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THE HILBERT-SCHMIDT PROPERTY FOR A PARTICULAR CLASS OF h-ADMISSIBLE FOURIER INTEGRAL OPERATORS

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Abstract. In this paper, we define a particular class of h-admissible Fourier integral operators F_h. These classes of Integral operators turn out to be bounded on $\mathcal{S}(\mathbb{R}^n)$ (or Schwartz space) and on its dual $\mathcal{S}'(\mathbb{R}^n)$. Moreover, we show that F_h can be extended as a Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$.

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Key words. h-admissible Fourier integral operators, symbol and phase, boundedness and compactness, Hilbert-Schmidt operators.

1. INTRODUCTION

A h-Fourier integral operator on $\mathbb{R}^n$, is an integral operator of the following form

$$I_h(a,\phi)\,u(x) = \iint \mathrm{e}^{\mathrm{i}h^{-1}\phi(x,\xi,y)} a(x,\xi,y)\, u(y)\,\mathrm{d}y\mathrm{d}\xi,$$

where ϕ is called the phase function, a is the symbol, and $h \in]0,h_0]$ is a semiclassical parameter of $I_h(a,\phi)$. In particular when $\phi(x,\xi,y) = \langle x-y,\xi\rangle$, $I_h(a,\phi) := Op_h(a)$ is called a h-pseudodifferential operator.

As it is well known, Fourier integral operators are used to express solution to Cauchy problems of hyperbolic equations as well as for obtaining asymptotic formulas for the Weyl eigenvalue function associated to geometric operators (see Hörmander [15, 16, 17], and Duistermaat and Hörmander [8])

According to the theory of Fourier integral operators developed by Hörmander [15], the symbols are considered satisfying estimates of the form

$$\sup_{(x,y)\in K} \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi,y)\right| \le C_{\alpha,\beta,K}\,(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for every compact subset K of $\mathbb{R}^{2n}$ (i.e. $a(x,\xi) \in S^m_{\rho,\delta}$), while the phase functions are real-valued, positively homogeneous of order 1 in the frequency

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variable ξ and smooth on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with the non-degeneracy condition

$$\det\left(\frac{\partial^2 \phi}{\partial x \partial \xi}\,|_{(x,\xi)}\right) \neq 0 \quad \forall\,(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\,.$$

That class of operators was initiated in the classical paper of L. Hörmander [15]. Furthermore, G. Eskin [10] (in the case $a \in S^0_{1,0}$) and Hörmander [15] (in the case $a \in S^0_{\rho,1-\rho}$, $\frac{1}{2} < \rho \leq 1$) proved the local boundedness on $L^2(\mathbb{R}^n)$ with non-degenerate phase functions.

Later on, Hörmander's local L^2 result was extended by R. Beals [6] and A. Greenleaf and G. Uhlmann [9] to the case of amplitudes in $S^0_{\frac{1}{2},\frac{1}{2}}$. On the other hand, other classes of symbols and phase functions were studied. In [18], B. Messirdi and A. Senoussaoui proved the L^2-boundedness and L^2-compactness of Fourier integral operator with symbol class just defined and $\phi(x,\xi,y) = S(x,\xi) - \langle y,\xi\rangle$. After then, this kind of operator was investigated by many authors [4, 5, 11, 12, 13, 20, 21, 22]. We would mention that the h-admissible Fourier integral operators were created by D. Robert and B. Helffer [14, 24], where symbol a belongs to Γ^μ_ρ (see below).

In this paper, h-admissible Fourier integral operators are defined and these class of operators turn out to be bounded on the spaces $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions (or Schwartz space) and $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions in Section 2.

Then, in Section 3, we gave some results about the composition of h-admissible Fourier integral operators with its L^2-adjoint. These allow to obtain results about the L^2-boundedness and the L^2-compactness.

The main part of the work consists of Section 4 in which basic properties of Hilbert-Schmidt operators are studied.

More precisely, we show that class of h-admissible Fourier integral operators can be extended as a Hilbert-Schmidt operators.

2. PRELIMINARIES

Definition 2.1. Let Ω be an open set in $\mathbb{R}^n, \mu \in \mathbb{R}$ and $\rho \in [0,1]$. A function $a \in C^\infty(\Omega)$ is called (μ,ρ)-weight symbol on Ω if

$$\forall \sigma \in \mathbb{N}^n,\, \exists C_\sigma > 0,\, \forall z \in \Omega \quad |\partial^\sigma_z a(z)| \leq C_\sigma \lambda^{\mu-\rho|\sigma|}(z),$$

where $\lambda^k(z) = \left(1+|z|^2\right)^{k/2}$ is called a weight function on $\mathbb{R}^n$.

