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IDEALIZABILITY OF SOME EXPANSIONS OF OPEN SETS

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Abstract. Recently, Abbas has introduced and investigated the notion of h-open sets. In this paper, to generalize h-open sets, we introduce hI-open sets in an ideal topological space (X, τ, I), and we obtain some properties of hI-open sets. We also introduce and investigate hI-continuous functions in an ideal topological space.

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1. INTRODUCTION

Open sets, weak and strong forms of open sets play an important role in general topology. They have been researched by many researchers worldwide from the past to the present. The structure of ideals in topological spaces was discussed by Kuratowski [7, 8] and Vaidyanathaswamy [13, 14]. Janković and Hamlett [5, 6] compared the compatibility of ideal topological spaces with topologies and extended many topologically important properties with idealizations.

Recently, Abbas [1] has introduced and investigated the notion of h-open sets in a topological space. Further, h-continuous functions, h-irresolute, and other functions have been introduced and studied.

In this paper, we introduce the notion of hI-open sets in an ideal topological space and obtain some properties of the hI-open sets. Furthermore, we investigate the notion of hI-continuous functions and hI-irresolute functions.

2. PRELIMINARIES

This section provides some basic notions to be needed for the following sections. Throughout this paper, (X, τ) denotes a topological space. Let A be a subset of X. By $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ we denote the closure and the interior of a subset A of X respectively. An ideal I on (X, τ) is defined as follows: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$ [8]. An ideal topological space is a topological space (X, τ)

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with an ideal I on X, denoted by (X, τ, I). For a subset $A \subseteq X$, $A^\star(I, \tau) = \{x \in X : V \cap A \notin I$ for all open neighbourhood V of $x\}$ is called the local function of A with respect to I and τ [8]. We write $A^\star$ instead of $A^\star(I, \tau)$ in case there is no chance for confusion. Note that $X^\star$ is often a proper subset of X. The hypothesis $X = X^\star$ [5] is equivalent to the hypothesis $\tau \cap I = \emptyset$ [5]. For each ideal topological space (X, τ, I), there exists a topology $\tau^\star(I) = \{U \subseteq X : \mathrm{Cl}^\star(X - U) = (X - U)\}$, finer than τ, generated by $\mathrm{Cl}^\star(A) = A \cup A^\star$ which defines a Kuratowski closure operator for $\tau^\star$.

We recall some well-known definitions.

Definition 2.1. A subset A of a topological space (X, τ) is said to be
i) *α-open* if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ [11].
ii) *pre-open* if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$ [10].
iii) *semi-open* if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$ [9].
iv) *β-open* if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ [2].

The α-interior, $\alpha\mathrm{int}(A)$, of A is defined as follows:

$$\alpha\mathrm{int}(A) = \bigcup\{U \subseteq X : U \subseteq A,\ U \textit{ is } \alpha\textit{-open}\}.$$

The pre-interior $\mathrm{pint}(A)$, semi-interior $\mathrm{sint}(A)$, β-interior $\beta\mathrm{int}(A)$ are similarly defined.

Definition 2.2 ([1]). A subset A of a topological space (X, τ) is a said to be h-open if $A \subseteq \int(A \cup V)$ for all non-empty open set V in X such that $V \neq X$. The family of all h-open sets in (X, τ) is denoted by τ^h or $hO(X)$.

3. *hI*-OPEN SETS WITH IDEALIZATION

Definition 3.1. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be hI-open if $A \subseteq \mathrm{int}(A \cup \mathrm{Cl}^\star(V))$ for all $V \in \tau$ such that $\emptyset \neq V \neq X$. The complement of an hI-open set is said to be hI-closed. The family of all hI-open sets of an ideal topological space (X, τ, I) is denoted by $hIO(X)$ or τ^{hI}.

Remark 3.2. In [1], every open set is h-open, but the converse is not always true. Also, it is shown in Theorem 2.1 of [3] that the family of all h-open sets is a topology.

