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ON THE CROSSED POLYSQUARE VERSION OF HOMOTOPY COKERNELS

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Abstract. In this paper, we define a generalized notion of semidirect hyperproduct of polygroups and use that to introduce a pushout construction for crossed polymodules. Our results extend the classical results of crossed squares to crossed polysquares. One of the main tools in the study to polygroups is the fundamental relations. Additionally we study of the crossed polysquare version of homotopy cokernels.

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Key words. Polygroup, polysquare, homotopy, fundamental relation.

1. INTRODUCTION

We remind you that Yang-Baxter equations play a very important role in various fields of applied mathematics. We have already mentioned some of these fields. Among its solutions, which are made in the name of braidings, the following can be mentioned:

(1) from Yetter-Drinfel'd modules over a Hopf algebra,
(2) from self-distributive structures,
(3) from crossed modules of groups.

Furthermore, in the abstract, we have mentioned a number of fields in which crossed modules are used in their study. Therefore, studying crossed modules and all kinds of automorphisms at least through this is very important. This is one of the motivations of recent half-century studies in this field. Crossed modules were defined by Whitehead [17].

We note that there are many and interesting applications of crossed modules, such as Actor, Pullback, Pushout, and Induced crossed modules [1, 2]. Nilpotent, Solvable, n-Complete, and Representations of crossed modules were studied by Dehghanizadeh and Davvaz [13, 14]. Polygroups were studied by Comer [10], also see [12]. In fact, Comer and Davvaz extended algebraic theory, to polygroups. Alp and Davvaz [3], expressed the concept of crossed polymodule of polygroups along with some properties and characteristics of

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it. Moreover, they introduce new important classes by the fundamental relations. The pushout and pullback crossed polymodules has been introduced by the Alp and Davvaz, and they described the structure of these two concepts in crossed polymodules. Arvasi et al. [4, 5, 6], introduce the notion of a 2-crossed module, which is generalizations of crossed modules, in addition to was defined by Brown et al. [7, 8, 9]. In [15], Dehghanizadeh et al. introduce the notion of crossed polysquare.

We remind you that one of several natural generalizations of groups theory, which is studied, is the theory of polygroups. Regarding the action on their elements, in any group, the combination of two elements is one element ,but in any polygroup, that is a set. In addition, we point out that polygroups have important uses in many fields, such as lattices, geometry, color scheme, and combinatorics. As a good source for study, including definition, suitable examples, and actually studying polygroups as a subclass of supergroups, it can be referred to [12]. In [15], Dehghanizadeh, Davvaz and Alp studied crossed polysquare version of homotopy kernels.

In this article, considering the importance of crossed squares, we will examine their application. In addition we study, version of homotopy cokernels of their.

2. HOMOTOPY IN CROSSED POLYSQUARES

There are two versions of the kernel of a morphisms of crossed polymodule, Davvaz and Alp in [3] introduced strict version.

$$\begin{tikzcd} P_1 \arrow[r, "\phi"] \arrow[d, "\partial"'] & P_1' \arrow[d, "\partial'"] \\ P_0 \arrow[r, "\Psi"'] & P_0' \end{tikzcd}$$

Fig. 2.1 – Diagram (1)

Definition 2.1 ([15]). (Fiber hyperproduct) Let P_1, P_2 and Q be polygroups, and let $\phi : P_1 \longrightarrow Q$ and $\psi : P_2 \longrightarrow Q$ be homorphisms. The fiber hyperproduct of P_1 and P_2 over Q, also known as a pullback, is the following subpolygroup of $P_1 \times P_2$:

$$P_1 \times_Q P_2 = \{(p_1, p_2) \mid \ (p_1, p_2) \in P_1 \times P_2,\ \phi(p_1) = \psi(p_2)\}.$$

If $\phi : P_1 \longrightarrow Q\}$ and $\psi : P_2 \longrightarrow Q$ are epimorphism, then this is a subdirect product.

In this case the objects of the kernel are of categorical of the pullback $P_0 \times_{P_0'} P_1'$.

Definition 2.2 ([15]). A braided crossed polymodule of polygroups $\partial : P_1 \longrightarrow P_0$ is a crossed polymodule with a braiding polyfunction $\{-,-\} : P_0 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$ satisfying the following axioms:
(i) $\{p_1, p_2p_3\} = \{p_1, p_2\}^{p_2}\{p_1, p_3\}$;
(ii) $\{p_1p_2, p_3\} =^{p_1} \{p_2, p_3\}\{p_1, p_3\}$;
(iii) $\partial\{p_1, p_2\} = p_1p_2p_1^{-1}p_2^{-1}$;
(iv) $\{\partial(\alpha), p\} = \alpha^p\alpha^{-1}$;
(v) $\{p, \partial(\alpha)\} =^p \alpha\alpha^{-1}$; for all $\alpha \in P_1$ and $p, p_1, p_2, p_3 \in P_0$.
If the braiding is symmetric, we also have:
(vi) $\{p_1, p_2\}\{p_2, p\} = 1$,
Then the crossed polymodule $\partial : P_1 \longrightarrow P_0$ is called symmetric crossed polymodule.

For continue thread, we need to the some of contents, which are as following:

Definition 2.3 ([16]). Let H be a semihypergroup. Then, we set

$$\gamma_1 = \{(x, x) \mid x \in H\}$$

and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \longleftrightarrow \exists(z_1, z_2, \ldots, z_n) \in H^n,\ \exists\sigma \in \mathbb{S}_n\ : x \in \prod_{i=1}^{n} z_i,\ y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

where $\mathbb{S}_n$ is the symmetric group of degree n. Obviously, for $n \geq 1$, the relations γ_n are symmetric, and the relation $\gamma = \bigcup_{n\geq 1} \gamma_n$ is reflexive and symmetric. Let γ^* be the transitive closure of γ. If H is a hypergroup, then $\gamma = \gamma^*$.