We note $\Gamma^\mu_\rho(\Omega)$ the space of (μ,ρ)-weight symbols.

Lemma 2.2. *If $k > n$, then $\lambda^{-k}(z) \in L^1(\mathbb{R}^n)$ for all $z \in \mathbb{R}^n$.*

Proof. The lemma can easily be proved by using polar coordinates. An alternative approach can be found in [23, Lemma 1.3]. $\square$

Now, we consider the following integral transformations

$$[I_h(a,\phi)u](x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}h^{-1}\phi(x,\xi,y)} a\,(x,\xi,y)\,u\,(y)\,\mathrm{d}y\mathrm{d}\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

where $a \in \Gamma^{\mu}_{\rho}(\Omega)$ (with Ω is an open subset of $\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y$) and $h \in]0, h_0]$.

Remark 2.3. When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y$, we denote $\Gamma^{\mu}_{\rho}(\Omega) = \Gamma^{\mu}_{\rho}$.

In general, the integral (1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander. The phase function ϕ is assumed to satisfy the following assumptions

($H1$) ϕ is a real function.

($H2$) $\phi \in \Gamma^2_1\left(\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y\right)$.

($H3$) $\exists C > 0$ such that

$$\lambda\,(\partial_y\phi, \partial_\xi\phi, y)\,\lambda^{-1}\,(x,\xi,y) \leq C, \quad \forall\,(x,\xi,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y.$$

($H'3$) $\exists C' > 0$ such that

$$\lambda\,(x, \partial_\xi\phi, \partial_x\phi)\,\lambda^{-1}\,(x,\xi,y) \leq C', \quad \forall\,(x,\xi,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y.$$

In order to generalize the notion of h-admissible operators [1, 24],we give the following definitions

Definition 2.4. We call h-admissible (μ,ρ)-weight symbol, every application $a :]0, h_0] \to \Gamma^{\mu}_{\rho}$, such that

$$a\,(h) = \sum_{j=0}^{N} h^j a_j + h^{N+1} r_{N+1}\,(h)\,, \quad \forall N \in \mathbb{N},$$

where $a_j \in \Gamma^{\mu-2\rho j}_{\rho}$, and $\{r_{N+1}\,(h)\,, h \in]0, h_0]\}$ is bounded in $\Gamma^{\mu-2\rho(N+1)}_{\rho}$.

Definition 2.5. We call h-admissible Fourier integral operator, every C^∞ application $A :]0, h_0] \to \mathcal{L}\left(\mathcal{S}\,(\mathbb{R}^n)\,; L^2\,(\mathbb{R}^n)\right)$(where $\mathcal{L}(X,Y)$ is the set of all linear continuous operators from X to Y), for which there exists a sequence $\{a_j\}_{j\in\{0,\ldots,N\}} \in \Gamma^{\mu}_{0}$ satisfying

$$A\,(h) = \sum_{j=0}^{N} h^j I_h\,(a_j, \phi) + h^{N+1} R_{N+1}\,(h)\,, \quad \forall N \in \mathbb{N}, \tag{2}$$

where

$$[I_h\,(a_j,\phi)\,u]\,(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}h^{-1}\phi(x,\xi,y)} a_j\,(x,\xi,y)\,u(y)\mathrm{d}y\mathrm{d}\xi,$$

$$\sup_{h\in[0,h_0]} \|R_{N+1}\,(h)\,\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < \infty.$$

To give a meaning to the right-hand side of equality (1), we consider $g \in \mathcal{S}\left(\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y\right)$ and $g(0,0,0) = 1$. If $a \in \Gamma^\mu_0$, we define

$$a_r(x,\xi,y) = g\left(\frac{x}{r},\frac{\xi}{r},\frac{y}{r}\right) a(x,\xi,y), \quad r > 0.$$

THEOREM 2.6. *Let ϕ be a phase function satisfying $(H1),(H2),(H3)$ and $(H'3)$. Then*

1. *For all $f \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{r\to+\infty}[I_h(a_r,\phi)f](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We then set*
$$[I_h(a,\phi)f](x) := \lim_{r\to+\infty}[I_h(a_r,\phi)f](x) \quad \forall x \in \mathbb{R}^n.$$
2. *$I_h(a,\phi) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I_h(a,\phi) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R})$ such that *supp* $\psi \subset [-2,2]$ and $\psi \equiv 1$ on $[-1,1]$. For all $\epsilon > 0$, we set

$$\omega_\epsilon(x,\xi,y) = \psi\left[\frac{\lambda^{-2}(x,\xi,y)}{\epsilon}\left(|\partial_y\phi|^2 + |\partial_\xi\phi|^2\right)\right].$$

