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A SEQUENCE OF POLYNOMIAL PAIRS ASSIGNED TO A GRAPH

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Abstract. In this note, we introduce a new degree-based descriptive parameter, namely, the degree polynomial-pair (DPP), for the edges of a simple graph. This notion leads to a concept, namely, the degree polynomial-pair sequence (DPPS) in graphs. We show that the DPPS of a graph gives more information about the graph than its degree polynomial sequence does, but it still does not identify the graph uniquely. We obtain the DPPS for some well-known graphs. Also we prove a theorem in which a necessary condition for the graphic realizability of a sequence of polynomial pairs is given. Several open problems concerning these subjects are given as well.

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Key words. Degree polynomial-pair, degree polynomial-pair sequence, degree polynomial, degree polynomial sequence, graphic realization.

1. INTRODUCTION

The degree sequence of a graph is an important invariant of the graph. In recent decades, this invariant and its applications in various branches of mathematics, network, cryptography, and many other sciences have been investigated by several mathematicians. For instance, see [2, 3, 5, 6, 7, 10, 11, 12].

The degree sequence of a graph is not the only descriptive parameter on the degrees of the vertices of that graph. In [1], Amanatidis, Green, and Mihail have introduced another parameter named the joint-degree matrix which is stronger.

Recently, the author has introduced another descriptive parameter on the degrees of a simple graph, named the degree polynomial sequence of the graph. This parameter is derived from a concept called the degree polynomial for the vertices of the graph. Some properties of this parameter and its behavior under graph operations have been studied (see [8] and [9]). The degree polynomial sequence gives more information about a graph than a degree sequence and also a degree-joint matrix does.

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In this note, first we introduce a new degree-based descriptive parameter called the degree polynomial-pair (DPP) for the edges of a simple graph. This notion leads to a concept called the degree polynomial-pair sequence (DPPS) in graphs. Then we show that the DPPS of a graph gives more information about the graph than its degree polynomial sequence does, but it still does not identify the graph uniquely. Also we obtain the DPPS for some well-known graphs. Finally, we prove a theorem in which a necessary condition for the graphic realizability of a sequence of polynomial pairs is given. Several open problems concerning these subjects are given as well.

2. PRELIMINARIES

In the sequel, we use [4] for the basic terminologies and notations in graph theory. Also all graphs are finite and simple.

In a graph G, for two vertices $u, v \in V(G)$, we write $u \sim v$, whenever u is adjacent to v.

Let G is a graph with n vertices. A non-increasing sequence of nonnegative integers $q = (d_1, \ldots, d_n)$ is said the degree sequence of G, whenever there exists an ordering $v_1, \ldots, v_n$ of the vertices of G, such that d_i is the degree of v_i, for $1 \leq i \leq n$. A sequence $q = (d_1, \ldots, d_n)$ of integers is graphic realizable, if there exists a graph G such that q be the degree sequence of G. Since adding a finite number of isolated vertices to a graph and deleting a finite number of such vertices from a nonempty graph makes no change in the degrees of the other vertices, one can consider only the case in which each $d_i, 1 \leq i \leq n$, is positive.

The degree polynomial of a graph G, denoted by $\mathrm{dp}(G)$, is the polynomial $\sum_i t_i x^i$ in $\mathbb{R}[x]$ in which t_i is the number of vertices of G, each of degree i (specially, t_0 is the number of isolated vertices of G). If Δ be the maximum degree of G, then $\mathrm{dp}(G)$ is of degree Δ.

For a polynomial $f(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{R}[x]$ with $a_n \neq 0$, The sum of a_i's for $1 \leq i \leq n$, is denoted by $\mathrm{sc}(f)$. Also $\mathrm{sec}(f)$ and $\mathrm{soc}(f)$ are used for the sum of a_i's for even i, and sum of a_i's for odd i, respectively. We define $\mathrm{sc}(0) = 0$ as well.