Definition 2.4 ([12]). Let$\mathcal{P}_1 =< P_1, \bullet, e_1, {}^{-1} >$ and $\mathcal{P}_2 =< P_2, *, e_2, {}^{-1} >$ be two polygroups. We consider the group $\mathrm{Aut}\, P_1$ and the group $\frac{P_2}{\gamma^*_{P_2}}$, let

$$\widehat{\ }: \frac{P_2}{\gamma^*_{P_2}} \longrightarrow \mathrm{Aut}\, P_1$$

$$\gamma^*_{P_2}(p_2) \longrightarrow \widehat{\gamma^*_{P_2}(p_2)} = \widehat{p}_2$$

be a homomorphism of groups. Then, in $\mathcal{P}_1 \times \mathcal{P}_2$ we define a hyperproduct as follows:

$$(p_1, p_2) \circ (p_1', p_2') = \{(x, y) \mid x \in p_1 \bullet \widehat{p}_2(p_1'),\ y \in p_2 * p_2'\}$$

and we call this the *semidirect hyperproduct* of polygroups $\mathcal{P}_1$ and $\mathcal{P}_2$.

Now, we define a generalized notion of semidirect hyperproduct of polygroups, and use that to introduce a pushout construction for crossed polymodules. Let P_1, Γ_1 and P_0 be polygroups, each equipped with a right polyaction of P_0, the one on P_0 itself being conjugation. We denote all the polyaction

$$\begin{tikzcd} P_1 \arrow[r, "p"] \arrow[d, "d"'] & \Gamma_1 \\ P_0 & \end{tikzcd}$$

Fig. 2.2 – Diagram (2)

by ${}^{-k}$. Assume we are given a P_0-equivariant diagram in which we require the compatibility condition $\gamma_1^{d(p_1)} = \gamma_1^{p(p_1)}$ is satisfied for every $p_1 \in P_1$ and $\gamma_1 \in \Gamma_1$.

Definition 2.5. The semidirect hyperproduct $P_0 \ltimes {}^{P_1}\Gamma_1$ of P_0 and Γ_1 along P_1 is defined to be $P_0 \ltimes \frac{\Gamma_1}{N}$, where

$$N = \{(d(p_1)^{-1}, p(p_1)) \mid p_1 \in P_1\}.$$

There are natural polygroup homomorphisms $p' : P_0 \longrightarrow P_0 \ltimes {}^{P_1}\Gamma_1$ and $d' : \Gamma_1 \longrightarrow P_0 \ltimes {}^{P_1}\Gamma_1$, making diagram (3) commute There is also an polyac-

$$\begin{tikzcd} P_1 \arrow[r, "p"] \arrow[d, "d"'] & \Gamma_1 \arrow[d, "d'"] \\ P_0 \arrow[r, "p'"'] & P_0 \ltimes {}^{P_1}\Gamma_1 \end{tikzcd}$$

Fig. 2.3 – Diagram (3)

tion of $P_0 \ltimes^{P_1} \Gamma_1$ on Γ_1 which makes the above diagram equivariant.
An element $(p_0, \gamma_1) \in P_0 \ltimes^{P_1} \Gamma_1$ on $\gamma_1' \in \Gamma_1$ by sending it to $\{x \mid x \in \gamma_1^{-1}\gamma_1'{}^{p_0}\gamma_1\}$. Indeed, $d' : \Gamma_1 \longrightarrow P_0 \ltimes^{P_1} \Gamma_1$ is a crossed polymodule.

Theorem 2.6. *If outer Diagram* (1) *is a crossed polysquare, then outer diagram Diagram* (4) *gives rise to a crossed polysquare with actions, polygroup homomorphism* ∂'' *and function* $\bar{\bar{h}} : (P_0 \ltimes {}^{P_1}\Gamma_1) \times (P_0 \times {}_{\Gamma_0}\Gamma_1) \longrightarrow \mathcal{P}^*(P_1)$ *defined as following:*

(i) *the polyaction of* Γ_0 *on* P_1 *is induced by the polyaction of crossed polymodule of* $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ *on* $\partial : P_1 \longrightarrow P_0$*;*

(ii) *the polyaction of* Γ_0 *on* $P_0 \ltimes {}^{P_1}\Gamma_1$ *is the polyaction of a crossed polymodule* $d : P_0 \ltimes {}^{P_1}\Gamma_1 \longrightarrow \Gamma_0$*;*

(iii) *the polyaction of* Γ_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$ *is defined by*

$${}^{\sigma}(p_2, \beta_2) = \{(x, y) \mid x \in {}^{\sigma}p_2,\ y \in {}^{\sigma}\beta_2\}$$

(*the same polyaction seen in the crossed polysquare Diagram* (2)*);*

$$\begin{tikzcd}
P_1 \arrow[r, equal, "id_{P_1}"] \arrow[d, "\bar{\partial}"'] \arrow[rrr, bend left, "\widetilde{p_1}"] & P_1 \arrow[r, "\bar{p_1}"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[r, "\partial''"] \arrow[d, "\partial'"] & P_0 \times^{P_1} \Gamma_1 \arrow[d, "d"] \\
P_0 \times_{\Gamma_0} \Gamma_1 \arrow[r, "p_{P_0}"] \arrow[rr, bend right, "\bar{\bar{p_0}}"'] \arrow[rrr, bend right=40, "\widetilde{p_0}"'] & P_0 \arrow[r, "\widetilde{p_0}"] & \Gamma_0 \arrow[r, equal, "id_{\Gamma_0}"] & \Gamma_0
\end{tikzcd}$$