Remark 3.3. Let (X, τ, I) be an ideal topological space. In Definition 3.1, for all $V \in \tau$ such that $\emptyset \neq V \neq X$, can we obtain a new type of h-open sets by taking the local function of V instead of the *-closure of V?

Theorem 3.4. *Let (X, τ, I) be an ideal topological space and A be a subset of X. If A is an h-open set, then A is hI-open.*

Proof. Let A be an h-open set. Since $\mathrm{Cl}^\star(V) = V^\star \cup V, A \subseteq \mathrm{int}(A \cup V) \subseteq \mathrm{int}(A \cup \mathrm{Cl}^\star(V))$. Therefore, every h-open set is hI-open. $\square$

The converse of Theorem 3.4 is not always true as shown by the following example.

EXAMPLE 3.5. $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $\tau^h = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. If we take $A = \{a, b\}$, $A \in \tau^{hI}$. But A is not h-open.

PROPOSITION 3.6. *Let (X, τ, I) be an ideal topological space. If $I = P(X)$, then $\tau^h = \tau^{hI}$.*

Proof. Let $I = P(X)$, then $S^\star = \emptyset$ for all subset S of X. Now, let A be any hI-open set of X. Then $A \subseteq \text{Int}(A \cup \text{Cl}^\star(V)) = \text{Int}(A \cup (V \cup V^\star)) = \text{Int}(A \cup V)$ and hence A is h-open. By Theorem 3.4, we obtain $\tau^h = \tau^{hI}$. □

THEOREM 3.7. *Let (X, τ, I) be an ideal topological space. Then the family τ^{hI} of all hI-open sets is a topology for X.*

Proof. (1) It is clear that $\emptyset, X \in \tau^{hI}$.

(2) We show that if $U, V \in \tau^{hI}$, then $U \cap V \in \tau^{hI}$. Let V_1 be any open set such that $\emptyset \neq V_1 \neq X$. Then, from Definition 3.1, $U \subseteq \int(U \cup \text{Cl}^\star(V_1))$ and $V \subseteq \int(V \cup \text{Cl}^\star(V_1))$. Therefore $U \cap V \subseteq \int(U \cup \text{Cl}^\star(V_1)) \cap \int(V \cup \text{Cl}^\star(V_1)) = \int((U \cup \text{Cl}^\star(V_1)) \cap (V \cup \text{Cl}^\star(V_1))) \subseteq \int((U \cap V) \cup \text{Cl}^\star(V_1))$ and hence $U \cap V \subseteq \int((U \cap V) \cup \text{Cl}^\star(V_1))$. This shows that $U \cap V \in \tau^{hI}$.

(3) We show that if $A_\alpha \in \tau^{hI}$ $\alpha \in \Delta$, then $\bigcup\{A_\alpha : \alpha \in \Delta\} \in \tau^{hI}$. Let $A_\alpha \in \tau^{hI}$ for all $\alpha \in \Delta$. Then $A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$ for all $\alpha \in \Delta$. For all $V \in \tau$ ($\emptyset \neq V \neq X$) and each $\alpha \in \Delta$, we get $A_\alpha \subseteq \int(A_\alpha \cup \text{Cl}^\star(V)) \subseteq Int[(\cup A_\alpha) \cup \text{Cl}^\star(V)]$. Hence, we have $\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \text{Int}[(\cup A_\alpha) \cup \text{Cl}^\star(V)]$. Therefore, $\bigcup_{\alpha\in\Delta} A_\alpha \in \tau^{hI}$. □

4. GENERALIZATIONS OF *hI*-OPEN SETS

DEFINITION 4.1. A subset A of an ideal topological space (X, τ, I) is said to be

i) $h\alpha$I-open if $A \subseteq \alpha\text{int}(A \cup \text{Cl}^\star(V))$, for all $V \in \tau$ such that $\emptyset \neq V \neq X$,
ii) hpI-open if $A \subseteq \text{pint}(A \cup \text{Cl}^\star(V))$, for all $V \in \tau$ such that $\emptyset \neq V \neq X$,
iii) hsI-open if $A \subseteq \text{sint}(A \cup \text{Cl}^\star(V))$, for all $V \in \tau$ such that $\emptyset \neq V \neq X$,
iv) $h\beta$I-open if $A \subseteq \beta\text{int}(A \cup \text{Cl}^\star(V))$, for all $V \in \tau$ such that $\emptyset \neq V \neq X$.