The total order $<_{\mathrm{pol}}$ on the set of all nonzero polynomials with coefficients in nonnegative integers is defined such that $<_{\mathrm{pol}}$ compares two distinct polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$ with nonnegative integer coefficients and with $a_n, b_m \neq 0$, as follows:

If $\mathrm{sc}(f) \neq \mathrm{sc}(g)$, then which one of f and g whose sum of coefficients is greater (as an integer), will be greater;

If $\mathrm{sc}(f) = sc(g)$, then supposing that $i_1 = \max\{i | \ a_i \neq 0 \ or \ b_i \neq 0\}$, if $a_{i_1} \neq b_{i_1}$, then whichever of f and g has greater coefficient in x^{i_1}, will be greater;

If $\mathrm{sc}(f) = \mathrm{sc}(g)$ and $a_{i_1} = b_{i_1}$, then supposing that $i_2 = \max\{i|\ i < i_1,\ a_i \neq 0\ \mathit{or}\ b_i \neq 0\}$, if $a_{i_2} \neq b_{i_2}$, then whichever of f and g has greater coefficient in x^{i_2}, will be greater;

Continue on.

Let G is a graph and v is a vertex of G. The degree polynomial of v denoted by $\mathrm{dp}(v)$, is a polynomial with nonnegative integer coefficients, in which the coefficient of x^i is the number of neighbors of v each of degree i; Especially, for an isolated vertex v, $\mathrm{dp}(v) = 0$.

Since adding a finite number of isolated vertices to a graph and deleting a finite number of such vertices from a nonempty simple graph makes no change in the degree polynomials of the other vertices, we will consider only the graphs which has no isolated vertices.

For a graph G of order n without any isolated vertex, a sequence $q = (f_1, f_2, \ldots, f_n)$ of polynomials is called the degree polynomial sequence (DPS) of G, whenever

(a) $f_1 \geq_{\mathrm{pol}} \cdots \geq_{\mathrm{pol}} f_n$,

(b) there exists an ordering $v_1, \ldots, v_n$ of the vertices of G, such that f_i is the degree polynomial of v_i, for $1 \leq i \leq n$.

We denote the degree polynomial sequence of G by $\mathrm{dps}(G)$.

For the definitions and notations above, see [8].

3. DEGREE POLYNOMIAL-PAIR AND DEGREE POLYNOMIAL-PAIR SEQUENCE

Before all, for convenience we introduce some notations.

Let Q is the set of all pairs (a, b) of nonzero polynomials with coefficients in nonnegative integers. We use the notation "$<_{\mathrm{polp}}$" for the total order R on Q for which $(a_1, b_1)R(a_2, b_2)$, whenever

$a_1 <_{\mathrm{pol}} a_2$, or

$a_1 = a_2$ and $b_1 <_{\mathrm{pol}} b_2$.

In fact, "$<_{\mathrm{polp}}$" is a lexicographic total order on Q.

Let t_i's, $1 \leq i \leq n$, are some elements in a set T. For positive integers $m_1, \ldots, m_n$, the notation $t_1^{<m_1>}, \ldots, t_n^{<m_n>}$ denotes the finite sequence

$$\underbrace{t_1, \ldots, t_1}_{m_1 \text{terms}}, \ldots, \underbrace{t_n, \ldots, t_n}_{m_n \text{terms}}.$$

If m_i is 1, we can ignore writing it, $1 \leq i \leq n$.

