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THE FRACTALS AND THE STONES FROM THE SINAI MOUNTAIN

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Abstract. In this paper we present some information about famous fractals that have been studied during time. A fractal example is provided and analyzed with the Diffusion-Limited Aggregation model and generated in Python code.
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Key words. Fractal, Sierpinski triangle, Cantor dust, Koch curve, Mandelbrot set, fractal dimension, Diffusion-Limited Aggregation model, Python.

1. INTRODUCTION

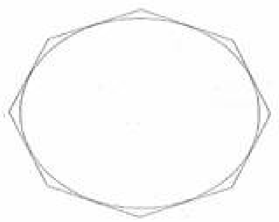
From early times, it has been known that Euclid proposed a geometry based on logic and intuitive truths. Almost two decades later, in 1600, René Descartes changed everything in Euclidean geometry, suggesting that the physical space can be split and measured with the help of three right angles axes, by locating every single point from the space in these three linear dimensions (see, for example [15, 17, 19]). The idea that the Universe can be imagined such as a multitude of small cubes became the modern science fundamental of the world.

A century later, Gottfried Wilhelm von Leibniz and Sir Isaac Newton continued this idea, by making a dangerous and revolutionary assumption. Initially, they couldn't give the mathematical proof: any curve is formed of an infinite number of segments, called tangents. So, the differential calculus was discovered. The main idea was that every single curve increased to infinite, and similar to a segment, has the limit of this process exactly that line.

Leibniz couldn't explain why the theory was giving correct results in most of the cases. Even if he had abandoned the idea of the infinite segment, this one remained in usage, because it was offering good results for the majority of the cases. So, the assumption that the curves are similar to the segments was kept over time. Also, there have been discovered other mathematical forms that were impossible to become linear.

In 1875, the German mathematician Karl Weierstrass described a continuous curve that couldn't be differentiated, even if it wasn't possible to have any tangent to it (see, as an example, Figure 1.1 - generated). After that, interesting curves appeared and were called *Monsters Gallery.*

Fig. 1.1 – Curves approximation with tangent lines

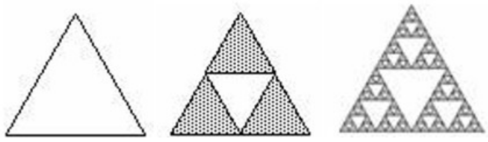


2. FAIMOUS FRACTALS

In the mind's eyes, a fractal is a way of seeing the infinite. was the phrase from where everything started. A lot of scientists wanted to discover new elements with useful properties (see, for example, [3, 9, 11]).

2.1. Sierpinski Triangle. The Polish mathematician Waclaw Sierpinski started his research by considering a triangle that was divided in four equal parts. Then, he applied the same division for the three marginal parts and continued in the same manner. The figure that he obtained is known as *Sierpinski Triangle* (Figure 2.2).

Fig. 2.2 – Sierpinski Triangle [5]



Another possible way to construct the same shape starts with an entire triangle, from where are cropped identical goals, instead of drawing lines. The result is the same, but this new form is called *Sierpinski Sieve* (Figure 2.3).

Fig. 2.3 – Sierpinski Triangle [5]

Serpienski Carpet has another shape and is represented in two ways, Figure 2.4 or Figure 2.5 (see, for example [7, 12]).

Related to these two figures the surface question arose. Due to the fact that there are only segments and mathematical there is no surface to calculate for them, the mathematicians have assumed that each figure should have the surface 0, but they couldn't prove it.

The Italian mathematician Giuseppe Peano, an extraordinary infinite calculus professor, at Torino University, proved that a continuous curve without

Fig. 2.4 – Sierpinski Carpet [7]

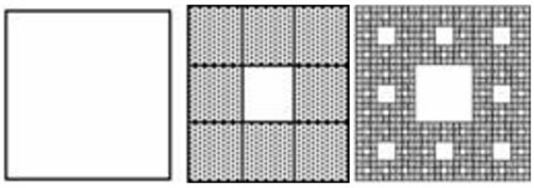


Fig. 2.5 – Sierpinski Carpet [12]

width and surface, can fill a space shape, by using the Sierpinnski Carpet. On the infinite, between the lines will not exist empty spaces (because of the space-filling curve). Consequently, the curve will have the surface of the border square, even if it is formed of segments.