From Definition 4.1, we have the following diagram:

$$\begin{array}{ccccccc} h\text{-open} & \rightarrow & hI\text{-open} & \rightarrow & h\alpha I\text{-open} & \rightarrow & hpI\text{-open} \\ & & & & \downarrow & & \downarrow \\ & & & & hsI\text{-open} & \rightarrow & h\beta I\text{-open} \end{array}$$

In the above definition, for all $V \in \tau$ such that $\emptyset \neq V \neq X$, can new types of h-open set be given by taking the local function of V instead of the *-closure of V and *-interior operator instead of interior operations (resp. αInt, pInt, sInt, βInt)?

REMARK 4.2. The converses of the implications in the diagram above may not be true.

EXAMPLE 4.3. It can be seen from Example 3.5 that not every hI-open set is an h-open set.

EXAMPLE 4.4. $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,d\}, \{a,c,d\}\}$ and $I = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$. Then, we have $\alpha(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,d\}, \{a,c,d\}, \{a,b,c\}\}$, $\tau^{hI} = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,d\}, \{a,c,d\}\}$ and we have $h\alpha I(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,d\}, \{a,c,d\}, \{a,b,c\}\}$. So, $\{a,b,c\} \in h\alpha I(X)$, but it is not in τ^{hI}.

EXAMPLE 4.5. $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{d\}, \{a,c\}, \{a,c,d\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $PO(X) = \{\emptyset, X, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}\}$, $SO(X) = \{\emptyset, X, \{d\}, \{a,c\}, \{b,d\}, \{a,c,d\}, \{a,b,c\}\}$ and $\alpha O(X) = \{\emptyset, X, \{d\}, \{a,c\}, \{a,c,d\}\}$. Therefore, we have $\{b,d\} \in hSIO(X) \setminus hPIO(X)$ and $\{c,d\} \in hPIO(X) \setminus hSIO(X)$. This shows that hsI-open sets and hpI-open sets are independent.

5. *hI*-INTERIOR AND *hI*-CLOSURE OPERATORS

DEFINITION 5.1. Let (X, τ, I) be an ideal topological space and a subset A of X. The hI-interior, $\mathrm{Int}_{hI}(A)$, is defined as follows:

$$\mathrm{Int}_{hI}(A) = \bigcup\{U : U \in \tau^{hI},\ U \subseteq A\}.$$

THEOREM 5.2. *Let (X, τ, I) be an ideal topological space and M, N be subsets of X. Then, the following statements are hold:*
i) *If $M \subseteq N$, then* $\mathrm{Int}_{hI}(M) \subseteq \mathrm{Int}_{hI}(N)$
ii) $\mathrm{Int}_{hI}(M) \subseteq M$
iii) $\mathrm{Int}_{hI}(\mathrm{Int}_{hI}(M)) = \mathrm{Int}_{hI}(M)$
iv) *M is hI-open if and only if* $M = \mathrm{Int}_{hI}(M)$
v) $\mathrm{Int}_{hI}(M) \cap \mathrm{Int}_{hI}(N) = \mathrm{Int}_{hI}(M \cap N)$
vi) $\mathrm{Int}_{hI}(M) \cup \mathrm{Int}_{hI}(N) \subseteq \mathrm{Int}_{hI}(M \cup N)$

Proof. It is clear. □

DEFINITION 5.3. Let (X, τ, I) be an ideal topological space and $x \in X$. Every hI-open set containing x is called an hI-open neighborhood of x.

