have $A_{\theta_2}(u)\leqslant 0$ and $\|u\|_{1,p}\geqslant z\left(\theta_2\right)$

for $t > 0$. Thus (14) gives

$$\ddot{\varphi}_1(t) = -2A(u) = 2(\theta_2 - 1)\|u\|_{1,p}^p - 2A_{\theta_2}(u) \geqslant 2(\theta_2 - 1) z^p(\theta_2) > 0, \quad t \geqslant 0,$$
$$\dot{\varphi}_1(t) \geqslant 2(\theta_2 - 1) z^p(\theta_2) t + \dot{\varphi}_1(0) \geqslant 2(\theta_p - 1) z^p(\theta_2) t, \quad t \geqslant 0,$$
$$\varphi_1(t) \geqslant (\theta_2 - 1) z^p(\theta_2) t^2 + \varphi_1(0) = (\theta_2 - 1) z^p(\theta_2) t^2, \quad t \geqslant 0.$$

Therefore for sufficiently large t we get

$$\frac{1}{2}(q-1)\varphi_1(t) > (q+1)\|u_0\|_{\mathcal{X}^2}^2,$$
$$\frac{1}{2}(q-1)\dot{\varphi}_1(t) > 2(q+1)B(u_0).$$

Hence from (15) we again obtain (16) for sufficiently large t. The remainder of the proof is similar to that in the proof of (i).

2.1. **Proof of Theorem 1.4.** By Theorem 1.2, we know that there exists a global weak solution to problem (1). Let $u(t)$ be any global weak solution of problem (1) with $B(u_0) < h$ and $A(u_0) > 0$. Consequently,

(17)
$$\langle u_t, v\rangle_{\mathcal{X}^2} + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g(u), v\rangle_0, \forall v \in W^{1,p}(\Omega), t \in (0,T).$$

Multiplying (17) by any $h(t) \in C[0,\infty)$, we have

$$\langle u_t, h(t)v\rangle_{\mathcal{X}^2} + \langle |u|^{p-2}u, h(t)v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla(h(t)v)\rangle = \langle g(u), h(t)v\rangle_0,$$

$\forall v \in W^{1,p}(\Omega)$, and $t \in (0,T)$, consequently

$$\begin{aligned} &\langle u_t, \varphi\rangle_{\mathcal{X}^2} + \langle |u|^{p-2}u, \varphi\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\ &= \langle g(u), \varphi\rangle_0, \forall \varphi \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right), \end{aligned} \tag{18}$$

and $t \in (0,T)$.

Setting $\varphi = u$, (18) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\mathcal{X}^2}^2 + A(u) = 0, \quad 0 \le t < \infty. \tag{19}$$

By $0 < B(u_0) < h, A(u_0) > 0$ and Lemma 2.5, we obtain $u(t) \in S_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 \le t < \infty$, where $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = B(u_0)$. Consequently, we get $A_\theta(u) \ge 0$ for $\theta_1 < \theta < \theta_2$ and $A_{\theta_1}(u) \ge 0$ for $0 \le t < \infty$. Then, (19) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\mathcal{X}^2}^2 + (1-\theta_1)\|u\|_{1,p}^p + A_{\theta_1}(u) = 0, \quad 0 \le t < \infty,$$

accordingly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\mathcal{X}^2}^2 + (1-\theta_1)\|u\|_{\mathcal{X}^2}^2 \le 0, \quad 0 \le t < \infty.$$

Finally, Gronwall's inequality leads to

$$\|u\|_{\mathcal{X}^2}^2 \le \|u_0\|_{\mathcal{X}^2}^2 \mathrm{e}^{-2(1-\theta_1)t}, \quad 0 \le t < \infty.$$

The proof of the theorem is now finished.

REFERENCES

[1] R.B. Anibal and T. Anas, *Nonlinear balance for reaction diffusion equations under nonlinear boundary conditions: Dissipativity and blow-up*, J. Differential Equations, **169** (2001), 332–372.