The hypothesis $(H3)$ implies that there exists $\gamma > 0$ such that we have on the support of ω_ϵ

$$\begin{aligned}
\lambda(x,\xi,y) &\leq \gamma\left(\lambda^{\frac{1}{2}}(y) + \epsilon^{\frac{1}{2}}\lambda(x,\xi,y)\right),\\
&\leq \gamma\lambda^{\frac{1}{2}}(y) + \gamma\epsilon^{\frac{1}{2}}\lambda(x,\xi,y),\\
&\leq \frac{\gamma\lambda^{\frac{1}{2}}(y)}{1-\gamma\epsilon^{\frac{1}{2}}}.
\end{aligned}$$

Therefore, there exists ϵ_0 and $\gamma_0 = \frac{\gamma}{1-\gamma\epsilon^{1/2}}$, such that for all $\epsilon \leq \epsilon_0$ we have the inequality $\lambda(x,\xi,y) \leq \gamma_0\lambda^{\frac{1}{2}}(y)$, on the support of ω_ϵ.

Now, we fix $\epsilon = \epsilon_0$. Then we have

$$\lim_{r\to+\infty}[I_h(\omega_{\epsilon_0}a_r,\phi)f](x) = [I_h(\omega_{\epsilon_0}a,\phi)f](x), \quad \forall x \in \mathbb{R}^n \tag{3}$$

using $(H2)$ we prove also that $I_h(\omega_{\epsilon_0}a,\phi) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$.

To study $\lim_{r\to+\infty}[I_h((1-\omega_{\epsilon_0})a_r,\phi)f](x)$ for all $x \in \mathbb{R}^n$, we create the operator

$$\Lambda_h = \Phi_h(x,\xi,y)\left(\sum_{j=1}^n \frac{\partial\phi}{\partial y_j}\frac{\partial}{\partial y_j} + \sum_{j=1}^N \frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j}\right),$$

where $\Phi_h(x,\xi,y) = -ih\left(|\nabla_y\phi|^2 + |\nabla_\xi\phi|^2\right)^{-1}$ and

$$\nabla_y\phi = \left(\frac{\partial\phi}{\partial y_1},\ldots,\frac{\partial\phi}{\partial y_n}\right)^t, \quad \nabla_\xi\phi = \left(\frac{\partial\phi}{\partial\xi_1},\ldots,\frac{\partial\phi}{\partial\xi_N}\right)^t,$$

are Gradient vectors. So, we have

$$\Lambda_h\left(\mathrm{e}^{\mathrm{i}h^{-1}\phi}\right)=\mathrm{e}^{\mathrm{i}h^{-1}\phi} \tag{4}$$

Indeed,

$$\begin{aligned}
\Lambda_h\left(\mathrm{e}^{\mathrm{i}h^{-1}\phi}\right) &= \Phi_h(x,\xi,y)\left(\sum_{j=1}^{n}\frac{\partial\phi}{\partial y_j}\frac{\partial}{\partial y_j}\left(\mathrm{e}^{\mathrm{i}h^{-1}\phi}\right)+\sum_{j=1}^{N}\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j}\left(\mathrm{e}^{\mathrm{i}h^{-1}\phi}\right)\right)\\
&= \Phi_h(x,\xi,y)ih^{-1}\mathrm{e}^{\mathrm{i}h^{-1}\phi}\left(\sum_{j=1}^{n}\frac{\partial\phi}{\partial y_j}\frac{\partial\phi}{\partial y_j}+\sum_{j=1}^{N}\frac{\partial\phi}{\partial\xi_j}\frac{\partial\phi}{\partial\xi_j}\right)\\
&= \Phi_h(x,\xi,y)ih^{-1}\mathrm{e}^{\mathrm{i}h^{-1}\phi}\left(\sum_{j=1}^{n}\left(\frac{\partial\phi}{\partial y_j}\right)^2+\sum_{j=1}^{N}\left(\frac{\partial\phi}{\partial\xi_j}\right)^2\right)\\
&= \Phi_h(x,\xi,y)ih^{-1}\mathrm{e}^{\mathrm{i}h^{-1}\phi}\left(|\nabla_y\phi|^2+|\nabla_\xi\phi|^2\right)\\
&= \Phi_h(x,\xi,y)ih^{-1}\left(|\nabla_y\phi|^2+|\nabla_\xi\phi|^2\right)\mathrm{e}^{\mathrm{i}h^{-1}\phi}\\
&= \mathrm{e}^{\mathrm{i}h^{-1}\phi}.
\end{aligned}$$

Let Ω_0 be the open subset of $\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$ defined by

$$\Omega_0=\left\{(x,\xi,y)\in\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n, |\nabla_y\phi|^2+|\nabla_\xi\phi|^2>\frac{\epsilon_0}{2}\lambda^2(x,\xi,y)\right\}.$$

We will use the next lemma.

