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STABILITY OF THE KDV EQUATION IN THE CASE OF MIXED INTERNAL AND BOUNDARY DAMPINGS WITH TIME-DEPENDENT DELAY

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Abstract. The aim of this paper is to consider the nonlinear Korteweg-de Vries equation with internal feedback without delay and a boundary feedback with time-dependent delay. We study the well-posedness of the system under some assumptions on the length of the spatial domain and on the delay using semigroups theory and we study the exponential stability of the equation considering a Lyapunov functional approach.

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1. INTRODUCTION

We want to study the stability of the Korteweg-de Vries equation (KdV) in the case of mixed internal and boundary dampings with time-varying delay in the boundary feedback. The KdV equation is a third-order quasilinear one-dimensional equation. It models the propagation of waves in water of shallow depth. This wave was observed for the first time by John Scott Russel in 1834 and the equation was introduced in [5]. The controllability and stabilization properties of the KdV equation have been studied by many authors, see [6, 11] and [2, 12].

Intensive research has been developed recently on stability problems with delay for partial differential equations due to the different applications in biology and engineering. The interest in considering a delay in an equation is due to the fact that the sensors act with delay in the control systems.

The problem of stabilization of nonlinear KdV equation with constant time-delay was studied in [1, 8, 13] using a Lyapunov functional approach or an observability inequality method. The stability problems with time-varying delays was analyzed in [7] for one-dimensional heat and wave equations. Recently, the problem of stability for the KdV equation with time-varying delay was studied using a Lyapunov functional approach (see [9]). This work is an

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open problem mentioned in [9] and in our best knowledge, there is no work dealing with this problem for the KdV equation with mixed boundary and internal dampings with time-varying delay.

In this work, we are inspired by the techniques developed in [9]. We consider the following system

$$
(1)\quad \begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)+u(x,t)u_x(x,t) & \\ \qquad +a(x)u(x,t)=0, & t>0,\ x\in(0,L), \\ u(0,t)=u(L,t)=0, & t>0, \\ u_x(L,t)=\beta u_x(0,t-\tau(t)), & t>0, \\ u(x,0)=u_0(x), & x\in(0,L), \\ u_x(0,t-\tau(0))=z_0(t-\tau(0)), & 0<t<\tau(0), \end{cases}
$$

where $u(x,t)$ is the amplitude of the water wave at position x at time t and $L>0$ is the length of the spatial domain. We assume that the time-varying delay τ is a function of time t, which satisfies the following conditions

$$
(2)\qquad 0<\tau_0\le\tau(t)\le M,\qquad \forall t\ge 0,
$$

$$
(3)\qquad \dot{\tau}(t)\le d<1,\qquad \forall t\ge 0,
$$

where $0\le d<1$, and

$$
(4)\qquad \tau\in W^{2,\infty}([0,T]),\qquad \forall T>0.
$$

We assume that the real constant β satisfies

$$
(5)\qquad 0<|\beta|<1-d,
$$

and $a=a(x)$ is nonnegative function belonging to $L^\infty(0,L)$.

In [1], the exponential stability problem of the nonlinear KdV equation with constant boundary time-delay feedback was studied using a Lyapunov functional method for any lengths $L<\pi\sqrt{3}$ of the spatial domain and an observability inequality method for any non-critical lengths

$$
L\notin\mathcal{N}=\left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}},k,l\in\mathbb{N}^*\right\}.
$$

In [13], the asymptotic stability of the quasilinear Korteweg-de Vries equation with constant time-delay internal feedback was studied. In [9], the problem of stability for the KdV equation with time-varying delay was studied using a Lyapunov functional approach under some assumptions on the weight of the feedbacks, on the time-varying delay and on the length of the spacial domain.

In this work we want to extend these results in the case of mixed internal and boundary dampings with time-varying delay in the boundary feedback.

This paper is organized as follows. In Section 2, we prove the well-posedness for system (1). In Section 3 we prove the exponential stability result for system (1).

2. WELL-POSEDNESS RESULTS

The goal of this section is to prove the well-posedness results of (1). We start by proving the well-posedness result of the linearization around 0 of (1). The next stage is devoted to the study of the linear system with a source term. To finish, we use the fixed-point argument to show the well-posedness of the nonlinear system.