[2] V.J. Below, M. Cuesta and G.P. Mailly, *Qualitative results for parabolic equations involving the p-Laplacian under dynamical boundary conditions*, North-West. Eur. J. Math., **4** (2018), 59–97.

[3] S. Bergman and M. Schiffer, *Kernel functions and elliptic differential equations in mathematical physics*, Courier Corporation, 2005.

[4] A. Bermudez, R. Rodriguez and D. Santamarina, *A finite element solution of an added mass formulation for coupled fluid-solid vibrations*, Numer. Math., **87** (2000), 201–227.

[5] J.I. Diaz, G. Hetzer and L.Tello, *An energy balance climate model with hysteresis*, Nonlinear Anal., **64** (2006), 2053–2074.

[6] J.I. Diaz and L. Tello, *On a climate model with a dynamic nonlinear diffusive boundary condition*, Discrete Contin. Dyn. Syst. Ser. S, **1** (2008), 253–262.

[7] A.H. Erhardt, *The stability of parabolic problems with nonstandard $p(x,t)$-growth*, Mathematics, **5** (2017), 1–14.

[8] E. Feireisl, *Mathematical analysis of fluids in motion: From well-posedness to model reduction*, Rev. Mat. Complut., **26** (2013), 299–340.

[9] J. Filo, *Difusivity versus absorption through the boundary*, J. Differential Equations, **99** (1992), 281–305.

[10] Y. Jingxue, J. Chunhua and Y. Ying, *Critical exponents of evolutionary p-Laplacian with interior and boundary sources*, Acta Math. Sci., **31** (2011), 778–790.

[11] A. Khan, T. Abdeljawad, J.G. Aguilar and H. Khan, *Dynamical study of fractional order mutualism parasitism food web module*, Chaos Solitons Fractals, **134** (2020), 1–15.

[12] A. Khan, J.G. Aguilar, T. Abdeljawad and H. Khan, *Stability and numerical simulation of a fractional order plantnectar-pollinator model* , Alexandria Engineering Journal, **59** (2019), 49–59.

[13] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, 2006.

[14] A. Lamaizi, A. Zerouali, O. Chakrone and B. Karim, *Global existence and blow-up of solutions for parabolic equations involving the Laplacian under nonlinear boundary conditions*, Turkish J. Math., **45** (2021), 2406–2418.

[15] A. Lamaizi, A. Zerouali, O. Chakrone and B. Karim, *Existence and blow up of solutions for a class of parabolic problems with dynamical boundary condition*, Stat. Optim. Inf. Comput., **11** (2023), 44–54.

[16] H.A. Levine and L.E. Payne, *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for porous medium equation backward in time*, J. Differential Equations, **16** (1974), 319–334.

[17] K. Li and B. You, *Uniform attractors for the non-autonomous p-Laplacian equations with dynamic flux boundary conditions*, Bound. Value Probl., **2013** (2013), 1–24.

[18] Y. Li and C. Xie, *Blow-up for p-Laplacian parabolic equations*, Electron. J. Differential Equations, **20** (2003), 1–12.

[19] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, 1969.

[20] K.S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley-Interscience, 1993.

[21] L.E. Payne and D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math., **22** (1975), 273–303.

[22] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, Fractional Differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.
[23] L.Yacheng, X. Runzhang and Y. Tao, *Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations*, Nonlinear Anal., **68** (2008), 3332–3348.
[24] Y. Zhou, J. Wang and L. Zhang, *Basic theory of fractional differential equations*, World Scientific, 2016.

Received February 11, 2024
Accepted May 26, 2024

Mohammed First University
Department of Mathematics
Oujda, Morocco
E-mail: lamaizi.anass@ump.ac.ma
https://orcid.org/0000-0003-4469-6460

E-mail: elahmadi.mahmoud@ump.ac.ma
https://orcid.org/0000-0002-6493-3219

Abdelmalek Essaadi University
Department of Mathematics, FSTH
Tetouan, Morocco
E-mail: o.darhouche@uae.ac.ma
https://orcid.org/0000-0003-2604-1700

E-mail: m.allaoui@uae.ac.ma
https://orcid.org/0000-0001-6732-9327