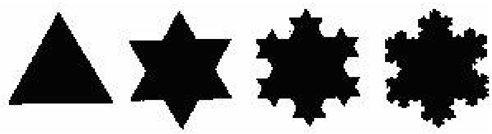
2.2. Cantor Dust. The german mathematician Georg Cantor created in 1877 a shape called *Cantor Dust.* This one has been constructed through dimensional segments fragmentation. In the end were obtained only points with 0 dimension, even if it was still formed of segments (see, for example Figure 2.6).

Fig. 2.6 – Cantor Dust [4]

2.3. Koch Curve. The Swedish mathematician Helge von Koch, fascinated by infinite problems, has constructed *Koch Coast Curve.* He started by considering a segment on which he drew an exterior triangle. On every single segment that resulted, in the same shape, he drew triangles, over and over again (see, for example [13]). Similarly, the Koch coast curve can be created by starting from a square or from an equilateral triangle on which are drawn the same type of triangles (see, for example, Figure 2.7). Koch was the one that introduced also the *Koch Snow Flake* (see, for example, Figure 2.8).

Fig. 2.7 – Koch Coast Curve [13]

Fig. 2.8 – Koch Snow Flake [13]]



The most famous fractal was introduced by Benoit Mandelbrot, which is considered the founder of fractal geometry. But, in reality, the fractal has been seen and used before, by other two mathematicians.

3. FRACTALS

Definition. The *fractals* can be defined as complex space representations obtained in a recursive method.

A fractal object is much more difficult to be obtained due to its complexity. It requires the observer to make an imagining effort and participate in the mental process. It becomes an infinite process that represents the essence of fractals (see, for example, [3, 9, 11]).

The fractal shapes remain the same even if the representation is highly increased. Generally, a fractal proves a limit; the fractal object is the process limit with an infinite number of operations (see, for example [15, 17, 19]).

The fractals can be seen to be complicated in addition to the classical geometric forms. Straight lines, arcs, curves, and polygons are all considered linear with respect to the differentiability. If one increase to infinite their borders, a tangent is obtained. In the case of the non-linear forms, by increasing their images to infinite, there are still complex details to be discovered.

Usually, the fractals are products of simple processes that generate complicated results. This property characterizes also the chaos theory: from a simple input can be obtained complicated results. Consequently, some mathematicians consider chaos as a part of the process that causes the complex results obtained.

Because the fractals can be considered auto-similar forms, the entire system structure can be reflected in every single part of it. A system is auto-similar when similar forces act at different levels. The auto-similar forms can be found very easily in nature: field lines, tree branches, mountain peaks, waves, and small clouds. So, the fractals can be found all over the surrounding environment.

3.1. The fractal dimension. The fractal dimension is essential for the fractals.

Definition. A set is considered *n-dimensional* if there are necessary n variables to describe the closeness of a point.

This notion is also called *topological set dimension.* Some misunderstandings may be understood from here for a figure dimension, like: a line with dimension 1, a space with dimension 2, a cube with dimension 3.

Often, a sphere is considered of dimension 3, because it can exist only in the space, not in the linear space. However, the sphere is bidimensional, because each section of it seems like a space zone and needs only two coordinates to represent a point.

Definition. A *fractal dimensional* is a complex auto-similar figure.

It has been introduced in 1919 by the mathematician Hausdorff, under the name *Hausdorff dimension* or *fractal dimension*. It measures the number of sets of small diameters need to cover a figure. If this number is an integer one, the dimension is topological, otherwise, it is a fractal dimension.

The mathematician Besicovitch extended Hausdorff ideas and proved that some forms can have fractional dimensions, such as 1.3 or 2.5. The Sierpinski and Koch curves were explained with this new dimension.

In practice, the Hausdorff / Besicovitch dimension is defined as a fraction between the logarithm of the copy numbers and the logarithm of the seed measure of the copies.

For the Koch coast curve applied to a triangle, the fractal dimension will be $\frac{\log_{10} 4}{\log_{10} 3} = 1.2618$, because there are 4 copies and each of it is a third part of the seed measure.

For the Koch coast curve applied to a square, the fractal dimension will be $\frac{\log_{10} 5}{\log_{10} 3} = 1.46$.