Let $q = (q_1, \ldots, q_m)$ is a finite sequence of polynomial pairs where for each $1 \leq i \leq m$, $q_i = (a_i, b_i)$ for nonzero polynomials a_i and b_i with coefficients in nonnegative integers. Let

$$P := \{a_i|\ 1 \leq i \leq m|\} \cup \{b_i|\ 1 \leq i \leq m\}.$$

Let the distinct elements of P in non-increasing ordering (with respect to $<_{\mathrm{pol}}$) are $p_1, \ldots, p_k$. We denote the sequence $p_1, \ldots, p_k$ by $\mathrm{p}(q)$. Also for $1 \leq j \leq k$, the numbers

$$|\{1 \leq i \leq m \mid \text{only one of the components } a_i \text{ and } b_i \text{ equals } p_j\}|$$

and

$$|\{1 \leq i \leq m \mid a_i \text{ and } b_i \text{ equal } p_j\}|$$

are denoted by $\mathrm{r}_j(q)$ and $\mathrm{s}_j(q)$, respectively.

Now we are ready to introduce our new notions.

Definition 3.1. Let G is a graph and $e = \{u, v\}$ is an edge of G. The degree polynomial pair (DPP) of e, denoted by $\mathrm{dpp}(e)$, is a pair (a, b) of nonzero polynomials with coefficients in nonnegative integers such that

if $\mathrm{dp}(u) \geq_{\mathrm{pol}} \mathrm{dp}(v)$, then $(a, b) = (\mathrm{dp}(u), \mathrm{dp}(v))$,

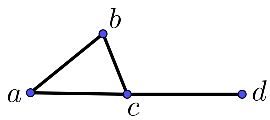
if $\mathrm{dp}(v) \geq_{\mathrm{pol}} \mathrm{dp}(u)$, then $(a, b) = (\mathrm{dp}(v), \mathrm{dp}(u))$.

Definition 3.2. For a graph G without any isolated vertex, a sequence $q = (q_1, \ldots, q_{|E(G)|})$ of polynomial pairs is called the degree polynomial-pair sequence (DPPS) of G, denoted by $\mathrm{dpps}(G)$, whenever

(a) $q_1 \geq_{\mathrm{polp}} \cdots \geq_{\mathrm{polp}} q_{|E(G)|}$,

(b) there exists an ordering $e_1, \ldots, e_{|E(G)|}$ of the edges of G, such that q_i is the degree polynomial-pair of e_i, for $1 \leq i \leq |E(G)|$.

Example 3.3. Consider the graph G with the following representation.



G

The DPPS of G is the sequence

$$(2x^2 + x, x^3 + x^2)^{<2>}, (2x^2 + x, x^3), (x^3 + x^2, x^3 + x^2).$$

4. MAIN RESULTS

Theorem 4.1. *If G is a graph without any isolated vertex, then* $\mathrm{dps}(G)$ *can be obtained from* $\mathrm{dpps}(G)$.

Proof. Let the DPPS of G be

$$q = ((a_1, b_1), \ldots, (a_{|E(G)|}, b_{|E(G)|})).$$

and let $\mathrm{p}(q)$ be $p_1, \ldots, p_k$. Put

$$V_j = \{v \in V(G) |\ \mathrm{dp}(v) = p_j\},\ 1 \leq j \leq k.$$

Since G has no isolated vertex, the set $\{V_j |\ 1 \leq j \leq k\}$ is a partition of $V(G)$. Consider an integer $1 \leq j \leq k$. Since for all $v \in V_j$, $\deg(v) = \mathrm{sc}(p_j)$, we have

$$|V_j|\mathrm{sc}(p_j) = \mathrm{r}_j(q) + 2\mathrm{s}_j(q).$$

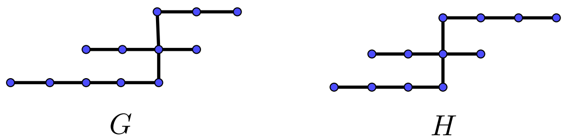
Thus

$$|V_j| = \frac{\mathrm{r}_j(q) + 2\mathrm{s}_j(q)}{\mathrm{sc}(p_j)}.$$

But the DPS of G is $p_1^{<|V_1|>}, \ldots, p_k^{<|V_k|>}$. $\square$

Upon Theorem 4.1, if two non-isomorphic graphs have the same DPPS, then they have the same DPS. But the converse of this matter is not established, as the following example shows.