2.1 WELL-POSEDNESS RESULT OF THE LINEAR SYSTEM

This part is devoted to the study of the linearization around 0 of (1), that is

$$\tag{6} \begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)+a(x)u(x,t)=0, & t>0,\ x\in(0,L),\\ u(0,t)=u(L,t)=0, & t>0,\\ u_x(L,t)=\beta u_x(0,t-\tau(t)), & t>0,\\ u(x,0)=u_0(x), & x\in(0,L),\\ u_x(0,t-\tau(0))=z_0(t-\tau(0)), & t\in(0,\tau(0)). \end{cases}$$

We introduce the following new variable (see, for instance, [7]): Let $z(\rho,t)=u_x(0,t-\tau(t)\rho)$ for $\rho\in(0,1)$ and $t>0$. We can show that z verifies the following transport equation

$$\tag{7} \begin{cases} \tau(t)z_t(\rho,t)+(1-\dot{\tau}(t)\rho)z_\rho(\rho,t)=0, & t>0,\ \rho\in(0,1),\\ z(0,t)=u_x(0,t), & t>0,\\ z(\rho,0)=z_0(-\tau(0)\rho), & \rho\in(0,1). \end{cases}$$

Define $\Psi=\begin{pmatrix}u\\ z\end{pmatrix}$, then Ψ satisfies

$$\Psi_t=\begin{pmatrix}u_t\\ z_t\end{pmatrix}=\begin{pmatrix}-u_x-u_{xxx}-au\\ \dfrac{\dot{\tau}(t)\rho-1}{\tau(t)}z_\rho\end{pmatrix}.$$

We can rewrite this problem as the following first-order system

$$\tag{8} \begin{cases} \Psi_t(t)=\mathcal{A}(t)\Psi(t), & t>0,\\ \Psi(0)=\begin{pmatrix}u_0\\ z_0(-\tau(0)\cdot)\end{pmatrix}=:\Psi_0, \end{cases}$$

where the time-dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix}u\\ z\end{pmatrix}=\begin{pmatrix}-u_x-u_{xxx}-au\\ \dfrac{\dot{\tau}(t)\rho-1}{\tau(t)}z_\rho\end{pmatrix},$$

with domain

$$D(\mathcal{A}(t))=\left\{\begin{pmatrix}u\\ z\end{pmatrix}\in\left(H^3(0,L)\cap H_0^1(0,L)\right)\times H^1(0,1),\ z(0)=u_x(0), \right. \\ \left. u_x(L)=\beta z(1)\right\}.$$

We see that $D(\mathcal{A}(t))$ is independent of time t.

We consider the Hilbert space $H = L^2(0, L) \times L^2(0, 1)$, equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} v \\ y \end{pmatrix} \right\rangle = \int_0^L uv\mathrm{d}x + \int_0^1 zy\mathrm{d}\rho,$$

and with the associated norm $\| \cdot \|_H$.

Following [7], we prove the well-posedness of (8). The proof is based on the following theorem which is proven in [3]:

Theorem 2.1. *Assume that*

(1) $\mathcal{Y} = D(\mathcal{A}(0))$ *is a dense subset of* H,
(2) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, *for all* $t > 0$,
(3) *for all* $t \in [0, T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* H *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)\Psi\|_H \leq C\mathrm{e}^{ms}\|\Psi\|_H$, *for all* $\Psi \in H$ *and* $s \geq 0$*),*
(4) $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)$ *belongs to* $L^\infty_*([0, T], B(\mathcal{Y}, H))$, *the space of equivalent classes of essentially bounded, strongly measure functions from* $[0, T]$ *into the set* $B(\mathcal{Y}, H)$ *of bounded operators from* $\mathcal{Y}$ *into* H.

Then, problem (8) *has a unique solution* $\Psi \in C([0, T], \mathcal{Y}) \cap C^1([0, T], H)$ *for any initial datum in* $\mathcal{Y}$.

The following theorem gives the existence and uniqueness results of the solution of the problem (8)

Theorem 2.2. *Assume that* (2)-(5) *hold. Let* $\Psi_0 \in H$, *then there exists a unique solution* $\Psi \in C([0, +\infty), H)$ *to* (8). *Moreover, if* $\Psi_0 \in D(\mathcal{A}(0))$ *then* $\Psi \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H)$.

Proof. We are going to prove the four assumptions of Theorem 2.1. We follow closely the methods used in [9]. The space $\mathcal{Y} = D(\mathcal{A}(0))$ is a dense subset of H and we have $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, for all $t > 0$ by definition. Now, we will prove the assumption 3 of theorem 2.1. First, we introduce the time-dependent inner product on H defined by

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} v \\ y \end{pmatrix} \right\rangle_t = \int_0^L uv\mathrm{d}x + |\beta|\tau(t) \int_0^1 zy\mathrm{d}\rho,$$

and with the associated norm $\| \cdot \|_t$.
From (2), the two norms $\| \cdot \|_t$ and $\| \cdot \|_H$ are equivalent in H. Indeed,
(9)
$$\forall t \geq 0,\ \forall (u, z) \in H, \quad (1 + |\beta|\tau_0)\|(u, z)\|^2_H \leq \|(u, z)\|^2_t \leq (1 + |\beta|M)\|(u, z)\|^2_H.$$