Fig. 2.4 – Diagram (4)

(iv) $\partial'' : \Gamma_1 \longrightarrow P_0 \ltimes {}^{P_1}\Gamma_1$ *is the canonical inclusion map of* Γ_1 *in* $P_0 \ltimes {}^{P_1}\Gamma_1$*;*

(v) $\bar{\bar{h}}\,((p_1, \beta_1), (p_2, \beta_2)) := \left\{h(\beta_1, p_1p_2p_1^{-1})h(\beta_2, p_1)^{-1}\right\}$ *where the function* h *is given by the crossed polysquare structure of Diagram* (1)*.*

Proof. $\tilde{p}_0 = \bar{\bar{p}}_0$ is a strong homomorphism, where $\bar{\bar{p}}_0$ is defined in Diagram (3). Then $\bar{p}_1$ is a strong homomorphism, and so $\tilde{p}_1(\alpha) = (1, \bar{p}_1(\alpha))$ is a strong homomorphism. Diagram (4) is commutative and the last map is a crossed polymodule, because it is easy to check that $d\tilde{p}_1 = \partial'\bar{p}_1 = \bar{p}_0\partial = \tilde{p}_0\bar{\partial}$.
But $\bar{\bar{h}}$ is well defined, in fact we have

$$\begin{aligned}
&\bar{\bar{h}}\left\{((x, y), (p_2, \beta_2)) \mid x \in \partial(\alpha)p_1,\ y \in \beta_1\bar{p}_1(\alpha)^{-1}\right\}\\
&= h\left\{(x, y) \mid x \in \beta_1\bar{p}_1(\alpha)^{-1},\ y \in \partial(\alpha)p_1p_2p_1^{-1}\partial(\alpha)\right\}\\
&\quad\ h\left\{(\beta_2, z)^{-1} \mid z \in \partial(\alpha)p_1\right\}\\
&= h\left\{(x, y) \mid x \in \beta_1\bar{p}_1(\alpha)^{-1},\ y \in {}^{\bar{p}_0\partial(\alpha)}(p_1p_2p_1^{-1})\right\}^{\partial(\alpha)}\\
&\quad\ h(\beta_2, p_1)^{-1}h\left\{(\beta_2, z)^{-1} \mid z \in \partial(\alpha)\right\}\\
&= {}^{\bar{p}_0\partial(\alpha)}h\left\{(x, y) \mid x \in {}^{\bar{p}_0\partial(\alpha)^{-1}}(\beta_1\bar{p}_1(\alpha)^{-1},\ y \in p_1p_2p_1^{-1})\right\}\\
&\quad\ \alpha h\left\{(\beta_2, p_1)^{-1}\,\alpha^{-1}\alpha^{\beta_2}\alpha^{-1}\right\}\\
&= \alpha h\left\{(x, y)\ \ x \in {}^{\partial'\bar{p}_1(\alpha)^{-1}}(\beta_1\bar{p}_1(\alpha)^{-1},\ y \in p_1p_2p_1^{-1}\right\}\alpha^{-1}\alpha h(\beta_2, p_1)^{-1\,\beta_2}\alpha^{-1}\\
&= \alpha^{\partial'\bar{p}_1(\alpha)^{-1}}h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\}h\left\{(\bar{p}_1(\alpha)^{-1}, y) \mid y \in p_1p_2p_1^{-1}\right\}\\
&\quad\ h(\beta_2, p_1)^{-1\,\beta_2}\alpha^{-1}\\
&= \alpha^{\partial'\bar{p}_1(\alpha)^{-1}}h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\}h\left\{(\bar{p}_1(\alpha)^{-1}, y) \mid y \in p_1p_2p_1^{-1}\right\}\\
&\quad\ h(\beta_2, p_1)^{-1\,\beta_2}\alpha^{-1}\\
&= \alpha\alpha^{-1}h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\}\alpha\alpha^{-1\,p_1p_2p_1^{-1}}\alpha h(\beta, p_1)^{-1\,\beta_2}\alpha^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} {}^{p_1\beta_2}({}^{p_1^{-1}}\alpha)h(\beta_2, p_1)^{-1}\,{}^{\beta_2}\alpha^{-1}\\
&= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}\,{}^{\beta_2p_1}({}^{p_1^{-1}}\alpha)\,{}^{\beta_2}\alpha^{-1}\\
&= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}\,{}^{\beta_2}\alpha\,{}^{\beta_2}\alpha^{-1}\\
&= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}.
\end{aligned}
$$

Outer Diagram (3) is crossed polysquare and so the equalities above consequences of the axioms of the crossed polysquare.

Now we want to check the five properties making Diagram (4) a crossed polysquare.

(i) the map $\tilde{p}_1$ preserves the polyactions of Γ_0; in fact

$$\tilde{p}_1({}^{\sigma}\alpha) = \{(1, x) \mid x \in \bar{p}_1({}^{\sigma}\alpha)\} = \{(1, x) \mid x \in {}^{\sigma}\bar{p}_1(\alpha)\} = {}^{\sigma}(1, \bar{p}_1(\alpha)) = {}^{\sigma}\tilde{p}_1(\alpha).$$

The map $\bar{\partial}$ preserves the polyactions of Γ_0. Also d is a crossed polymodule and $\tilde{p}_0$ is a crossed polymodule because $\bar{\bar{p}}_0$ is.