DEFINITION 5.4. Let (X, τ, I) be an ideal topological space and $x \in X$. A subset N of X is called an hI-neighborhood of x if there exists an $U \in \tau^{hI}$ such that $x \in U \subseteq N$.

REMARK 5.5. Every hI-open neighborhood is an hI-neighborhood, but the converse is not true, as shown by the following example.

EXAMPLE 5.6. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$ and $I = \{\emptyset, \{a\}\}$. Then, for $x = a \in X$, {a, c} is an hI-neighborhood of x but it is not an hI-open neighborhood of x.

Definition 5.7. Let (X, τ, I) be an ideal topological space and a subset A of X. The hI-closure of A, $\mathrm{cl}_{hI}(A)$, is defined as follows:

$$\mathrm{cl}_{hI}(A) = \bigcap\{F : F \text{ is } hI\text{-closed}, A \subseteq F\}.$$

A point x of an ideal topological space (X, τ, I) is called an hI-cluster point of a subset A of X if $V \cap A \neq \emptyset$ for each hI-open set V containing x. The set of all hI-cluster points of A equals the hI-closure of A.

Theorem 5.8. *Let (X, τ, I) be an ideal topological space and M, N be subsets of X. Then the following properties hold:*
i) *If $M \subseteq N$, then $\mathrm{cl}_{hI}(M) \subseteq \mathrm{cl}_{hI}(N)$,*
ii) *$M \subseteq \mathrm{cl}_{hI}(M)$,*
iii) *$\mathrm{cl}_{hI}(\mathrm{cl}_{hI}(M)) = \mathrm{cl}_{hI}(M)$,*
iv) *M is hI-closed if and only if $M = \mathrm{cl}_{hI}(M)$,*
v) *$\mathrm{cl}_{hI}(M \cap N) \subseteq \mathrm{cl}_{hI}(M) \cap \mathrm{cl}_{hI}(N)$,*
vi) *$\mathrm{cl}_{hI}(M) \cup \mathrm{cl}_{hI}(N) = \mathrm{cl}_{hI}(M \cup N)$.*

Theorem 5.9. *Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties hold:*
i) *X-$\mathrm{cl}_{hI}(A) = \mathrm{Int}_{hI}(X - A)$,*
ii) *X-$\mathrm{Int}_{hI}(A) = \mathrm{cl}_{hI}(X - A)$.*

Proof. Straightforward. □

6. *hI*-CONTINUOUS FUNCTIONS

Definition 6.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be h-continuous [1] if for each open set $V \in \sigma$, $f^{-1}(V)$ is h-open in X.

Definition 6.2. A function $f:(X, \tau, I) \to (Y, \sigma)$ is said to be hI-continuous [3] if for each open set $V \in \sigma$, $f^{-1}(V)$ is hI-open in (X, τ, I).

Theorem 6.3. *Every h-continuous function is hI-continuous.*

Proof. By Theorem 3.4, every h-open set is hI-open, and the proof is obvious. □

Example 6.4. Let $X = \{p, q, r\}$, $\tau = \{\emptyset, X, \{q\}, \{q, r\}\}$, $I = \{\emptyset, \{p\}\}$ and $\sigma = \{\emptyset, X, \{p, q\}\}$. Let $f : (X, \tau, I) \to (X, \sigma)$ be the identity function. Then, it is a hI-continuous function but not h-continuous. We have the collection $\tau^h = \{\emptyset, X, \{q\}, \{r\}, \{p, r\}, \{q, r\}\}$. For $\{p, q\} \in \sigma$, $f^{-1}(\{p, q\}) = \{p, q\} \in hIO(X)$ but $\{p, q\} \notin \tau^h$. Therefore f is hI-continuous but is not h-continuous.