Lemma 2.7. *Let $q\in\mathbb{N}_0$, and $b\in C^\infty\left(\mathbb{R}^N_\xi\times\mathbb{R}^n_y\right)$. We have*

$$\left({}^t\Lambda_h\right)^q\left((1-\omega_{\epsilon_0})\,b\right)=\sum_{|\alpha|+|\beta|\le q}k^q_{\alpha,\beta}\partial^\alpha_\xi\partial^\beta_y\left((1-\omega_{\epsilon_0})\,b\right),$$

where $k^q_{\alpha,\beta}\in\Gamma_0^{-q}(\Omega_0)$ and depend only on ϕ.

Remark 2.8. $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and ${}^t\Lambda_h$ is the transpose of Λ_h.

Proof. We prove the lemma by recurrence. For $q=0$ it is evident. Now we see easily that

$$ {}^t\Lambda_h=\sum_{j=1}^{n}A_{h,j}\frac{\partial}{\partial y_j}+\sum_{k=1}^{N}B_{h,k}\frac{\partial}{\partial\xi_k}+E_h \tag{5}$$

where

$$\begin{aligned}
A_{h,j}\in\Gamma_0^{-1}(\Omega_0)\,, &\quad \forall j\in\{1,\dots,n\}\\
B_{h,k}\in\Gamma_0^{-1}(\Omega_0)\,, &\quad \forall k\in\{1,\dots,N\}
\end{aligned}$$

and $E_h\in\Gamma_0^{-2}(\Omega_0)$ (which results from the hypothesis $(H2)$). Therefore, the recurrence is immediately proved. □

We have from (4)

(6)
$$[I_h((1-\omega_{\epsilon_0})a_r,\phi)f](x)=\iint e^{ih^{-1}\phi(x,\xi,y)}\left({}^t\Lambda_h\right)^q((1-\omega_{\epsilon_0})a_rf(y))\,dyd\xi,$$

for all $q\in\mathbb{N}_0$.

Now $\left({}^t\Lambda_h\right)^q((1-\omega_{\epsilon_0}))a_rf$ described (when r varies) a bound of $\Gamma_0^{\mu-q}$, and

$$\lim_{r\to+\infty}\left({}^t\Lambda_h\right)^q((1-\omega_{\epsilon_0})a_rf)(x,\xi,y)=\left({}^t\Lambda_h\right)^q((1-\omega_{\epsilon_0})af)(x,\xi,y),\tag{7}$$

for all $(x,\xi,y)\in\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$.

Finally, we have

$$\lambda^{s-n-N}(x)\iint\lambda^{-s}(x,\xi,y)\,d\xi dy\le\gamma_s,\tag{8}$$

for all $s>n+N$.

It results so from $(6)-(8)$ and using Lebesgue's dominated convergence Theorem show us that

$$\lim_{r\to+\infty}[I_h((1-\omega_{\epsilon_0})a_r,\phi)f](x)=\iint e^{ih^{-1}\phi(x,\xi,y)}\delta_{1,q}(x,\xi,y)dyd\xi,\tag{9}$$

where $\delta_{1,q}(x,\xi,y)=\left({}^t\Lambda_h\right)^q((1-\omega_{\epsilon_0})af(y))$ and $q>n+N+\mu$.

From (3) and (9) we can show the first part of the theorem.

Now let us prove that $I_h((1-\omega_{\epsilon_0})a,\phi)$ is bounded. Taking account of (5) and (9), we find

$$[I_h((1-\omega_{\epsilon_0})a,\phi)f](x)=\sum_{|\gamma|\le q}\iint\delta_{2,q}(x,\xi,y)\partial_y^\gamma f(y)\,dyd\xi,\tag{10}$$

where $\delta_{2,q}(x,\xi,y)=e^{ih^{-1}\phi(x,\xi,y)}b_\gamma^{(q)}(x,\xi,y)$ and $b_\gamma^{(q)}\in\Gamma_0^{\mu-q}$.