The fractional dimension of Cantor Dust will be $\frac{\log_{10} 2}{\log_{10} 3} = 0.63$, so this object has the dimension higher than the point (0) and less than the line (1).

The Sierpinski dimension applied to a triangle will be $\frac{\log_{10} 3}{\log_{10} 2} = 1.58$.

Definition. A *form* is considered *fractal* if it has the Hausdorff-Besicovitch dimension greater than its topological dimension.

In the surrounding environment, the mountains, the clouds, the tree, and the flowers, have their dimension between 2 and 3. So, the fractal dimension (as was later called by Mandelbrot) became a new instrument for measuring the space.

3.2. Mandelbrot Set. Due to John Hubbard (1985), the *Mandelbrot Set* is *the most complex mathematical object.*

The Mandelbrot Set was created in the complex space having each point included or not in the space. Besides its complexity, the set was determined by a recursive formula $Z^2 + C$, where Z, C are complex numbers. Each point from the complex space was introduced in the formula, and so, $Z_1 = Z_0^2 + Z_0$, $Z_2 = Z_1^2 + Z_0$, ..., $Z_{n+1} = Z_n^2 + Z_0$. The first part of the formula was previous obtained, and the second part was using the same initial number Z_0. The general formula became $Z_{n+1} = Z_n^2 + Z_0$.

Iterating this formula, the results will tend to infinite or will remain finite. If the result is finite, then the original point belongs to the Mandelbrot Set. Otherwise, the point does not belong to the Mandelbrot Set. Performing this iteration for every single point from the complex space, and coloring the points from inside with black and the others with different colors (based on the iteration number performed until the belonging check), we will obtain the figure called *Mandelbrot Fractal*.

In what follows, we will introduce the algorithm, from where the Mandelbrot fractal will be generated. We proceed with an example, in which we clear some assumptions: on which base did we say that the point belongs or not to the complex space; which part of the complex space do we sight; which is the necessary iteration number.

4. MANDELBROT FRACTAL EXAMPLE

In what follows, we provide an example for the Mandelbrot Fractal. The steps performed include the recognition of the image that we start with, the association of it with the fractals, and the code that generates the corresponding fractal.

4.1. Image recognition. In this part, we consider the image from Figure 4.9 and analyze it with the help of Artificial Intelligence. The stone from the image was found in the Sinai Mountain and brought to us, such that, this analysis could have been performed. In the same mountain, could be found multiple stones like this one.

Fig. 4.9 – Dentritic Mineral

Artificial Intelligence recognizes this image as a dendritic mineral that is formed due to the manganese or iron oxides. This structure is similar to the one of the trees that grow on the rocks. These natural models are known as *dendrites* and are rarely used. Mostly, they are the result of the mineral rainfalls through the cracks and porous layers of the rock during the time (see, for example [1, 6, 8]).

4.2. Association of the dendritic mineral model with the fractals. Next, we proceed and associate the dendritic mineral model with the fractal, which can be determined by Artificial Intelligence applied to the natural models.

4.2.1. *Diffusion-Limited Aggregation.* Diffusion-Limited Aggregation, or DLA, is a model that can be used for the natural processes, and can generate a model similar to a fractal. The DLA models are created when the particles follow random paths and are stocked together to form splitter structures, similar to the ones of the trees. The formula or the algorithm used to generate such a fractal involves a random process and provides the structured fractal (see, for example, [10, 18]).

The mathematical model can be obtained with the help of DLA, fractal dimension, and through the system approximation.

Diffusion-Limited Aggregation

- Start with an initial seed point, that is a a fixed point in $2D$ or $3D$.
- The *walker* particles are introduced and are moving in a random order, following the equation: (Random Walk Equation)

$$x_{n+1} = x_n + \Delta x,$$

$$y_{n+1} = x_n + \Delta y,$$

  where Δ_x, Δ_y are small and random steps.
- The particle paste happens when the walker is at the smallest distance to an existing clustered.
- During time, the repetitive aggregation conducts to a dendritic fractal structure.

Fractal Dimension The fractal dimension of the DLA structure is usually a number between 1.6 and 1.8 in $2D$ and around 2.5 in $3D$. This one is calculated by the box-counting method.

$$D = \lim_{\epsilon \to 0} \frac{log_{10} N(\epsilon)}{log_{10}(\frac{1}{\epsilon})},$$

where $N(\epsilon)$ is the number of tapes of dimension ϵ that needs to cover the structure.