For $t \in [0,T]$ fixed, we start by proving that $\mathcal{A}(t)$ is dissipative. We Take $\Psi = \begin{pmatrix} u \\ z \end{pmatrix} \in D(\mathcal{A}(0))$ and we calculate $\langle \mathcal{A}(t)\Psi, \Psi\rangle_t$:

$$\langle \mathcal{A}(t)\Psi, \Psi\rangle_t = \left\langle \begin{pmatrix} u_t \\ z_t \end{pmatrix}, \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t = \left\langle \begin{pmatrix} -u_x - u_{xxx} - au \\ \dfrac{\dot\tau(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t$$
$$= \int_0^L (-u_x - u_{xxx} - au) u \mathrm{d}x + |\beta| \int_0^1 (\dot\tau(t)\rho - 1) z_\rho z \mathrm{d}\rho.$$

By integrating by parts in space and in ρ, we obtain

$$\langle \mathcal{A}(t)\Psi, \Psi\rangle_t = \frac{1}{2}[u_x^2(x,t)]_0^L - \int_0^L a(x)u^2(x,t)\mathrm{d}x + \frac{|\beta|}{2}[(\dot\tau(t)\rho - 1)z^2(\rho,t)]_0^1 - \frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho,$$

then

$$\langle \mathcal{A}(t)\Psi, \Psi\rangle_t = \frac{1}{2}(u_x^2(L,t) - u_x^2(0,t)) - \int_0^L a(x)u^2(x,t)\mathrm{d}x + \frac{|\beta|}{2}z^2(0,t) + \frac{|\beta|}{2}(\dot\tau(t) - 1)z^2(1,t) - \frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho.$$

Using the boundary conditions, we get

$$\langle \mathcal{A}(t)\Psi, \Psi\rangle_t = \frac{1}{2}(\beta^2 z^2(1,t) - z^2(0,t)) - \int_0^L a(x)u^2(x,t)\mathrm{d}x + \frac{|\beta|}{2}z^2(0,t) + \frac{|\beta|}{2}(\dot\tau(t) - 1)z^2(1,t) - \frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho,$$

hence

$$\langle \mathcal{A}(t)\Psi, \Psi\rangle_t = \frac{1}{2}(\beta^2 + |\beta|(\dot\tau(t) - 1))z^2(1,t) + \frac{1}{2}(|\beta| - 1)z^2(0,t)) - \int_0^L a(x)u^2(x,t)\mathrm{d}x - \frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho.$$

Finally we obtain

$$\begin{aligned}\langle \mathcal{A}(t)\Psi, \Psi\rangle_t &\le \frac{1}{2}(\beta^2 + |\beta|(d - 1))z^2(1,t) + \frac{1}{2}(|\beta| - 1)z^2(0,t)) \\ &\quad - \int_0^L a(x)u^2(x,t)\mathrm{d}x - \frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho \\ &\le -\frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2(\rho,t)\mathrm{d}\rho,\end{aligned}$$

since we have $\dot\tau(t) \le d < 1$ and $|\beta| < 1 - d$.

We set $\kappa(t)=\frac{(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)}$, then

$$\langle \mathcal{A}(t)\Psi,\Psi\rangle_t-\kappa(t)\langle\Psi,\Psi\rangle_t\le -\frac{|\beta|}{2}(\dot\tau(t)+(\dot\tau(t)^2+1)^{1/2})\int_0^1 z^2(\rho,t)\mathrm{d}\rho\le 0,$$

which implies that the operator $\widetilde{\mathcal{A}}(t):=\mathcal{A}(t)-\kappa(t)I$ is dissipative.
Now, we prove that the adjoint of $\widetilde{\mathcal{A}}(t)$, denoted by $\widetilde{\mathcal{A}}(t)^*$ is dissipative. We can show that the adjoint of $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)^*\begin{pmatrix}u\\ z\end{pmatrix}=\begin{pmatrix}u_x+u_{xxx}-au\\ \dfrac{1-\dot\tau(t)\rho}{\tau(t)}z_\rho-\dfrac{\dot\tau(t)}{\tau(t)}z\end{pmatrix},$$

with domain

$$\begin{aligned}D(\mathcal{A}(t)^*)=\Bigg\{\begin{pmatrix}u\\ z\end{pmatrix}\in\big(H^3(0,L)\cap H_0^1(0,L)\big)\times H^1(0,1),\ u_x(0)=|\beta|z(0),\\ z(1)=\frac{\beta}{|\beta|(1-\dot\tau(t))}u_x(L)\Bigg\}.\end{aligned}$$

Let $\Psi=\begin{pmatrix}u\\ z\end{pmatrix}\in D(\mathcal{A}(t)^*)$, then

$$\langle\mathcal{A}(t)^*\Psi,\Psi\rangle_t=\left\langle\begin{pmatrix}u_x+u_{xxx}-au\\ \dfrac{1-\dot\tau(t)\rho}{\tau(t)}z_\rho-\dfrac{\dot\tau(t)}{\tau(t)}z\end{pmatrix},\begin{pmatrix}u\\ z\end{pmatrix}\right\rangle$$

$$=\int_0^L(u_x+u_{xxx}-au)u\mathrm{d}x+|\beta|\tau(t)\int_0^1(\frac{1-\dot\tau(t)\rho}{\tau(t)}z_\rho-\frac{\dot\tau(t)}{\tau(t)}z)z\mathrm{d}\rho.$$