Definition 6.5. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $h\alpha I$-continuous (resp. hpI-continuous, hsI-continuous, $h\beta I$-continuous) if for each $V \in \sigma$, $f^{-1}(V)$ is $h\alpha I$-open (resp. hpI-open, hsI-open, $h\beta I$-open) in (X, τ, I).

By Definitions 6.1, 6.2, and 6.5, the following implications hold:

$$\begin{array}{ccccccc} h\text{-continuous} & \rightarrow & hI\text{-continuous} & \rightarrow & h\alpha I\text{-continuous} & \rightarrow & hpI\text{-continuous} \\ & & & & \downarrow & & \downarrow \\ & & & & hsI\text{-continuous} & \rightarrow & h\beta I\text{-continuous} \end{array}$$

REMARK 6.6. The examples below show that none of the implications in the above diagram is reversible.

EXAMPLE 6.7. Let $X = \{p,q,r,s\}$, $\tau = \{\emptyset, X, \{q\}, \{r,s\}, \{q,r,s\}\}$, $I = \{\emptyset, \{q\}\}$ and $\sigma = \{\emptyset, X, \{p,q\}\}$. The identity function $f : (X,\tau,I) \to (X,\sigma)$ is $h\alpha I$-continuous but it is not hI-continuous. We find the collection $\alpha O(X) = \{\emptyset, X, \{q\}, \{r,s\}, \{q,r,s\}\}$. By Definition 6.5, since $f^{-1}(\{p,q\}) = \{p,q\} \in h\alpha IO(X)$ but $\{p,q\} \notin \tau^{hI}$. Hence f is $h\alpha I$-continuous but is not hI-continous.

EXAMPLE 6.8. Let $X = \{p,q,r,s\}$, $\tau = \{\emptyset, X, \{s\}, \{p,r\}, \{p,r,s\}\}$, $I = \{\emptyset, \{p\}\}$ and $\sigma = \{\emptyset, X, \{p,r\}\}$. The function $f : (X,\tau,I) \to (X,\sigma)$ is defined by $f(\{p\}) = \{q\}$, $f(\{q\}) = \{r\}$, $f(\{r\}) = \{s\}$, $f(\{s\}) = \{p\}$. For $\{p,r\} \in \sigma$, $f^{-1}(\{p,r\}) = \{q,s\} \in hSIO(X)$ but $\{q,s\} \notin hPIO(X)$. Hence f is hsI-continuous but not hpI-continuous.

EXAMPLE 6.9. Let the (X,τ,I) ideal topological space be as in Example 6.8 and let $\sigma = \{\emptyset, X, \{r,s\}\}$. The identity function $f : (X,\tau,I) \to (X,\sigma)$ is hpI-continuous but it is not hsI-continuous. For $\{r,s\} \in \sigma$, $f^{-1}(\{r,s\}) = \{r,s\} \in hPIO(X)$ but $\{r,s\} \notin hSIO(X)$.

COROLLARY 6.10. *A function $f : (X,\tau,I) \to (Y,\sigma)$ is hI-continuous if and only if a function $f : (X,\tau^{hI}) \to (Y,\sigma)$ is continuous.*

Proof. This is an immediate consequence of Theorem 3.7. □

THEOREM 6.11. *A function $f : (X,\tau,I) \to (Y,\sigma)$ is hI-continuous and $g : (Y,\sigma) \to (R,\eta)$ is continuous, then $gof : (X,\tau,I) \to (R,\eta)$ is hI-continuous.*

Proof. It is clear. □

By using hI-neighborhoods, hI-open sets, hI-closed sets, hI-interior and hI-closure, we obtain characterizations of hI-continuous functions.