(ii) We want to prove that

$$\tilde{p}_1\left(\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2))\right) = (p_1, \beta_1)^{(p_2,\beta_2)}(p_1, \beta_1)^{-1}$$

and we develop the two members separately:

$$
\begin{aligned}
\tilde{p}_1\left(\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2))\right) &= \tilde{p}_1\left(h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2, p_1)^{-1}\right)\\
&= \left(1, \bar{p}_1(h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2, p_1)^{-1}\right)\\
&= \{(1, y) \mid y \in \beta_1\,{}^{p_1p_2p_1^{-1}}\beta_1^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\};
\end{aligned}
$$

and

$$
\begin{aligned}
& (p_1, \beta_1)\,{}^{(p_2,\beta_2)}(p_1, \beta_1)^{-1}\\
= \;& (p_1, \beta_1)\,{}^{\tilde{p}_0(p_2,\beta_2)}(p_1, \beta_1)^{-1}\\
= \;& (p_1, \beta_1)\,{}^{\tilde{p}_0(p_2)}(p_1^{-1}, {}^{p_1^{-1}}\beta_1^{-1})\\
= \;& (p_1, \beta_1)\{(x, y) \mid x \in p_2p_1^{-1}p_2^{-1},\, y \in {}^{p_2p_1^{-1}}\beta_1^{-1}\}\\
= \;& \{(u, v) \mid u \in p_1p_2p_1^{-1}p_2^{-1},\, v \in \beta_1\,{}^{p_1p_2p_1^{-1}}\beta_1^{-1}\}\\
= \;& \{(r, s) \mid r \in \partial h(\beta_2, p_1)^{-1}1,\, s \in \beta_1\,{}^{p_1p_2p_1^{-1}}\beta_1^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\bar{p}_1h(\beta_2, p_1)\}.
\end{aligned}
$$

Now we want to prove that

$$\bar{\partial}\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2)) = {}^{(p_1,\beta_1)}(p_2, \beta_2)(p_2, \beta_2)^{-1};$$

and we develop the two members separately:

$$
\begin{aligned}
& \bar{\partial}\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2))\\
= \;& \bar{\partial}(h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}h(\beta_2, p_1)^{-1})\\
= \;& \{(\partial h(\beta_1, y)\, \partial h(\beta_2, p_1)^{-1}, \bar{p}_1h(\beta_1, y)\bar{p}_1h(\beta_2, p_1)^{-1}) \mid y \in p_1p_2p_1^{-1}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1)^{-1}p_1p_2^{-1}p_1^{-1}p_1\,{}^{\beta_2}p_1^{-1},\ v \in \beta_1\,{}^{p_1p_2p_1^{-1}}\beta_1^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\} \\
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1^{-1})p_1p_2^{-1}\,{}^{\bar{p}_0(p_2)}p_1^{-1}, v \in \beta_1\,{}^{p_1}({}^{\partial'(\beta_2)}({}^{p_1^{-1}}\beta_1^{-1}))\,{}^{p_1}\beta_2\beta_2^{-1}\} \\
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1^{-1})p_1p_2^{-1}p_2p_1^{-1}p_2^{-1},\ v \in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\,{}^{p_1}\beta_2^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\} \\
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1^{-1})p_2^{-1},\ v \in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\beta_2^{-1}\};
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{(p_1,\beta_1)}(p_2,\beta_2)(p_2,\beta_2)^{-1} = {}^{\partial'(\beta_1)\bar{p}_0(p_1)}(p_2,\beta_2)(p_2^{-1},\beta_2^{-1}) \\
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1^{-1}),\ v \in \beta_1\,{}^{g_1}\beta_2\beta_1^{-1}\}(p_2^{-1},\beta_2^{-1}) \\
&= \{(u,v) \mid u \in {}^{\beta_1}(p_1p_2p_1^{-1})p_2^{-1},\ v \in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\beta_2^{-1}\}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\bar{\bar{h}}(\tilde{p}_1(\alpha),(p_2,\beta_2)) &= \bar{\bar{h}}((1,\bar{p}_1(\alpha)),(p_2,\beta_2)) \\
&= h(\bar{p}_1(\alpha),p_2)h(\beta_2,1)^{-1} = \alpha\,{}^{p_2}\alpha^{-1} = \alpha\,{}^{\bar{p}_0(p_2)}\alpha^{-1} \\
&= \alpha^{\bar{p}_0(p_2,\beta_2)}\alpha^{-1} = \alpha\,{}^{(p_2,\beta_2)}\alpha^{-1};
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\bar{h}}((p_1,\beta_1),\bar{\partial}(\alpha)) &= \bar{\bar{h}}((p_1,\beta_1),(\partial(\alpha),\bar{p}_1(\alpha))) \\
&= h\{(\beta_1,y) \mid y \in p_1\partial(\alpha)p_1^{-1}\}\,h(\bar{p}_1(\alpha),p_1)^{-1} \\
&= h\{(\beta_1,y) \mid y \in \partial({}^{p_1}\alpha)\}\,h(\bar{p}_1(\alpha),p_1)^{-1} \\
&= {}^{\beta_1}({}^{p_1}\alpha)\,{}^{p_1}\alpha^{-1}\,{}^{p_1}\alpha\alpha^{-1} = {}^{\beta_1}({}^{p_1}\alpha)\alpha^{-1} \\
&= {}^{\partial'(\beta_1)\bar{p}_0(p_1)}\alpha\alpha^{-1} = {}^{d(p_1,\beta_1)}\alpha\alpha^{-1} = {}^{(p_1,\beta_1)}\alpha\alpha^{-1}.
\end{aligned}
$$