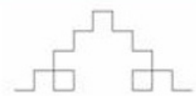
System Approximation The approximation is performed with the help of the Lindenmayer System (L-System). This one approximates branched structures that are similar to dendritic fractals.

- In the context of a L-System, start with the axiom, that represent the movement point or the initial state of the system. It is a series of F symbols that determine the manner in which the structure is generated.

The symbol F is often used to indicate a straight movement, by drawing a line on a graphic (see, Figure 4.10, where the rule involves that each line (or F) is replaced by the symbols sequel $F - F + F + F - F$).

- Apply rules: $F \to F[+F][-F]F$, where $[+\backslash]$ and $[-\backslash]$ refers branches of the specified angles.

Fig. 4.10 – The increase Koch fractal [13]

Iteration	*Fractal Image*	*Descriptor series*
Axiom		F
First Iteration		F-F+F+F-F
Second Iteration		F-F+F+F-F-F-F+F+F-F+F-F+F+F- F+F-F+F+F-F-F-F+F+F-F
Third Iteration		F-F+F+F-F-F-F+F+F-F+F-F+F+F-F+F-F+F+F-F-F-F+F+F-F- F-F+F+F-F-F- F+F+F-F+F-F+F+F-F+F-F+F+F-F-F-F+F+F-F+F-F+F+F-F-F- F+F+F-F+F- F+F+F-F+F-F+F+F-F-F-F+F+F-F+F-F+F+F-F-F-F+F+F-F+F- F+F+F-F+F-

4.3. Python code to generate a fractal. In this part is presented a Python code through which is generated a fractal similar to the one from the surrounding environment (see, for example [2, 14, 16]).

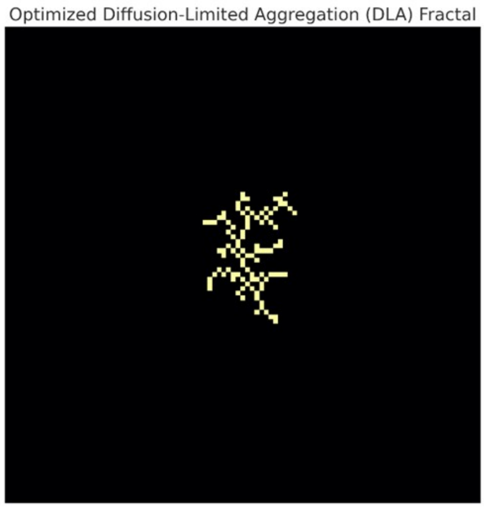
Python code:

```
import numpy as np
import matplotlib as plt
def generate_dla_optimized(size, num_particles, stick_distance=1):
grid = np.zeros((size, size), dtype=int)
center = size // 2
grid[center, center] = 1
for _ in range(num_particles):
angle = np.random.uniform(0, 2 * np.pi)
x, y = int(center + (size // 2 - 1) * np.cos(angle)), int(center + (size // 2 -
1) * np.sin(angle))
while True:
x += np.random.choice([-1, 0, 1])
y += np.random.choice([-1, 0, 1])
if x ≤ 0 or x ≥ size − 1 or y ≤ 0 or y ≥ size − 1:
```

```
break
if np.any(grid[x-stick_distance:x+stick_distance+1, y-stick_distance:y+stick_distance+1]
== 1):
grid[x, y] = 1
break
return grid
grid_size_optimized = 101 (the network size)
num_particles_optimized = 2000 (the number of particles)
dla_pattern_optimized = generate_dla_optimized(grid_size_optimized, num_particles_optimized)
plt.figure(figsize=(8, 8))
plt.imshow(dla_pattern_optimized, cmap='inferno', origin='lower')
plt.title("Optimized Diffusion-Limited Aggregation (DLA) Fractal")
plt.axis('off')
plt.show()
```

The generated image follows in Figure 4.11.

Fig. 4.11 – Fractal generated by Python code



5. CONCLUSION

The dendritic models from the rocks can be efficiently modeled by using fractals and their specific techniques, like, Diffusion-Limited Aggregation, and L-System Approximations. These models prove that rules or simple processes can generate complex and auto-similar structures that exist in the surrounding environment. In the future, we want to provide a more accurate analysis with multiple examples from the surrounding environment.

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