By integrating by parts in space and in ρ, we obtain

$$\begin{aligned}\langle\mathcal{A}(t)^*\Psi,\Psi\rangle_t=-\frac12[u_x^2(x,t)]_0^L-\int_0^L a(x)u^2(x,t)\mathrm{d}x+\frac{|\beta|}{2}[(1-\dot\tau(t)\rho)z^2(\rho,t)]_0^1\\ +\frac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2\mathrm{d}\rho-|\beta|\dot\tau(t)\int_0^1 z^2\mathrm{d}\rho,\end{aligned}$$

then

$$\begin{aligned}\langle\mathcal{A}(t)^*\Psi,\Psi\rangle_t=-\frac12(u_x^2(L,t)-u_x^2(0,t))-\int_0^L a(x)u^2(x,t)\mathrm{d}x-\frac{|\beta|}{2}z^2(0,t)\\ +\frac{|\beta|}{2}(1-\dot\tau(t))z^2(1,t)-\frac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2\mathrm{d}\rho.\end{aligned}$$

Using the boundary conditions, we have

$$\begin{aligned}\langle\mathcal{A}(t)^*\Psi,\Psi\rangle_t=-\frac12((1-\dot\tau(t))^2z^2(1,t)-\beta^2z^2(0,t))-\int_0^L a(x)u^2(x,t)\mathrm{d}x\\ +\frac{|\beta|}{2}(1-\dot\tau(t))z^2(1,t)-\frac{|\beta|}{2}z^2(0,t)-\frac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2\mathrm{d}\rho.\end{aligned}$$

Finally we obtain

$$\begin{array}{c}\langle \mathcal{A}(t)^*\Psi,\Psi\rangle_t = \dfrac{1-\dot\tau(t)}{2}(|\beta|+\dot\tau(t)-1)z^2(1,t)+\dfrac{1}{2}(\beta^2-|\beta|)z^2(0,t))\\ -\displaystyle\int_0^L a(x)u^2(x,t)\mathrm{d}x-\dfrac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2\mathrm{d}\rho \le -\dfrac{|\beta|\dot\tau(t)}{2}\int_0^1 z^2\mathrm{d}\rho,\end{array}.$$

since we have $\dot\tau(t)\le d<1$ and $|\beta|<1-d$.

Hence

$$\langle \mathcal{A}(t)^*\Psi,\Psi\rangle_t-\kappa(t)\langle\Psi,\Psi\rangle_t \le -\frac{|\beta|}{2}(\dot\tau(t)+(\dot\tau(t)^2+1)^{1/2})\int_0^1 z^2(\rho,t)\mathrm{d}\rho\le 0,$$

then the operator $\widetilde{\mathcal{A}}(t)^*:=\mathcal{A}(t)^*-\kappa(t)I$ is also dissipative.

As $\widetilde{\mathcal{A}}(t)$ and $\widetilde{\mathcal{A}}(t)^*$ are dissipative and $\widetilde{\mathcal{A}}(t)$ is a densely defined closed linear operator, then $\widetilde{\mathcal{A}}(t)$ is the infinitesimal generator of a C_0 semigroup of contraction on H for any fixed $t\in[0,T]$ (see [10]).

Following the proof of Theorem 2.2 in [9], we can show that.

$$\frac{\|\Psi\|_t}{\|\Psi\|_s}\le \mathrm{e}^{\frac{c}{2\tau_0}|t-s|},\quad \forall t,s\in[0,T], \tag{10}$$

where $\Psi=(u,z)\in H$ and c is a positive constant.

Hence, for all $t\in[0,T]$, $\widetilde{\mathcal{A}}(t)$ generates a strongly continuous semigroup on H and the family $\widetilde{\mathcal{A}}=\{\widetilde{\mathcal{A}}(t):t\in[0,T]\}$ is stable with stability constants C and m independent of t (see Proposition 3.4 of [3]). So the third assumption of Theorem 2.1 is satisfied.

We can also prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{A}}(t)\in L^\infty_*([0,T],B(D(\mathcal{A}(0)),H)).$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{A}}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)-\dot\kappa(t)I,$$

where from the condition (4), $\dot\kappa(t)=\frac{\dot\tau(t)\ddot\tau(t)}{2\tau(t)(\dot\tau(t)^2+1)^{1/2}}-\frac{\dot\tau(t)(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T>0$ and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t)\Psi=\begin{pmatrix}0\\ \dfrac{\tau(t)\ddot\tau(t)\rho-(\dot\tau(t)\rho-1)\dot\tau(t)}{\tau(t)^2}z_\rho\end{pmatrix},$$

where from the assumption (4), $\dfrac{\tau(t)\ddot\tau(t)\rho-(\dot\tau(t)\rho-1)\dot\tau(t)}{\tau(t)^2}$ is bounded on $[0,T]$.