On the other hand, we have

$$x^\alpha\partial_x^\beta\delta_{2,q}(x,\xi,y)\in\Gamma_0^{\mu-q+|\alpha|+|\beta|}.\tag{11}$$

We deduce from (10) and (11) that, for all $q>n+N+\mu+|\alpha|+|\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$\left|x^\alpha\partial_x^\beta I_h((1-\omega_{\epsilon_0})a,\phi)f(x)\right|\le C_{\alpha,\beta,q}\sup_{|\gamma|\le q}\sup_{x\in\mathbb{R}^n}|\partial_x^\gamma f(x)|,$$

which proves $I_h((1-\omega_{\epsilon_0})a,\phi)\in\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$.

The boundedness of $I_h(a,\phi)$ on the space $\mathcal{S}'(\mathbb{R}^n)$ is obtained by the same way via the condition $(H'3)$. $\square$

3. L^2-BOUNDEDNESS AND L^2-COMPACTNESS OF F_H

In this section we will study a particular case on the phase function ϕ, which is very useful for solving Cauchy problems [19]. Let $\phi(x,y,\xi) = S(x,\xi) - \langle y,\xi\rangle$, where $S \in C^\infty\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi ; \mathbb{R}\right)$ satisfying

(G_1) $S \in \Gamma^2_1\left(\mathbb{R}^{2n}_{x,\xi}\right)$

(G_2) There exists $\delta_0 > 0$ such that

$$\inf_{(x,\xi)\in\mathbb{R}^{2n}} \left| det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi)\right| \geq \delta_0;$$

Proposition 3.1. *If S satisfies (G_1) and (G_2). Then the phase function ϕ satisfies (H_1), (H_2), (H_3) and (H'_3).*

Proof. (H_1) and (H_2) are trivially satisfied.

By the global implicit function theorem and using the conditions (G_1) and (G_2), we can easily see that the mappings h_1 and h_2 defined by $h_1 : (x,\xi) \to (x, \partial_x S(x,\xi))$ and $h_2 : (x,\xi) \to (\xi, \partial_\xi S(x,\xi))$ are global diffeomorphism of $\mathbb{R}^{2n}$.

Indeed,

$$h'_1(x,\xi) = \begin{pmatrix} I_n & \frac{\partial^2 S}{\partial x^2}(x,\xi) \\ 0 & \frac{\partial^2 S}{\partial x\partial \xi}(x,\xi)\end{pmatrix},$$

$$h'_2(x,\xi) = \begin{pmatrix} 0 & \frac{\partial^2 S}{\partial x\partial \xi}(x,\xi) \\ I_n & \frac{\partial^2 S}{\partial x^2}(x,\xi)\end{pmatrix}.$$

and

$$\left|\det h'_1(x,\xi)\right| = \left|\det h'_2(x,\xi)\right| = \left|\det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi)\right| \geq \delta_0 > 0 \quad \forall (x,\xi) \in \mathbb{R}^{2n}$$

Then

$$\left\| \left(h'_i(x,\xi)\right)^{-1} \right\| = \left|\det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi)\right|^{-1} \left\| {}^t Co\left[h'_i(x,\xi)\right]\right\|, \quad \forall i \in \{1,2\}$$

where $\mathrm{Co}\left[h'_i(x,\xi)\right]$ is the cofactor matrix of $h'_i(x,\xi)$ for all $i \in \{1,2\}$. By (G_1), we know that $\left\|{}^t Co\left[h'_i(x,\xi)\right]\right\|$ are uniformly bounded for all $i \in \{1,2\}$.

The mappings $\mathbb{R}^n \ni \xi \to \pi_{1,x}(\xi) = \partial_x S(x,\xi)$ and $\mathbb{R}^n \ni x \to \pi_{2,\xi}(x) = \partial_\xi S(x,\xi)$ are global diffeomorphisms of $\mathbb{R}^n$. From (G_1) and (G_2), it follows that $\left\|\left(h_i^{-1}\right)'\right\|$, $\left\|\left(\pi^{-1}_{1,x}\right)'\right\|$ and $\left\|\left(\pi^{-1}_{2,\xi}\right)'\right\|$ are uniformly bounded. Thus (G_2) and the Taylor's theorem lead to the existence of $M, N > 0$, such that for all $(x,\xi), (x',\xi') \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\xi| &= \left|\pi^{-1}_{1,x}\left(\pi_{1,x}(\xi)\right) - \pi^{-1}_{1,x}\left(\pi_{1,x}(0)\right)\right|, \\ &\leq M\left|\partial_x S(x,\xi) - \partial_x S(x,0)\right|, \\ &\leq C_1 \lambda\left(x, \partial_x S\right), \end{aligned} \tag{12}$$