LEMMA 6.12. *Let (X,τ,I) be an ideal topological space. A subset B is hI-closed if and only if* $\mathrm{cl}(B \cap \mathrm{Int}^\star(F)) \subseteq B$ *for all closed set F of X such that $\emptyset \neq F \neq X$.*

Proof. B is hI-closed if and only if $X - B$ is hI-open. By Definition 3.1, $(X - B) \subseteq$ Int[((X-B)) $\cup$Cl$^\star(V)$] for all $V \in \tau$ such that $\emptyset \neq V \neq X$. This is equivalent to X-Int$[((X - B)) \cup \mathrm{Cl}^\star(V)] \subseteq B$. Now, we have X-Int$[((X - B)) \cup \mathrm{Cl}^\star(V)] = \mathrm{cl}(X - [((X - B)) \cup \mathrm{Cl}^\star(V)])$=cl$(B \cap (X - \mathrm{Cl}^\star(V))$ =cl$(F \cap Int^\star(X - V))$. Therefore, we obtain $\mathrm{cl}(B \cap \mathrm{Int}^\star(F)) \subseteq B$ for all closed set F of X such that $\emptyset \neq F \neq X$. □

THEOREM 6.13. *For a function $f : (X,\tau,I) \to (Y,\sigma)$, the following properties are equivalent:*

i) *f is hI-continuous;*

ii) *For each point $x \in X$ and each open $V \subseteq Y$ containing $f(x)$, there exists $W \in hIO(X)$ such that $x \in W$, $f(W) \subseteq V$;*

iii) *For each point $x \in X$ and each open set V of Y containing $f(x)$, there exists an hI-neighorhood U of x such that $f(U) \subseteq V$;*

iv) *The inverse image of each closed set in Y is hI-closed;*

v) *For each closed set B of Y, $\mathrm{cl}(f^{-1}(B) \cap \mathrm{Int}^\star(F)) \subseteq f^{-1}(B)$ for all closed set in X such that $\emptyset \neq F \neq X$;*

vi) *For each subset B of Y, $\mathrm{cl}(f^{-1}(\mathrm{cl}(B)) \cap \mathrm{Int}^\star(F)) \subseteq f^{-1}(\mathrm{cl}(B))$ for all closed set F in X such that $\emptyset \neq F \neq X$;*

vii) *For each subset A of X, $f(\mathrm{Cl}[A \cap \mathrm{Int}^\star(F)]) \subseteq \mathrm{cl}(f(A))$ for all closed set F in X such that $\emptyset \neq F \neq X$;*

viii) *For each subset B of Y, $\mathrm{cl}_{hI}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$;*

ix) *For each subset B of Y, $f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{Int}_{hI}(f^{-1}(B))$.*

Proof. i) $\Rightarrow$ ii): Let $x \in X$ and V be any open set of Y containing $f(x)$. Set $W = f^{-1}(V)$, then by Definition 6.5, W is an hI-open set containing x and $f(W) \subseteq V$.

ii) $\Rightarrow$ iii): Every hI-open set containing x is an hI-neighborhood of x and the proof is obvious.

iii) $\Rightarrow$ i): Let V be any open set in Y. For each $x \in f^{-1}(V)$, $f(x) \in V \in \sigma$. By (iii) there exists an hI-neighborhood N of x such that $f(N) \subseteq V$; hence $x \in N \subseteq f^{-1}(V)$. There exists $U_x \in \tau^{hI}$ such that $x \in U_x \subseteq N \subseteq f^{-1}(V)$. Hence $f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\} \in \tau^{hI}$. This shows that f is hI-continuous.

i) $\Rightarrow$ iv) $\Rightarrow$ v) $\Rightarrow$ i): By Lemma 6.12, the proof is obvious.

v) $\Rightarrow$ vi): For each subset B of Y, $\mathrm{cl}(B)$ is closed in Y and the proof is obvious.

vi) $\Rightarrow$ vii): Let A be any subset of X. Set $B = f(A)$, then by (vi)

$$\mathrm{Cl}[A \cap \mathrm{Int}^\star(F)] \subseteq \mathrm{Cl}[f^{-1}(\mathrm{cl}(f(A))) \cap \mathrm{Int}^\star(F)] \subseteq f^{-1}(\mathrm{cl}(f(A)))$$

for all closed set F in X such that $\emptyset \neq F \neq X$. Therefore, we obtain for each subset A of X,

$$f(\mathrm{Cl}[A \cap \mathrm{Int}^\star(F)]) \subseteq \mathrm{cl}(f(A))$$

for all closed set F in X such that $\emptyset \neq F \neq X$.