As the four conditions of Theorem 2.1 are verified, then the problem

$$\begin{cases}\tilde\Psi_t(t)=\tilde{\mathcal{A}}(t)\tilde\Psi(t),\\ \tilde\Psi(0)=\Psi_0,\end{cases}$$

has a unique solution $\tilde{\Psi} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H)$ for $\Psi_0 \in D(\mathcal{A}(0))$. We have

$$\tilde{\mathcal{A}}(t)\tilde{\Psi}(t) = \tilde{\Psi}_t(t),$$

then

$$\mathcal{A}(t)\tilde{\Psi}(t) - \kappa(t)\tilde{\Psi}(t) = \tilde{\Psi}_t(t),$$

so

$$\mathcal{A}(t)\tilde{\Psi}(t) = \tilde{\Psi}_t(t) + \kappa(t)\tilde{\Psi}(t),$$

we multiply this equation by $\mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}$ to obtain

$$\mathcal{A}(t)\mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}(t) = \mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}_t(t) + \kappa(t)\mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}(t),$$

hence

$$\mathcal{A}(t)\mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}(t)),$$

The solution of (8) is then given by $\Psi(t) = \mathrm{e}^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{\Psi}(t)$, which finishes the proof. □

2.2 WELL-POSEDNESS OF THE LINEAR SYSTEM WITH A SOURCE TERM

We consider now the linear equation (6) with a source term f

$$(11) \quad \begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + a(x)u(x,t) = f(x,t), & \\ \qquad\qquad t > 0,\ x \in (0,L) & \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u_x(L,t) = \beta u_x(0, t - \tau(t)), & t > 0, \\ u(x,0) = u_0(x), & x \in (0,L), \\ u_x(0, t-\tau(0)) = z_0(t - \tau(0)), & t \in (0, \tau(0)). \end{cases}$$

Let $T > 0$ and introduce the space $B{=}C([0,T], L^2(0,L)){\cap}L^2((0,T), H^1(0,L))$.

Proposition 2.3. *Assume that the conditions* (2)-(5) *hold. Let* $\Psi_0 = \begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in H$ *and* $f \in L^1((0,T), L^2(0,L))$. *Then there exists a unique solution* $\Psi = \begin{pmatrix} u \\ u_x(0, t-\tau(t).) \end{pmatrix} \in B \times C([0,T], L^2(0,1))$ *to* (11). *Moreover, there exists* $K > 0$ *such that*

$$(12) \qquad \|(u,z)\|_{C([0,T],H)} \leq K\left(\|\Psi_0\|_H + \|f\|_{L^1((0,T),L^2(0,L))}\right),$$

$$(13) \qquad \|u_x\|_{L^2((0,T),L^2(0,L))} \leq K\left(\|\Psi_0\|_H + \|f\|_{L^1((0,T),L^2(0,L))}\right).$$

Proof. We can write the system (11) as $\Psi_t(t) = \mathcal{A}(t)\Psi(t) + \begin{pmatrix} f \\ 0 \end{pmatrix}$. Using [4, Th 2] we can show that if $\Psi_0 \in H$ and $f \in L^1((0,T), L^2(0,L))$, then there exists a unique solution $\Psi \in C([0,T), H)$. Furthermore, $\Psi \in$

$C([0,T),D(\mathcal{A}(0)))\cap C^1([0,T),H)$ if $\Psi_0\in D(\mathcal{A}(0))$ and $f\in C([0,T),L^2(0,L))$ $\cap L^1((0,T),D(\mathcal{A}(0)))$.

We take $\Psi=(u,z)$ a classical solution of (11) (it exists if $\Psi_0\in D(\mathcal{A}(0))$) . Let us consider the following energy

$$E(t)=\int_0^L u^2(x,t)\mathrm{d}x+|\beta|\tau(t)\int_0^1 u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho. \tag{14}$$

Differentiating (14), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=2\int_0^L u(x,t)u_t(x,t)\mathrm{d}x+|\beta|\dot\tau(t)\int_0^1 u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho\\&\quad+2|\beta|\tau(t)\int_0^1 u_x(0,t-\tau(t)\rho)u_{xt}(0,t-\tau(t)\rho)\mathrm{d}\rho\\&=2\int_0^L u(x,t)(-u_x-u_{xxx}-au+f(x,t))\mathrm{d}x+|\beta|\dot\tau(t)\int_0^1 u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho\\&\quad+2|\beta|\int_0^1(\dot\tau(t)\rho-1)u_x(0,t-\tau(t)\rho)u_{x\rho}(0,t-\tau(t)\rho)\mathrm{d}\rho.\end{aligned}$$