(iv) We want to prove that:

$$
\bar{\bar{h}}((p_1,\beta_1)(p_1',\beta_1'),(p_2,\beta_2)) = {}^{(p_1,\beta_1)}\bar{\bar{h}}((p_1',\beta_1'),(p_2,\beta_2))\,\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2))
$$

and we develop the two members separately:

$$
\begin{aligned}
&\bar{\bar{h}}((p_1,\beta_1)(p_1',\beta_1'),(p_2,\beta_2)) \\
&= \bar{\bar{h}}\{((x,y),(p_2,\beta_2)) \mid x \in p_1p_1',\ y \in \beta_1\,{}^{p_1}\beta_1'\} \\
&= h\{(y,z) \mid y \in \beta_1\,{}^{p_1}\beta_1',\ z \in p_1p_1'p_2{p_2'}^{-1}p_1^{-1}\}\,h\{(\beta_2,r)^{-1} \mid r \in p_1p_1'\} \\
&= {}^{\beta_1}h\{(s,z) \mid s \in {}^{p_1}\beta_1',\ z \in p_1p_1'p_2{p_1'}^{-1}p_1^{-1}\}\,h\{(\beta_1,z) \mid z \in p_1p_1'p_2{p_1'}^{-1}p_1^{-1}\} \\
&= {}^{\beta_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1} \\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\,h\{(\beta_1,u) \mid u \in p_1p_1'\,{}^{\bar{p}_0(p_2)}(p_1p_1')^{-1}p_2\} \\
&= {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1} \\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\,h\{(\beta_1,\partial h(\beta_2,r)^{-1}p_2 \mid r \in p_1p_1^{-1}\} \\
&\qquad {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1} \\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1}h\{(\beta_2,r)^{-1} \mid r \in p_1p_1'\}
\end{aligned}
$$

$$
\begin{aligned}
& h(\beta_1,p_2)h\{(\beta_2,r)\mid r\in p_1p_1^{-1}\}\,{}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
= \;& {}^{\beta_1 p_1}h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1}h\{(\beta_2,r)^{-1}\mid r\in p_1p_1^{-1}\}\\
& h(\beta_1,p_2)h(\beta_2,p_1)\,{}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
= \;& {}^{\beta_1 p_1}h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1}h\{(\beta_2,r)^{-1}\mid r\in p_1p_1^{-1}\}\,h(\beta_1,p_2);
\end{aligned}
$$

and

$$
\begin{aligned}
& {}^{(p_1,\beta_1)}\bar{\bar{h}}((p_1',\beta_1'),(p_2,\beta_2))\,\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2))\\
&={}^{\beta_1p_1}\,[h\{(\beta_1',t)\mid t{\in}p_1'p_2{p_1'}^{-1}\}\,h(\beta_2,p_1')^{-1}]h\{(\beta_1,y)\mid y\in p_1p_2p_1^{-1}\}\,h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w)\mid w\in p_1{}^{\bar{p}_0(p_2)}p_1^{-1}p_2\}\\
&h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w_1)\mid w_1\in p_1{}^{\partial'(\beta_2)}p_1^{-1}p_2\}\\
&h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t{\in}p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w_2)\mid w_2{\in}\partial h(\beta_2,p_1)^{-1}p_2\}\\
&h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t{\in}p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1p_1}h(\beta_2,p_1')^{-1}\,{}^{\beta_1}h(\beta_2,p_1)^{-1}h(\beta_2,p_1)^{-1}h(\beta_1,p_2)\\
&h(\beta_2,p_1)\,h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1}[{}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}]h(\beta_1,p_2)\\
&={}^{\beta_1p_1}\,h\{(\beta_1',t)\mid t\in p_1'p_2{p_1'}^{-1}\}\,{}^{\beta_1}h\{(\beta_2,r)^{-1}\mid r\in p_1p_1'\}\,h(\beta_1,p_2).
\end{aligned}
$$

(v)

$$
\begin{aligned}
\bar{\bar{h}}({}^{\sigma}(p_1,\beta_1),{}^{\sigma}(p_2,\beta_2)) &= \bar{\bar{h}}(({}^{\sigma}p_1,{}^{\sigma}\beta_1),({}^{\sigma}p_2,{}^{\sigma}\beta_2))\\
&= h\{({}^{\sigma}\beta_1,x)\mid x\in {}^{\sigma}p_1\,{}^{\sigma}p_2\,{}^{\sigma}p_1^{-1}\}\,h({}^{\sigma}\beta_2,{}^{\sigma}p_1)^{-1}\\
&= h\{({}^{\sigma}\beta_1,x)\mid x\in {}^{\sigma}(p_1p_2p_1^{-1})\}\,h({}^{\sigma}\beta_2,{}^{\sigma}p_1)^{-1}\\
&= {}^{\sigma}h\{(\beta_1,y)\mid y\in p_1p_2p_1^{-1}\}\,{}^{\sigma}h(\beta_2,p_1)^{-1}\\
&= {}^{\sigma}[h\{(\beta_1,y)\mid y\in p_1p_2p_1^{-1}\}\,h(\beta_2,p_1)^{-1}]\\
&= {}^{\sigma}\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2).
\end{aligned}
$$

□

THEOREM 2.7. *If Diagram (1) is a crossed polysquare, then the outer diagram(Diagram (5)) is a crossed square with actions and function,*

$$
\bar{h}:\frac{P_0\ltimes^{P_1}\Gamma_1}{\beta^*_{P_0\ltimes^{P_1}\Gamma_1}}\times\frac{P_0\times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0\times_{\Gamma_0}\Gamma_1}}\longrightarrow\frac{P_1}{\beta^*_{P_1}}
$$

defined as follows:

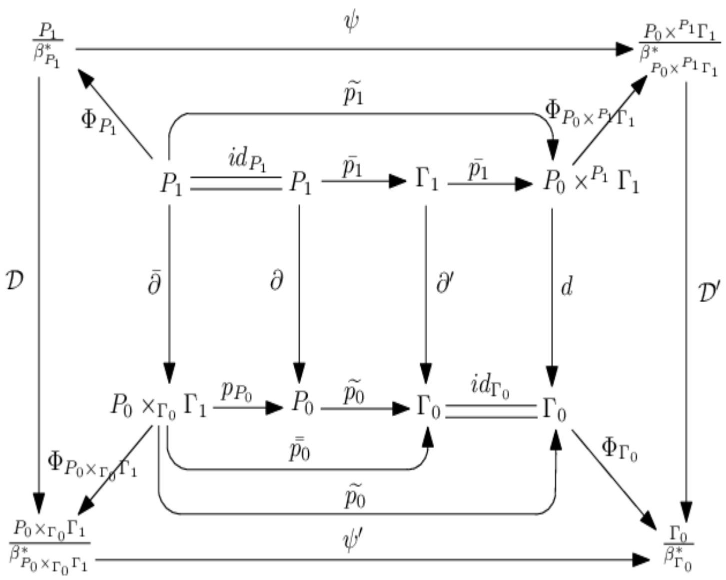

Fig. 2.5 – Diagram (5)

(a) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_1}{\beta^*_{P_1}}$ *is induced by the polyaction of* Γ_0 *on* P_1*;*

(b) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0\ltimes^{P_1}\Gamma_1}{\beta^*_{P_0\ltimes^{P_1}\Gamma_1}}\ltimes^{P_1}\Gamma_1$ *by the polyaction of* Γ_0 *on* $P_0\ltimes^{P_1}\Gamma_1$*;*

(c) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0\times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0\times_{\Gamma_0}\Gamma_1}}$ *is induced by the polyaction of* Γ_0 *on* $P_0\times_{\Gamma_0}\Gamma_1$*;*

(d) *the map* $\bar{h}:\frac{P_0\ltimes^{P_1}\Gamma_1}{\beta^*_{P_0\ltimes^{P_1}\Gamma_1}}\times\frac{P_0\times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0\times_{\Gamma_0}\Gamma_1}}\longrightarrow\frac{P_1}{\beta^*_{P_1}}$ *is*

$$\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1),(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1'))=\beta^*_{P_1}(h((p_0,\gamma_1),(p_0',\gamma_1'))).$$

Proof. The action $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ on $\frac{P_0\times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0\times_{\Gamma_0}\Gamma_1}}$ and $\frac{P_0\ltimes^{P_1}\Gamma_1}{\beta^*_{P_0\ltimes^{P_1}\Gamma_1}}$ and $\frac{P_1}{\beta^*_{P_1}}$ is well defined. ψ' is a group homomorphism. We now want to check the five properties making this diagram a crossed square.

(i) the map ψ preserves the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ because Diagram (1) is a crossed polysquare. The map $\mathcal{D}$ preserves the action of $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$:

$$\begin{aligned}
\mathcal{D}({}^{\sigma}\beta^*_{P_1}(p_1)) &= \left(\mathcal{D}({}^{\sigma}\beta^*_{P_1}(p_1)),\psi({}^{\sigma}\beta^*_{P_1}(p_1))\right)\\
&= \left({}^{\sigma}\mathcal{D}(\beta^*_{P_1}(p_1)),{}^{\sigma}\psi(\beta^*_{P_1}(p_1))\right)\\
&= {}^{\sigma}\left(\mathcal{D}(\beta^*_{P_1}(p_1)),\psi(\beta^*_{P_1}(p_1))\right)\\
&= {}^{\sigma}\mathcal{D}(\beta^*_{P_1}(p_1)).
\end{aligned}$$

$\mathcal{D}'$ is a crossed module. We want to prove that ψ' is a crossed module. The pre-crossed module property holds because $\bar{p}_0$ satisfies the pre-crossed polymodule property. It also holds the Peiffer condition:

$$\begin{aligned}
& {}^{\psi'(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1)) \\
= \; & {}^{\psi'|_{P_0}(\beta^*_{P_0}(p_0))}(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1)) \\
= \; & \left({}^{\psi'|_{P_0}(\beta^*_{P_0}(p_0))}\beta^{*'}_{P_0}(p_0),\ {}^{\psi'|_{P_0}(\beta^*_{P_0}(p_0))}\beta^{*'}_{\Gamma_1}(\gamma_1)\right) \\
= \; & \left(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\ {}^{\mathcal{D}'(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^{*'}_{\Gamma_1}(\gamma_1)\right) \\
= \; & \left(\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\right);
\end{aligned}$$

also

$$\begin{aligned}
& (\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1} \\
= \; & (\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^{*'}_{P_0}(p_0),\beta^{*'}_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{-1}) \\
= \; & (\beta^*_{P_0}(p_0)\beta^{*'}_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)\beta^{*'}_{\Gamma_1}(\gamma_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}).
\end{aligned}$$

$\psi'\mathcal{D} = \mathcal{D}'\psi$ is a crossed module.