with $C_1 > 0$,

$$
\begin{aligned}
|x| &= \left|\pi_{2,\xi}^{-1}\left(\pi_{2,\xi}(x)\right) - \pi_{2,\xi}^{-1}\left(\pi_{2,\xi}(0)\right)\right|, \\
&\leq N\left|\partial_\xi S(x,\xi) - \partial_\xi S(0,\xi)\right|, \qquad (13)\\
&\leq C_2 \lambda\left(x, \partial_x S\right),
\end{aligned}
$$

with $C_2 > 0$,

$$
\begin{aligned}
\left|(x,\xi) - \left(x',\xi'\right)\right| &= \left|h_2^{-1}\left(h_2(x,\xi)\right) - h_2^{-1}\left(h_2\left(x',\xi'\right)\right)\right| \\
&\leq C\left|\left(\xi, \partial_\xi S(x,\xi)\right) - \left(\xi', \partial_\xi S\left(x',\xi'\right)\right)\right|,
\end{aligned}
$$

and from (13) and (14) we have

$$\lambda(x,y,\xi) \leq \lambda(x,\xi) + \lambda(y) \leq C_3\left(\lambda\left(\xi, \partial_\xi S\right) + \lambda(y)\right),$$

with $C_3 > 0$.

Also, we have $\partial_{y_j}\phi = -\xi_j$; and $\partial_{\xi_j}\phi = \partial_{\xi_j} S - y_j$, so

$$\lambda\left(\xi, \partial_\xi S\right) = \lambda\left(\partial_y \phi, \partial_\xi \phi + y\right) \leq 2\lambda\left(\partial_y \phi, \partial_\xi \phi, y\right),$$

which finally gives for some $C_4 > 0$,

$$\lambda(x,\xi,y) \leq C_3\left(2\lambda\left(\partial_y \phi, \partial_\xi \phi, y\right)\right) \leq \frac{1}{C_4}\lambda\left(\partial_y \phi, \partial_\xi \phi, y\right).$$

The second inequality in (H_3) is a consequence of (13).

By the same argument we can show (H_3'). □

Example 3.2. Consider the function given by

$$S(x,\xi) = \alpha_1 x^2 + \alpha_2 x\xi + \alpha_3 \xi^2 \quad \forall x,\xi \in \mathbb{R}$$

where $\alpha_i \in \mathbb{R}$ for all $i \in \{1,2,3\}$, $S(x,\xi)$ verifies (G_1) and (G_2).

Theorem 3.3. *Let F_h be the h-admissible Fourier integral operator defined by*

$$F_h u(x) = \iint e^{ih^{-1}(S(x,\xi) - \langle y,\xi\rangle)} a(x,\xi)\, u(y)\, dy\, \widehat{d\xi} \tag{14}$$

where $\widehat{d\xi} = (2\pi h)^{-n}\, d\xi$, $a \in \Gamma_k^m\left(\mathbb{R}_{x,\xi}^{2n}\right)$, $k \in \{0,1\}$ and S satisfies (G_1) and (G_2). Then $F_h F_h^*$ and $F_h^* F_h$ are h-pseudodifferential operators with symbol in $\Gamma_k^{2m}\left(\mathbb{R}^{2n}\right)$, $k \in \{0,1\}$, given by

$$\sigma\left(F_h F_h^*\right)\left(x, \partial_x S(x,\xi)\right) \equiv |a(x,\xi)|^2 \left|\left(\det \frac{\partial^2 S}{\partial\xi \partial x}\right)^{-1}(x,\xi)\right|,$$

$$\sigma\left(F_h^* F_h\right)\left(\partial_\xi S(x,\xi), \xi\right) \equiv |a(x,\xi)|^2 \left|\left(\det \frac{\partial^2 S}{\partial\xi \partial x}\right)^{-1}(x,\xi)\right|,$$

we denote here $a \equiv b$ for $a,b \in \Gamma_k^{2p}\left(\mathbb{R}^{2n}\right)$ if $(a-b) \in \Gamma_k^{2p-2}\left(\mathbb{R}^{2n}\right)$ and σ stands for the symbol.