We use (11) and integrations by parts, to have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=(|\beta|-1)u_x^2(0,t)+(\beta^2-|\beta|(1-\dot\tau(t))u_x^2(0,t-\tau(t))\\&\quad-2\int_0^L a(x)u^2(x,t)\mathrm{d}x+2\int_0^L u(x,t)f(x,t)\mathrm{d}x.\end{aligned} \tag{15}$$

From (2)-(5) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le 2\int_0^L f(x,t)u(x,t)\mathrm{d}x.$$

Using the Cauchy-Schwarz inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le 2\|f(t)\|_{L^2(0,L)}\|u(t)\|_{L^2(0,L)}.$$

Now we can take $0\le t\le T$ and integrate the above expression on $[0,t]$ to get

$$E(t)-E(0)\le 2\int_0^t\|f(s)\|_{L^2(0,L)}\|u(s)\|_{L^2(0,L)}\mathrm{d}s. \tag{16}$$

Thus, by the definition of the energy, we get

$$\|(u(\cdot,t),z(\cdot,t))\|_H^2\le\|\Psi_0\|_H^2+2\|f\|_{L^1((0,T),L^2(0,L))}\|(u,z)\|_{C([0,T],H)}.$$

Using the Young inequality and taking the maximum for $t\in[0,T]$, then there exists $K>0$ such that

$$\|(u,z)\|_{C([0,T],H)}\le K\left(\|\Psi_0\|_H+\|f\|_{L^1((0,T),L^2(0,L))}\right),$$

yielding (12). From (15), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + (|\beta|(1-\dot{\tau}(t)) - \beta^2)u_x^2(0,t-\tau(t)) \leq 2\int_0^L f(x,t)u(x,t)\mathrm{d}x,$$

and from (2)-(5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + (|\beta|(1-d) - \beta^2)u_x^2(0,t-\tau(t)) \leq 2\int_0^L f(x,t)u(x,t)\mathrm{d}x.$$

We can take $0 \leq t \leq T$ and integrate the above expression on $[0,t]$ to get

$$\begin{aligned} E(t) - E(0) + (|\beta|(1-d) - \beta^2)\int_0^t u_x^2(0,t-\tau(t))\mathrm{d}t \\ \leq 2\int_0^t \|f(s)\|_{L^2(0,L)}\|u(s)\|_{L^2(0,L)}\mathrm{d}s. \end{aligned} \tag{17}$$

If we take $t = T$ in (17), there exists $K > 0$ such that

$$\int_0^T u_x^2(0,t-\tau(t))\mathrm{d}t \leq K\left(\|\Psi_0\|_H^2 + \|f\|_{L^1((0,T),L^2(0,L))}^2\right). \tag{18}$$

Now multiplying the first equation of (11) by xu and integrations by parts on $(0,T)\times(0,L)$, we obtain

$$\begin{aligned} \frac{1}{3}\int_0^L xu^2(x,T)\mathrm{d}x + \int_0^T\int_0^L u_x^2\mathrm{d}x\mathrm{d}t + \frac{2}{3}\int_0^T\int_0^L a(x)xu^2\mathrm{d}x\mathrm{d}t = \frac{1}{3}\int_0^L xu_0^2\mathrm{d}x \\ +\frac{1}{3}\int_0^T\int_0^L u^2\mathrm{d}x\mathrm{d}t + \frac{1}{3}\int_0^T Lu_x^2(L,t)\mathrm{d}t + \frac{2}{3}\int_0^T\int_0^L xfu\mathrm{d}x\mathrm{d}t, \end{aligned}$$

since we have $a(x) > 0$, $u_x(L,t) = \beta u_x(0,t-\tau(t))$ and $2fu \leq u^2 + f^2$, then, there exists $K > 0$ such that

$$\begin{aligned} \|u_x\|_{L^2((0,L)\times(0,T))}^2 \leq K\bigg(\int_0^L u_0^2\mathrm{d}x + \int_0^T\int_0^L u^2\mathrm{d}x\mathrm{d}t + \int_0^T u_x^2(0,t-\tau(t))\mathrm{d}t \\ + \int_0^T\int_0^L f^2\mathrm{d}x\mathrm{d}t\bigg). \end{aligned}$$

Using (18) we obtain (13).

□

2.3 WELL-POSEDNESS OF THE NONLINEAR SYSTEM

The last step is to prove the local well-posedness result for the nonlinear system (1).