(ii) we want to prove that

$$\begin{aligned}
& \psi(\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1')))) \\
& = (\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1'))}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1}
\end{aligned}$$

and we develop the two members separately:

$$\begin{aligned}
& \psi(\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1')))) \\
= \; & \psi(h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma_1'),\beta^*_{P_0}(p_0))^{-1}) \\
= \; & (1,\psi(h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma_1'),\beta^*_{P_0}(p_0))^{-1})) \\
= \; & (1,\beta^*_{\Gamma_1}(\gamma_1)^{\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1}}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\ {}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma_1')\beta^*_{\Gamma_1}(\gamma_1')^{-1})
\end{aligned}$$

also

$$\begin{aligned}
& (\beta^*_{P_1}(p_1),\beta^*_{\Gamma_1}(\gamma_1))^{(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1'))}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1} \\
& = (\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{\psi'(\beta^*_{P_0}(p_0'),\beta^*_{\Gamma_1}(\gamma_1'))}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))^{-1} \\
& = (\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{\Gamma_1}(\gamma_1')\beta^*_{P_0}(p_0)^{-1}\beta^*_{\Gamma_1}(\gamma_1')^{-1},\ {}^{\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1}}\beta^*_{\Gamma_1}(\gamma_1)^{-1}) \\
& = (\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1}\beta^*_{P_0}(p_0')^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0')\beta^*_{P_0}(p_0)^{-1}}\beta^*_{\Gamma_1}(\gamma_1)^{-1})
\end{aligned}$$

Now we want to prove that

$$\begin{aligned}
&\mathcal{D}(\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))))\\
= \; & {}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))^{-1}
\end{aligned}$$

and we develop the two members separately:

$$\begin{aligned}
&\mathcal{D}(\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))))\\
&=\mathcal{D}(h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1})\\
&=\big(\psi|_{P_1}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\psi|_{P_1}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1},\\
&\beta^*_{\Gamma_1}(\gamma_1)^{\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1}}\beta^*_{\Gamma_1}(\gamma_1)^{-1}\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\big)\\
&=\big({}^{\beta^*_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\\
&\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)^{-1}\beta^*_{P_0}(p_0)^{-1}\beta^*_{P_0}(p_0)^{\beta^*_{\Gamma_1}(\gamma'_1)}\\
&\beta^*_{P_0}(p_0)^{-1},\beta^*_{\Gamma_1}(\gamma_1)^{\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1}}\\
&\beta^*_{\Gamma_1}(\gamma_1)^{-1}\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\big)\\
&=\big({}^{\beta^*_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\\
&\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)^{-1}\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1}\beta^*_{P_0}(p'_0)^{-1},\\
&\beta^*_{\Gamma_1}(\gamma_1)\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\;{}^{\beta^*_{P_0}(p_0)}\\
&\beta^*_{\Gamma_1}(\gamma'_1)^{-1}\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma'_1)^{-1}\big).\\
&=\Big({}^{\beta^*_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\beta^*_{P_0}(p'_0)^{-1},\\
&\beta^*_{\Gamma_1}(\gamma_1)\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\beta^*_{\Gamma_1}(\gamma'_1)^{-1}\Big)
\end{aligned}$$

also

$$\begin{aligned}
&{}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))^{-1}\\
= \; & \Big({}^{\beta^*_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1}),\beta^*_{\Gamma_1}(\gamma_1)\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\Big)\\
& \quad \big(\beta^*_{P_0}(p'_0)^{-1},\beta^*_{\Gamma_1}(\gamma'_1)^{-1}\big)\\
= \; & \Big({}^{\beta^*_{\Gamma_1}(\gamma_1)}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\beta^*_{P_0}(p'_0)^{-1},\\
& \beta^*_{\Gamma_1}(\gamma_1)\;{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma'_1)\beta^*_{\Gamma_1}(\gamma_1)^{-1}\beta^*_{\Gamma_1}(\gamma'_1)^{-1}\Big).
\end{aligned}$$

(iii)

$$\begin{aligned}
\bar{h}(\psi(\beta^*_{P_1}(p_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1)) &= \beta^*_{P_1}(p_1)\;{}^{\beta^*_{P_0}(p'_0)}\beta^*_{P_1}(p_1)^{-1}\\
&= \beta^*_{P_1}(p_1)\;{}^{(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))}\beta^*_{P_1}(p_1)^{-1};
\end{aligned}$$

also

$$
\begin{aligned}
& \bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),\mathcal{D}(\beta^*_{P_1}(p_1))) \\
= \; & {}^{\beta^*_{\Gamma_1}(\gamma_1)}({}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1))\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)^{-1}\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1)\beta^*_{P_1}(p_1)^{-1} \\
= \; & {}^{\beta^*_{\Gamma_1}(\gamma_1)}({}^{\beta^*_{P_0}(p_0)}\beta^*_{P_1}(p_1))\beta^*_{P_1}(p_1)^{-1} \\
= \; & {}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\beta^*_{P_1}(p_1)\beta^*_{P_1}(p_1)^{-1}.
\end{aligned}
$$

(iv) we want to prove that:

$$
\begin{aligned}
& \bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p''_0),\beta^*_{\Gamma_1}(\gamma''_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))) \\
& = {}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\bar{h}((\beta^*_{P_0}(p''_0),\beta^*_{\Gamma_1}(\gamma''_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))) \\
& \bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))),
\end{aligned}
$$

and we develop two members separately:

$$
\begin{aligned}
& \bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))(\beta^*_{P_0}(p''_0),\beta^*_{\Gamma_1}(\gamma''_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))) \\
& = \bar{h}((\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0),\beta^*_{\Gamma_1}(\gamma_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma''_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))) \\
& = h(\beta^*_{\Gamma_1}(\gamma_1)\,{}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}\beta^*_{P_0}(p_0)^{-1}) \\
& h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))^{-1} \\
& = {}^{\beta^*_{\Gamma_1}(\gamma_1)}h({}^{\beta^*_{P_0}(p_0)}\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}\beta^*_{P_0}(p_0)^{-1}) \\
& h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p_0)^{-1}\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}\beta^*_{P_0}(p_0)^{-1}) \\
& {}^{\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1} \\
& = {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1)^{-1},\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}) \\
& {}^{\beta^*_{\Gamma_1}(\gamma_1)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0)) \\
& h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))\,{}^{\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1} \\
& = {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}) \\
& {}^{\beta^*_{\Gamma_1}(\gamma_1)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0))h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)) \\
& {}^{\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))\,{}^{\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1} \\
& = {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1}) \\
& {}^{\beta^*_{\Gamma_1}(\gamma_1)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0));
\end{aligned}
$$

