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COUNTING FORMULAS FOR CERTAIN p-SUBGROUPS OF $GL_n(\mathbb{F}_p)$

NOUREDDINE SNANOU

Abstract. Let p be a prime number and $\mathbb{F}_p$ a finite field of order p. Let $GL_n(\mathbb{F}_p)$ denote the general linear group and let U_n denote the unitriangular group of $n\times n$ upper triangular matrices with ones on the diagonal, over the finite field $\mathbb{F}_p$. This is a finite group of order $p^{\frac{n(n-1)}{2}}$ and a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$. In this work, we characterize some p-subgroups of $GL_n(\mathbb{F}_p)$ with respect to a given property. By the Sylow theorems, every p-subgroup of $GL_n(\mathbb{F}_p)$ is contained in some Sylow p-subgroup of $GL_n(\mathbb{F}_p)$ and then it is conjugate to a p-subgroup of U_n, which is why we characterize the p-subgroups of U_n. More precisely, we compute the number of T-invariant p-subgroups of U_n, where T is the diagonal subgroup of $GL_n(\mathbb{F}_p)$. Furthermore, for $n \leq p$, we obtain an interesting formula which computes the number of abelian p-subgroups of order p^t in U_n where $t \leq \left[\frac{n^2}{4}\right]$.

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Key words. General linear group, unitriangular group, T-invariant p-subgroup, elementary abelian subgroup, conjugacy classes.

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Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Department of Mathematics
Fez, Morocco
E-mail: noureddine.snanou@usmba.ac.ma
https://orcid.org/0000-0002-9437-8332