THEOREM 2.4. *Let $T > 0$, $L > 0$ and assume that* (2)-(5) *hold. Then there exist $r > 0$ and $K > 0$ such that for every $(u_0, z_0) \in H$ satisfying $\|(u_0,z_0)\|_H \leq r$, there exists a unique solution $u \in B$ of the system* (1) *such that $\|u\|_B \leq K\|(u_0,z_0)\|_H$.*

Proof. Let $(u_0, z_0) \in H$ and $r > 0$ chosen small enough such that $\|(u_0, z_0)\|_H \leq r$. Take $v \in B$ and consider the map $Q : B \to B$, defined by $Q(v) = u$, where u is the solution of

$$(19) \qquad \begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) & \\ +a(x)u(x,t) = -v(x,t)v_x(x,t), & t > 0, \ x \in (0,L), \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u_x(L,t) = \beta u_x(0, t - \tau(t)), & t > 0, \\ u(x,0) = u_0(x), & x \in (0,L), \\ u_x(0, t - \tau(0)) = z_0(t - \tau(0)), & 0 < t < \tau(0). \end{cases}$$

We see that $u \in B$ is a solution of (1) if and only if u is a fixed point of Q. We can prove similarly to the proof of Theorem 2.6 in [9] that the map Q is a contraction on the closed ball $\{v \in B, \ \|v\| \leq R\}$. Finally, by applying the Banach fixed point theorem, we deduce that the map Q has a unique fixed point which is the solution of the nonlinear system (1) and then the well-posedness result is proven. □

3. EXPONENTIAL STABILITY RESULTS

In this section, we prove the exponential stability results for the nonlinear system (1). We recall that the energy of (1) is defined by

$$(20) \qquad E(t) = \int_0^L u^2(x,t)\mathrm{d}x + |\beta|\tau(t)\int_0^1 u_x^2(0, t - \tau(t)\rho)\mathrm{d}\rho.$$

We are going to show that for a solution of (1) this energy is a decreasing function of time.

PROPOSITION 3.1. *Suppose that* (2)-(5) *be satisfied. Then for all regular solution of* (1)*, the energy defined by* (20) *is decreasing and satisfies*

$$(21) \qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) = (|\beta| - 1)u_x^2(0,t) + (\beta^2 - |\beta|(1 - \dot{\tau}(t))) \\ \times u_x^2(0, t - \tau(t)) - 2\int_0^L a(x)u^2\mathrm{d}x \leq 0. \end{aligned}$$

Proof. The proof is the same as the proof of Proposition 2.3 and for the nonlinear term we have for $u \in H_0^1(0,L)$, $\int_0^L u^2(x,t)u_x(x,t)\mathrm{d}x = \frac{1}{3}(u^3(L,t) - u^3(0,t)) = 0$. □

Following [9], we consider the Lyapunov functionnal defined by

$$(22) \qquad V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t),$$

where E is defined by (20), μ_1, $\mu_2 > 0$ and for any regular solution of (1), we define V_1 and V_2 by

$$(23) \qquad V_1(t) = \int_0^L x u^2(x,t)\mathrm{d}x,$$

$$V_2(t) = \tau(t)\int_0^1 (1-\rho)u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho.$$

In this work we choose μ_1 and μ_2 such that $0 < \mu_1 < \dfrac{|\beta|(1-d)-\beta^2}{L\beta^2}$ and $0 < \mu_2 < 1-|\beta|$ to guarantee the decrease of the energy of the system.

Theorem 3.2. *Suppose that* (2)-(5) *are satisfied and assume that the length L fulfills $L < \pi\sqrt{3}$. Then, there exists $r > 0$ such that, for every $(u_0, z_0) \in H$ satisfying $\|(u_0, z_0)\|_H \le r$, the energy of the system* (1) *decays exponentially and so there exist two positive constants γ and K such that*

$$E(t) \le K\mathrm{e}^{-2\gamma t}E(0),\ \forall t > 0, \tag{24}$$

where, for $0 < \mu_1 < \dfrac{|\beta|(1-d)-\beta^2}{L\beta^2}$ and $0 < \mu_2 < 1-|\beta|$,

$$\gamma \le \min\left\{\frac{(9\pi^2 - 2\pi^2 L^{3/2}r - 3L^2)\mu_1}{6L^2(\mu_1 L + 1)}, \frac{\mu_2(1-d)}{2M(|\beta|+\mu_2)}\right\}, \tag{25}$$

and

$$K \le 1 + \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\}.$$

Proof. The function V is equivalent to the energy E. Indeed we can easily check for every $t > 0$ that

$$E(t) \le V(t) \le E(t)\left(1 + \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\}\right). \tag{26}$$

Then, it suffices to show that V decays exponentially. Our goal is to prove that $\dfrac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t) \le 0$ for $\gamma > 0$ to fix later. Let u solution of (1) with $(u_0, z_0) \in D(\mathcal{A}(0))$ such that $\|(u_0, z_0)\|_0 \le r$ with $r > 0$ chosen later.