Proof. See [2] or [3]. □

Corollary 3.4. *Let F_h be the h-admissible Fourier integral operator defined as (14). Then, we have*
(1) *For any m such that $m \leq 0$, F_h can be extended as a bounded linear mapping on $L^2(\mathbb{R}^n)$.*
(2) *For any m such that $m < 0$, F_h can be extended as a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. It follows from theorem (3.3) that $F_h^* F_h$ is an h-admissible operator with symbol in $\Gamma_0^{2m}\left(\mathbb{R}^{2n}\right)$.
(1) If $m \leq 0$, the weight $\lambda^{2m}(x,\theta)$ is a bounded, so we can apply the Calderon-vaillancourt theorem [7] for $F_h^* F_h$ and obtain the existence of a positive constant $\gamma(n)$ and a integer $k(n)$ such that

$$\|(F_h^* F_h)\, u\|_{L^2(\mathbb{R}^n)} \leq \gamma(n)\, Q_{k(n)}\left(\sigma\left(F_h F_h^*\right)\right) \|u\|_{L^2(\mathbb{R}^n)}\,,\ \forall u \in \mathcal{S}(\mathbb{R}^n)\,, \tag{15}$$

where

$$Q_{k(n)}\left(\sigma\left(F_h F_h^*\right)\right) = \sum_{|\alpha|+|\beta|\leq k(n)} \sup_{(x,\theta)\in\mathbb{R}^n} \left|\partial_x^\alpha \partial_\theta^\beta \sigma\left(F_h F_h^*\right)\left(\partial_\theta S(x,\theta),\theta\right)\right|.$$

Hence, for all $v \in S(\mathbb{R}^n)$,

$$\begin{aligned} \|F_h u\|^2_{L^2(\mathbb{R}^n)} &= \langle F_h u, F_h u\rangle_{L^2(\mathbb{R}^n)}, \\ &= \langle F_h^* F_h u, u\rangle_{L^2(\mathbb{R}^n)}, \\ &\leq \|F_h^* F_h u\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}\,, \end{aligned}$$

from (15), we obtain

$$\|F_h u\|_{L^2(\mathbb{R}^n)} \leq \left(\gamma(n)\, Q_{k(n)}\left(\sigma\left(F_h F_h^*\right)\right)\right)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}\,.$$

Thus F_h is also a bounded linear operator on $L^2(\mathbb{R}^n)$.
(2) If $m < 0$, $\lim_{|x|+|\theta|\to\infty} \lambda^m(x,\theta) = 0$, and the compactness theorem show that the operator $F_h^* F_h$ can be extended as a compact operator on $L^2(\mathbb{R}^n)$. Thus, the Fourier integral operator F_h is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\left\| F_h^* F_h - \sum_{j=1}^{n} \langle \varphi_j, .\rangle\, F_h^* F_h \varphi_j \right\| \to 0 \text{ as } n \to \infty.$$

Since F_h is bounded, for all $\psi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
&\left\| F_h\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle F_h\varphi_j \right\|^2_{L^2(\mathbb{R}^n)} \\
= &\left\langle F_h\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle F_h\varphi_j, F_h\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle F_h\varphi_j \right\rangle_{L^2(\mathbb{R}^n)}, \\
= &\left\langle F_h \left(\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right), F_h \left(\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right) \right\rangle_{L^2(\mathbb{R}^n)}, \\
= &\left\langle F_h^* F_h \left(\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right), \psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right\rangle_{L^2(\mathbb{R}^n)}, \\
\leq &\left\| F_h^* F_h \left(\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right) \right\|_{L^2(\mathbb{R}^n)} \left\| \psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right\|_{L^2(\mathbb{R}^n)}, \\
\leq &\left\| F_h^* F_h\psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle F_h^* F_h\varphi_j \right\|_{L^2(\mathbb{R}^n)} \left\| \psi - \sum_{j=1}^{n} \langle \varphi_j, \psi \rangle \varphi_j \right\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

it follows that

$$\left\| F_h\psi - \sum_{j=1}^{n} \langle \varphi_j, . \rangle F_h\varphi_j \right\|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } n \to \infty.$$

Finally, F_h is a compact linear operator on $L^2(\mathbb{R}^n)$. □

4. THE HILBERT-SCHMIDTNESS OF F_H

This section concerns another important set of bounded operators, namely the Hilbert-Schmidt operators. The class of Hilbert-Schmidt operators has a natural Hilbert space structure.

Definition 4.1. Let H_1 and H_2 be two Hilbert spaces. A bounded linear operator $A : H_1 \to H_2$ is called a Hilbert-Schmidt operator if for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in H_1 we have

$$\sum_{n=0}^{\infty} \|Ae_n\|^2_{H_2} < +\infty \tag{16}$$

The set of all Hilbert-Schmidt operators $A : H_1 \to H_2$ is denoted by $\mathcal{C}_2(H_1, H_2)$, or $\mathcal{C}_2(H)$ in case $H_1 = H_2 = H$.

Remark 4.2. The square root of the left-hand side of (16) is called the Hilbert-Schmidt norm or the Frobenius norm of the operator A and denoted by $\|.\|_{HS}$.