Example 4.2. Consider the graphs G and H with the following representations.



The degree polynomial sequence of both G and H is

$$3x^2 + x, (x^4 + x^2)^{<2>}, x^4 + x, (2x^2)^{<2>}, (x^2 + x)^{<2>}, x^4, (x^2)^{<3>}$$

but the DPPS of G is

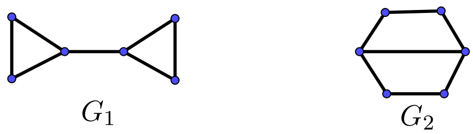
$$(3x^2 + x, x^4 + x^2)^{<2>}, (3x^2 + x, x^4 + x), (3x^2 + x, x^4), (x^4 + x^2, 2x^2),$$
$$(x^4 + x^2, x^2 + x), (x^4 + x, x^2), (2x^2, 2x^2), (2x^2, x^2 + x), (x^2 + x, x^2)^{<2>}$$

while the DPPS of H is

$$(3x^2 + x, x^4 + x^2)^{<2>}, (3x^2 + x, x^4 + x), (3x^2 + x, x^4), (x^4 + x^2, 2x^2)^{<2>},$$
$$(x^4 + x, x^2), (2x^2, x^2 + x)^{<2>}, (x^2 + x, x^2)^{<2>}.$$

From what we said above, it yields that the DPPS of a graph gives more information about the graph than its DPS does. But it still does not identify the graph uniquely, as the following example shows.

Example 4.3. Consider the graphs G_1 and G_2 with the following representations.



The graphs G_1 and G_2 are non-isomorphic, but the DPPS for both of them is

$$(x^3 + 2x^2, x^3 + 2x^2), (x^3 + 2x^2, x^3 + x^2)^{<4>}, (x^3 + x^2, x^3 + x^2)^{<2>}.$$

Now we calculate the DPPS of some well-known graphs.

Proposition 4.4. *A graph G with no isolated vertex is r-regular, if and only if each term of its* DPPS *is in the form (rx^r, rx^r).*

Proof. Is clear. □

Proposition 4.5. *Let G is a graph with no isolated vertex.*
(1) *If G is a complete graph, K_n, then its* DPPS *is*

$$((n-1)x^{n-1}, (n-1)x^{n-1})^{<\frac{n(n-1)}{2}>}.$$

(2) *If G is a cycle, C_n, then its* DPPS *is*

$$(2x^2, 2x^2)^{<n>}.$$

(3) *If G is a path, P_n, then*
if $n = 2$, its DPPS *is (x, x),*
if $n = 3$, its DPPS *is $(2x, x^2)^{<2>}$,*
if $n = 4$, its DPPS *is $(x^2 + x, x^2 + x), (x^2 + x, x^2)^{<2>}$,*
if $n = 5$, its DPPS *is $(2x^2, x^2 + x)^{<2>}, (x^2 + x, x^2)^{<2>}$,*
finally, if $n \geq 6$, its DPPS *is $(2x^2, 2x^2)^{<n-4>}, (2x^2, x^2 + x)^{<2>}, (x^2 + x, x^2)^{<2>}$.*
(4) *If G is a complete bipartite graph, $K_{r,s}$ where $r \geq s$, then its* DPPS *is*

$$(rx^s, sx^r)^{<rs>}.$$

Proof. Is clear. □

The following theorem give a necessary condition for the graphic realizability of a sequence of polynomial pairs.