also

$$
\begin{aligned}
& {}^{(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1))}\bar{h}((\beta^*_{P_0}(p''_0),\beta^*_{\Gamma_1}(\gamma''_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1)))\\
& \bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1)))\\
= & {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}\big[h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\\
& \beta^*_{P_0}(p''_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}\big]\\
& \quad h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\\
= & {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1})\\
& \quad {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}\,{}^{\beta^*_{\Gamma_1}(\gamma_1)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\\
& \quad h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0))h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\\
= & {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1})\\
& \quad {}^{\beta^*_{\Gamma_1}(\gamma_1)}\left[{}^{\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\right]\\
& h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0))\\
= & {}^{\beta^*_{\Gamma_1}(\gamma_1)\beta^*_{P_0}(p_0)}h(\beta^*_{\Gamma_1}(\gamma''_1),\beta^*_{P_0}(p''_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p''_0)^{-1})\\
& \quad {}^{\beta^*_{\Gamma_1}(\gamma_1)}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p''_0))^{-1}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p'_0)).
\end{aligned}
$$

(v)

$$
\begin{aligned}
& \bar{h}({}^{\sigma}(\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),{}^{\sigma}(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))\\
= & \bar{h}(({}^{\sigma}\beta^*_{P_0}(p_0),{}^{\sigma}\beta^*_{\Gamma_1}(\gamma_1)),({}^{\sigma}\beta^*_{P_0}(p'_0),{}^{\sigma}\beta^*_{\Gamma_1}(\gamma'_1)))\\
= & h({}^{\sigma}\beta^*_{\Gamma_1}(\gamma_1),{}^{\sigma}\beta^*_{P_0}(p_0)\,{}^{\sigma}\beta^*_{P_0}(p'_0)\,{}^{\sigma}\beta^*_{P_0}(p_0)^{-1})h({}^{\sigma}\beta^*_{\Gamma_1}(\gamma'_1),{}^{\sigma}\beta^*_{P_0}(p_0))^{-1}\\
= & h({}^{\sigma}\beta^*_{\Gamma_1}(\gamma_1),{}^{\sigma}(\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1}))h({}^{\sigma}\beta^*_{\Gamma_1}(\gamma'_1),{}^{\sigma}\beta^*_{P_0}(p_0))^{-1}\\
= & {}^{\sigma}h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})\,{}^{\sigma}h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\\
= & {}^{\sigma}\left(h(\beta^*_{\Gamma_1}(\gamma_1),\beta^*_{P_0}(p_0)\beta^*_{P_0}(p'_0)\beta^*_{P_0}(p_0)^{-1})h(\beta^*_{\Gamma_1}(\gamma'_1),\beta^*_{P_0}(p_0))^{-1}\right)\\
= & {}^{\sigma}\bar{h}((\beta^*_{P_0}(p_0),\beta^*_{\Gamma_1}(\gamma_1)),(\beta^*_{P_0}(p'_0),\beta^*_{\Gamma_1}(\gamma'_1))).
\end{aligned}
$$

□

Now in our conclusion, we consider the image of a crossed polymodule [3] and we will prove another result showing the analogy between crossed polymodules and crossed polysquares.

PROPOSITION 2.8. [3] *Let* $\chi = (C, P, \partial, \alpha)$ *be a crossed polymodule. Then,* $\partial(C)$ *is a normal subpolygroup of* P.

PROPOSITION 2.9. *Let Diagram(1) be a crossed polysquare, the subcrossed polymodule* $\partial'|_{Im\bar{p}_1} : Im\bar{p}_1 \longrightarrow Im\bar{p}_0$ *of* $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ *is normal.*

Proof. (i) $\mathrm{Im}\,\bar{p}_0$ is a normal subpolygroup of Γ_0, because $\bar{p}_0 : P_0 \longrightarrow \Gamma_0$ is a crossed polymodule;

(ii) for all $\sigma \in \Gamma_0$ and $\bar{\beta} \in \operatorname{Im} \bar{p}_1$, that is there exists $\bar{\alpha} \in P_1$ such that $\bar{p}_1(\bar{\alpha}) = \bar{\beta}$, we have ${}^{\sigma}\bar{\beta} = {}^{\sigma}\bar{p}_1(\bar{\alpha}) = \bar{p}_1({}^{\sigma}\bar{\alpha})$, so ${}^{\sigma}\bar{\beta} \subseteq \operatorname{Im} \bar{p}_1$.

(iii) for all $\bar{\sigma} \in \operatorname{Im} \bar{p}_0$ that is there exists ${p'}_0 \in P_0$ such that $\bar{p}_0(p'_0) = \bar{\sigma}$ and $\beta \in \Gamma_1$, we have ${}^{\bar{\sigma}}\beta\beta^{-1} = {}^{\bar{p}_0(p'_0)}\beta\beta^{-1} = {}^{\bar{p}_0}\beta\beta^{-1} = \bar{p}_1 h(\beta, {p'}_0)$, so ${}^{\sigma}\beta\beta^{-1} \subseteq \operatorname{Im} \bar{p}_1$.

□

3. CONCLUSION

In this paper, we defined a generalized notion of semidirect hyperproduct of polygroups and use that to introduce a pushout construction for crossed polymodules. Our results extended the classical results of crossed squares to crossed polysquares. One of the main tools in the study of polygroups is the fundamental relations. Additionally we studeid on crossed polysquare version of homotopy cokernels.

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