vii) $\Rightarrow$ i): Let V be any open set of Y. Then, $Y - V$ is closed in Y. Set $A = f^{-1}(Y - V)$, then by (vii)

$$f(\mathrm{Cl}[f^{-1}(Y - V) \cap \mathrm{Int}^\star(F)]) \subseteq \mathrm{cl}(f(f^{-1}(Y - V)))) = Y - V$$

for all closed set F in X such that $\emptyset \neq F \neq X$. Therefore, we have

$$\begin{aligned}\mathrm{Cl}[f^{-1}(Y - V) \cap \mathrm{Int}^\star(F)] &\subseteq f^{-1}(f(\mathrm{Cl}[f^{-1}(Y - V) \cap \mathrm{Int}^\star(F)]) \\ &\subseteq f^{-1}(Y - V) = X - f^{-1}(V).\end{aligned}$$

Therefore,

$$\begin{aligned} f^{-1}(V) \subseteq X - \mathrm{Cl}[f^{-1}(Y-V) \cap \mathrm{Int}^\star(F)] &= \mathrm{Int}[X - (f^{-1}(Y-V) \cap \mathrm{Int}^\star(F))] \\ &= \mathrm{Int}[f^{-1}(V) \cup (X - \mathrm{Int}^\star(F))] \\ &= \mathrm{Int}[f^{-1}(V) \cup \mathrm{Cl}^\star(X-F)] \\ &= \mathrm{Int}[f^{-1}(V) \cup \mathrm{Cl}^\star(U)] \end{aligned}$$

for all open set U of X such that $\emptyset \neq U \neq X$.

iv) $\Rightarrow$ viii): Let B be any subset of Y. By (iv) $f^{-1}(\mathrm{cl}(B))$ is hI-closed in X and $f^{-1}(B) \subseteq f^{-1}(\mathrm{cl}(B))$. Therefore, $\mathrm{cl}_{hI}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$.

viii) $\Rightarrow$ ix): Let B be any subset of Y. Then, $f^{-1}(\int(B)) = f^{-1}(Y - \mathrm{cl}(Y - B)) = X - f^{-1}(\mathrm{cl}(Y-B)) \subseteq X - \mathrm{cl}_{hI}(f^{-1}(Y-B)) = X - \mathrm{cl}_{hI}(X - f^{-1}(B)) = \mathrm{Int}_{hI}(f^{-1}(B))$.

ix) $\Rightarrow$ i): Let V be any open set of Y. By (ix), $f^{-1}(V) \subseteq \mathrm{Int}_{hI}(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore, we have $\mathrm{Int}_{hI}(f^{-1}(V)) = f^{-1}(V)$ and hence f is hI-continuous. □

Definition 6.14. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be hI-irresolute if for each hJ-open set V in Y, $f^{-1}(V)$ is hI-open in X.

Theorem 6.15. *If a function $f : (X, \tau, I) \to (Y, \sigma, J)$ is hI-irresolute, then f is hI-continuous.*

The converse of Theorem 6.15 is not always true, as shown by the following example.

Example 6.16. Let $X = \{p, q, r, s\}$, $\tau = \{\emptyset, X, \{q\}, \{r, s\}, \{q, r, s\}\}$, I=$\{\emptyset, \{q\}\}$, $Y = \{p, q, r\}$, $\sigma = \{\emptyset, Y, \{p\}, \{q, r\}\}$, $J = \{\emptyset, \{p\}\}$ and a function $f : (X, \tau, I) \to (Y, \sigma, J)$ be defined as below: $f(p) = q$, $f(q) = p$, $f(r) = r$, $f(s) = q$. Then $\tau^{hI} = \{\emptyset, X, \{q\}, \{p, r\}, \{p, r, s\}\}$ and $\sigma^{hJ} = \{\emptyset, Y, \{p\}, \{p, r\}, \{q, r\}\}$. For $\{p, r\} \in \sigma^{hJ}$, $f^{-1}(\{p, r\}) = \{q, r\} \notin \tau^{hI}$. Therefore, f is hI-continuous but not hI-irresolute.