Theorem 4.6. *Let $q = (q_1, \ldots, q_m)$ is a finite non-increasing (with respect to $<_{\mathrm{polp}}$) sequence of polynomial pairs $q_i = (a_i, b_i)$, $1 \leq i \leq m$ and let $\mathrm{p}(q) = (p_1, \ldots, p_k)$. If q is graphic realizable (is the* DPPS *of a graph), then*
(1) *$a_i \geq_{\mathrm{pol}} b_i$, for every $1 \leq i \leq m$,*
(2) *$\frac{\mathrm{r}_j(q)+2\mathrm{s}_j(q)}{\mathrm{sc}(p_j)}$ is an integer, for every $1 \leq j \leq k$,*
(3) *the sequence*

$$p_1^{<\frac{\mathrm{r}_1(q)+2\mathrm{s}_1(q)}{\mathrm{sc}(p_1)}>}, \ldots, p_k^{<\frac{\mathrm{r}_k(q)+2\mathrm{s}_k(q)}{\mathrm{sc}(p_k)}>}$$

of polynomials is graphic realizable (is the DPS *of a graph).*

Proof. By the definition of DPPS, (1) is clear.

Let the graph G is a realization of the sequence q. By Theorem 4.1 and its proof, the DPS of G is

$$p_1^{<\frac{\mathrm{r}_1(q)+2\mathrm{s}_1(q)}{\mathrm{sc}(p_1)}>}, \ldots, p_k^{<\frac{\mathrm{r}_k(q)+2\mathrm{s}_k(q)}{\mathrm{sc}(p_k)}>}.$$

Thus $\frac{\mathrm{r}_j(q)+2\mathrm{s}_j(q)}{\mathrm{sc}(p_j)}$ is an integer, for every $1 \leq j \leq k$, and the sequence

$$p_1^{<\frac{\mathrm{r}_1(q)+2\mathrm{s}_1(q)}{\mathrm{sc}(p_1)}>}, \ldots, p_k^{<\frac{\mathrm{r}_k(q)+2\mathrm{s}_k(q)}{\mathrm{sc}(p_k)}>}$$

of polynomials is realized by the graph G. Therefore (2) and (3) hold. □

5. SOME OPEN PROBLEMS

The following open problems can be raised:

(1) The characterization of all graphic realizable sequences of polynomial pairs.

(2) The characterization of all sequences of polynomial pairs which realize uniquely.

(3) The characterization of all realizable sequences of polynomial pairs which have at least one connected realization.

(4) The characterization of all realizable sequences of polynomial pairs whose all realizations are connected.

REFERENCES

[1] Y. Amanatidis, B. Green and M. Mihail, *Graphic realizations of joint-degree matrices*, 2015, 1–18, arXiv:1509.07076.

[2] E.A. Bertram, *Some applications of graph theory to finite groups*, Discrete Math., **44** (1989), 31–43.

[3] E.A. Bertram, *The effect of random edge removal on network degree sequence*, Electron. J. Combin., **19** (2012), 1–19.

[4] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, Macmillan, London, 1976.

[5] P. Erdös and T. Gallai, *Graphs with prescribed degree of vertices* (in Hungarian), Köz. Mat. Lapok, **11** (1960), 264–274.

[6] S.L. Hakimi, *On the realizability of a set of integers as degrees of the vertices of a graph*, SIAM J. Appl. Math., **10** (1962), 492–506.

[7] V. Havel, *A remark on the existence of finite graphs* (in Czech), Casopic Pest Mat., **80** (1955), 477–480.

[8] R. Jafarpour-Golzari, *Degree polynomial for vertices in a graph and its behavior under graph operations*, Comment. Math. Univ. Carolin., **63** (2022), 397–413.

[9] R. Jafarpour-Golzari, *On the degree polynomial of vertices in some classes of graphs*, Preprint, 2023.

[10] M. Ritchie, *Generation and analysis of networks with a prescribed degree sequence and subgraph family: higher-order structure matters*, J. Complex Netw., **5** (2011), 1–31.

[11] A. Tripathi ans H. Tyagi, *A simple criterion on degree sequence of graphs*, Discrete Math. Appl., **156** (2008), 3513–3517.

[12] I.E. Zrerovich and V.E. Zrerovich, *Contribution to the theory of graphic sequences*, Discrete Math., **105** (1992), 293–303.

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