Differentiating V_1 and using integration by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) &= \int_0^L u^2(x,t)\mathrm{d}x - 3\int_0^L u_x^2(x,t)\mathrm{d}x + L\beta^2 u_x^2(0,t-\tau(t))\\ &+\frac{2}{3}\int_0^L u^3(x,t)\mathrm{d}x - 2\int_0^L a(x)xu^2(x,t)\mathrm{d}x.\end{aligned} \tag{27}$$

Now, we differentiate V_2 to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) &= +2\tau(t)\int_0^1 (1-\rho)u_x(0,t-\tau(t)\rho)\partial_t u_x(0,t-\tau(t)\rho)\mathrm{d}\rho\\ &\dot{\tau}(t)\int_0^1 (1-\rho)u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho.\end{aligned}$$

We have $\tau(t)\partial_t u_x(0,t-\tau(t)\rho) = (\dot{\tau}(t)\rho - 1)\partial_\rho u_x(0,t-\tau(t)\rho)$ and after some integrations by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = u_x^2(0,t) - \int_0^1 (1-\dot{\tau}(t)\rho)u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho. \tag{28}$$

From (21), (27) and (28) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t)\\
&\leq (\beta^2 - |\beta|(1-d) + \mu_1 L\beta^2)u_x^2(0,t-\tau(t)) + (|\beta| - 1 + \mu_2)u_x^2(0)\\
&-3\mu_1\int_0^L u_x^2(x,t)\mathrm{d}x + \frac{2}{3}\mu_1\int_0^L u^3(x,t)\mathrm{d}x + (\mu_1 + 2\gamma + 2L\gamma\mu_1)\int_0^L u^2(x,t)\mathrm{d}x\\
&+(2\gamma|\beta|M + 2M\gamma\mu_2 - \mu_2(1-d))\int_0^1 u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho.
\end{aligned}$$

We have

$$\int_0^L u^3(x,t)\mathrm{d}x \leq L^{3/2}r\|u_x\|^2_{L^2(0,L)}.$$

Then we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t) \leq &\left(\frac{L^2}{\pi^2}(\mu_1 + 2\gamma + 2L\mu_1\gamma) + \frac{2}{3}L^{3/2}r\mu_1 - 3\mu_1\right)\int_0^L u_x^2(x,t)\mathrm{d}x\\
&+(|\beta| - 1 + \mu_2)u_x^2(0,t) + (\beta^2 - |\beta|(1-d) + \mu_1 L\beta^2)u_x^2(0,t-\tau(t))\\
&+(2\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d))\int_0^1 u_x^2(0,t-\tau(t)\rho)\mathrm{d}\rho.
\end{aligned}$$

We take μ_1 and μ_2 small enough to have $\beta^2 - |\beta|(1-d) + \mu_1 L\beta^2 < 0$ and $|\beta| - 1 + \mu_2 < 0$, then, $\mu_1 < \dfrac{|\beta|(1-d) - \beta^2}{L\beta^2}$ and $\mu_2 < 1 - |\beta|$.
Following [1], since $L < \pi\sqrt{3}$, we can choose r sufficiently small to have $r < \frac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$. Hence, we can choose $\gamma > 0$ such that

$$\begin{aligned}
&\tfrac{L^2}{\pi^2}(\mu_1 + \gamma + 2L\mu_1\gamma) + \frac{2}{3}L^{3/2}r\mu_1 - 3\mu_1 \leq 0,\\
&2\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d) \leq 0.
\end{aligned}$$

We obtain

$$\gamma \leq \frac{(9\pi^2 - 2\pi^2 L^{3/2}r - 3L^2)\mu_1}{6L^2(\mu_1 L + 1)},$$

and

$$\gamma \leq \frac{\mu_2(1-d)}{2M(|\beta| + \mu_2)},$$

then

$$\gamma \leq \min\left\{\frac{(9\pi^2 - 2\pi^2 L^{3/2} r - 3L^2)\mu_1}{6L^2(\mu_1 L + 1)}, \frac{\mu_2(1-d)}{2M(|\beta| + \mu_2)}\right\}. \tag{29}$$

Finally we get $\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\gamma V(t) \leq 0$ and by solving this equation we obtain $V(t) \leq V(0)\mathrm{e}^{-2\gamma t}$ for all $t > 0$. Using (26) we obtain

$$E(t) \leq K\mathrm{e}^{-2\gamma t}E(0), \ \forall t > 0,$$

Since $D(\mathcal{A}(0))$ is dense in H, we can take $(u_0, z_0) \in H$. □

4. CONCLUSION

In this work, we presented some well-posedness and stability results for the nonlinear KdV equation with internal feedback without delay and a boundary feedback with time-dependent delay. We take some assumptions on the weights of the feedbacks, on the length of the spatial domain and on the time-dependent delay in order to prove the exponential stability results, using a Lyapunov functional approach. We can mention a possible future research on the study of the well-posedness and the stability of the nonlinear KdV equation in the case of a boundary feedback without delay and an internal feedback with delay (constant or variable).

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