But every hI-irresolute function is not always continuous, as shown by the following example.

Example 6.17. Let $X = \{p, q, r\}$, $\tau = \{\emptyset, X, \{q\}, \{q, r\}\}$, $I = \{\emptyset, \{q, r\}\}$, $Y = \{p, q\}$, $\sigma = \{\emptyset, Y, \{p\}\}$, $J = \{\emptyset, \{q\}\}$ and a function $f : (X, \tau, I) \to (Y, \sigma, J)$ is identity. Then $\tau^{hI} = \{\emptyset, X, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$ and $\sigma^{hJ} = \{\emptyset, Y, \{p\}, \{q\}\}$. Therefore, f is hI-irresolute but not continuous.

It is shown in [1, Theorem 3.5] that every continuous function is h-irresolute. But this theorem is false, as demonstrated by the following example.

Example 6.18. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{a, c\}\}$, and $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then $\tau^h = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma^h = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Therefore, f is continuous but not h-irresolute.

REMARK 6.19. By Example 6.17, 6.18, and Proposition 3.6, we obtain that continuity and hI-irresolute are independent.

COROLLARY 6.20. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is hI-irresolute if and only if $f : (X, \tau^{hI}) \to (Y, \sigma^{hJ})$ is h-continuous.*

Proof. This is an immediate consequence of Theorem 3.7. □

DEFINITION 6.21. A function $f : (X, \tau) \to (Y, \sigma, J)$ is said to be hI-open if $f(T)$ is hI-open in Y for all open set T in X.

PROPOSITION 6.22. *Every open function is hI-open.*

REMARK 6.23. As seen from Example 4.4, the converse of Proposition 6.22 may not always be true.

THEOREM 6.24. *A function $f : (X, \tau) \to (Y, \sigma, J)$ is hI-open if and only if for each subset $W \subseteq Y$ and each closed set F of X containing $f^{-1}(W)$, there exists an hI-closed set $H \subseteq Y$ containing W such that $f^{-1}(H) \subseteq F$.*

Proof. Necessity. Let $H = Y - f(X - F)$. Since $f^{-1}(W) \subseteq F$, we have $f(X - F) \subseteq Y - W$. Since f is hI-open, then H is hI-closed and $f^{-1}(H) = X - f^{-1}(f(X - F)) \subseteq X - (X - F) = F$.

Sufficiency. Let U be any open set of X and $W = Y - f(U)$. Then $f^{-1}(W) = X - f^{-1}(f(U)) \subseteq X - U$ and $X - U$ is closed. By the hypothesis, there exists an hI-closed set H of Y containing W such that $f^{-1}(H) \subseteq X - U$. Then, we have $H \subseteq Y - f(U)$. Therefore, we obtain $Y - f(U) \subseteq H \subseteq Y - f(U)$ and $f(U)$ is hI-open in Y. This shows that f is hI-open. □

PROPOSITION 6.25. *A function $f : (X, \tau) \to (Y, \sigma)$ is open and $g : (Y, \sigma) \to (Z, \eta, J)$ is hI-open, then $gof : (X, \tau) \to (Z, \eta, J)$is hI-open.*

7. CONCLUSION

As a generalization of h-open sets, we defined hI-open sets in an ideal topological space (X, τ, I). And we obtained some properties of hI-open sets. We introduced and investigated hI-continuous functions on an ideal topological space. Moreover, we examine the notion of hI-continuous functions and hI-irresolute functions. Furthermore, we offer two open problems in this study.

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