Proposition 4.3. *Let $A \in \mathcal{C}_2(H)$.*

1. *The Hilbert-Schmidt norm $\|.\|_{HS}$ is independent of the choice of orthonormal basis*
2. $\|A^*\|_{HS} = \|A\|_{HS}$
3. $\|A\| \leq \|A\|_{HS}$, *where $\|.\|$ is the usual operator norm.*
4. *every operator $A \in \mathcal{C}_2(H_1, H_2)$ is compact.*

Lemma 4.4. *If $T \in \mathcal{L}(H)$ then $AT, TA \in \mathcal{C}_2(H)$ and*

$$\max\{\|AT\|_{HS}, \|TA\|_{HS}\} \leq \|T\|.\|A\|_{HS} \tag{17}$$

Now let $\mathbb{R}^n$ be a space with positive measure and $H_1 = H_2 = L^2(\mathbb{R}^n)$. In this situation, the operators $A \in \mathcal{C}_2(H_1, H_2)$ are described as follows

Theorem 4.5. *The operators $A \in \mathcal{C}_2(L^2(\mathbb{R}^n))$ are exactly those which can be represented as*

$$Au(x) = \int_{\mathbb{R}^n} k(x,y)\, u(y)\, \mathrm{d}y, \tag{18}$$

with a kernel $k \in L^2(\mathbb{R}^{2n})$. We then also have

$$\|A\|_{HS} = \|k\|_{L^2(\mathbb{R}^n)}. \tag{19}$$

Now, we have the following result concerning the Hilbert-Schmidt class of h-admissible Fourier integral operators

Proposition 4.6. *Let us recall that F_h on $\mathbb{R}^n$, are integral operators of the form*

$$F_h u(x) = (2\pi h)^{-n} \iint \mathrm{e}^{\mathrm{i}h^{-1}(S(x,\xi) - \langle y,\xi\rangle)} a(x,\xi)\, u(y)\, \mathrm{d}y\mathrm{d}\xi, \tag{20}$$

where $a \in \Gamma_0^m\left(\mathbb{R}^{2n}_{x,\xi}\right)$, and S satisfies (G_1) and (G_2). Then,

For any m such that $m \leq -n$, F_h can be extended as a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$.

Proof. First, let us observe that h-admissible Fourier integral operator F_h, can be written as

$$F_h u(x) = \int \mathrm{e}^{\mathrm{i}h^{-1}S(x,\xi)} a(x,\xi)\, \mathcal{F}_h u(\xi)\, \mathrm{d}\xi,$$

where $\mathcal{F}_h$ is semi-classical Fourier transform defined by

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-ih^{-1}\langle y,\xi\rangle} u(y)\, \mathrm{d}y.$$

We put

$$F_h = A_h(S)\, \mathcal{F}_h, \tag{21}$$

clearly we have

$$A_h(S)\,u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}h^{-1}S(x,\xi)} a(x,\xi)\,u(\xi)\,\mathrm{d}\xi. \tag{22}$$

It results so from (21) and using lemma 4.4 we have

$$\|F_h\|_{HS} = \|A_h(S)\,\mathcal{F}_h\|_{HS} \leq \|A_h(S)\|_{HS}\,\|\mathcal{F}_h\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

Now, it is enough to prove $A_h(S) \in \mathcal{C}_2\left(L^2(\mathbb{R}^n)\right)$.First, let us observe that $A_h(S)$ has a integral representation just as (18) with kernel $k_h(x,\xi)$. In fact, straightforward computation shows us that

$$A_h(S)\,u(x) := \int_{\mathbb{R}^n} k_h(x,\xi)\,u(\xi)\,\mathrm{d}\xi, \tag{23}$$

where

$$k_h(x,\xi) := \mathrm{e}^{\mathrm{i}h^{-1}S(x,\xi)} a(x,\xi). \tag{24}$$

Now let us show that $k_h \in L^2\left(\mathbb{R}^{2n}\right)$:

$$|k_h(x,\xi)| = \left|\mathrm{e}^{\mathrm{i}h^{-1}S(x,\xi)} a(x,\xi)\right| = |a(x,\xi)|\,, \leq C_{0,0}\lambda^m(x,\xi).$$

Hence $|k_h(x,\xi)|^2 \leq C_{0,0}^2\lambda^{2m}(x,\xi)$. We deduce from lemma 2.2 that, for all $m < -n$, $k_h \in L^2\left(\mathbb{R}^{2n}\right)$, and from (19) we have

$$\|A_h(S)\|_{HS} = \|k_h\|_{L^2(\mathbb{R}^{2n})} < +\infty, \tag{25}$$

which proves that $F_h \in \mathcal{C}_2\left(L^2(\mathbb{R}^n)\right)$. □

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