DIDACTICA MATHEMATICA, Vol. 43 (2025) , pp. 122–125

ON AN INTEGRAL INEQUALITY

Mihai Iancu

Abstract. We give a consequence of a Hardy type inequality. The proofs are elementary.

MSC 2000. 26D10.
Key words. Integral inequality.

1. PROBEM

The following problem can be proposed to a freshman studying Analysis. It is also suitable for Romanian twelfth-grade high school students preparing for mathematical competitions. The problem involves an integral inequality in the spirit of Hardy (see [1, Sections 9.8–9.9]) and, to the best of our knowledge, is new.

PROBLEM 1. Let $p > 0$, $p \neq 1$, and let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that

$$\big(xf(x)\big)^p \leq \int_0^x f(t)\mathrm{d}t, \quad \forall x \geq 0.$$

Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

2. SOLUTIONS

We give two solutions.

SOLUTION 1. Assume that we don't have $\lim\limits_{x\to\infty} f(x) = 0$.

Case $p \in (0, 1)$:

Since $f(x)^p \leq x^{1-p}\dfrac{1}{x}\displaystyle\int_0^x f(t)\mathrm{d}t$, $\forall x > 0$, and $\lim\limits_{x\searrow 0} \dfrac{1}{x}\displaystyle\int_0^x f(t)\mathrm{d}t = f(0)$, we have $f(0) = 0$. The assumption implies that f is not identically zero, so we can consider (f being continuous)

$$x_0 = \max\{x \geq 0 : f(t) = 0, \forall t \in [0, x]\} < \infty.$$

Since f is continuous and $f(x_0) = 0$, we can choose $\delta \in (0, 1)$ such that $f(x) < 1$, $\forall x \in [x_0, x_0 + \delta]$. We deduce that there exists $x_1 \in (x_0, x_0 + \delta]$ such that $\max\limits_{x\in[x_0,x_0+\delta]} f(x) = f(x_1) \in (0, 1)$.

Taking into account the above, we have $(x_1 - x_0)f(x_1) \in (0, 1)$ and

$$\int_0^{x_1} f(t)\mathrm{d}t = \int_{x_0}^{x_1} f(t)\mathrm{d}t \leq (x_1 - x_0)f(x_1) < ((x_1 - x_0)f(x_1))^p \leq (x_1 f(x_1))^p,$$

in contradiction with the inequality of the problem.

Case $p > 1$:

By the assumption, f is unbounded. Indeed, if f is bounded above by $M > 0$, then $f(x)^p \le x^{-p} \int_0^x f(t)\mathrm{d}t \le Mx^{1-p}$, $\forall x > 0$, and thus $\lim_{x\to\infty} f(x)^p = 0$.

Let $m_0 = 1 + \max_{x\in[0,1]} f(x)$. Since f is unbounded and continuous, we can consider

$$x_0 = \max\{x \ge 0 : f(t) \le m_0, \forall t \in [0,x]\} < \infty.$$

Taking into account the above, $\max_{x\in[0,x_0]} f(x) = f(x_0) = m_0 \ge 1$ and $x_0 > 1$. So, $x_0 f(x_0) > 1$ and $\int_0^{x_0} f(t)\mathrm{d}t \le x_0 f(x_0) < (x_0 f(x_0))^p$, in contradiction with the inequality of the problem.

Solution 2. Assume that we don't have $\lim_{x\to\infty} f(x) = 0$.

Let $F : [0,\infty) \to [0,\infty)$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $x \ge 0$. Since f is continuous, F is differentiable on $(0,\infty)$ with $F'(x) = f(x)$, $x > 0$, and $\lim_{x\searrow 0} \frac{F(x)}{x} = f(0)$.

Case $p \in (0,1)$:

We have $f(x)^p \le \frac{F(x)}{x^p}$, $\forall x > 0$. Taking into account the above limit, we deduce that $f(0) = 0$. The assumption implies that f is not identically zero, so we can consider (f being continuous)

$$x_0 = \max\{x \ge 0 : f(t) = 0, \forall t \in [0,x]\} < \infty.$$

We have $F(x) > 0, \forall x > x_0$, and $\frac{1}{x} - F'(x)F(x)^{-\frac{1}{p}} \ge 0, \forall x > x_0$.

Let $q = \frac{1}{p} - 1 > 0$ and let $G : (x_0,\infty) \to \mathbb{R}$, $G(x) = q\ln x + F(x)^{-q}$, $x > x_0$. The above inequality implies that G is increasing.

Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in (x_0,∞) such that $\lim_{n\to\infty} x_n = x_0$. Using the Mean Value Theorem for integrals and the definition of x_0, we can choose, for every $n \in \mathbb{N}^*$, $y_n \in (x_0, x_n]$ such that $F(x_n) = x_n f(y_n) > 0$.

We have $\lim_{n\to\infty} x_n^q \ln x_n = \begin{cases} x_0^q \ln x_0, & x_0 > 0 \\ 0, & x_0 = 0 \end{cases}$, because, applying the l'Hospital rule, $\lim_{x\searrow 0} \frac{\ln x}{x^{-q}} = 0$. Moreover, $\lim_{n\to\infty} f(y_n) = f(x_0) = 0$. Hence,

$$\lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} x_n^{-q}\left(qx_n^q \ln x_n + f(y_n)^{-q}\right) = \infty,$$

in contradiction with G being increasing.

Case $p > 1$:

The assumption implies that f is not identically zero, thus there exists $x_0 \geq 0$ such that $F(x) > 0$, $\forall x > x_0$. We have $F'(x)F(x)^{-\frac{1}{p}} - \frac{1}{x} \leq 0, \forall x > x_0$.

Let $q = 1 - \frac{1}{p} > 0$ and let $G : (x_0, \infty) \to \mathbb{R}$, $G(x) = F(x)^q - q \ln x$, $x > x_0$. The above inequality implies that G is decreasing.

By the assumption, there exists a sequence $(x_n)_{n \in \mathbb{N}^*}$ of positive numbers such that $\lim\limits_{n\to\infty} x_n = \infty$ and $\lim\limits_{n\to\infty} f(x_n) \in (0, \infty]$.

The inequality of the problem implies $F(x_n)^q \geq (x_n f(x_n))^{p-1}$, $\forall n \in \mathbb{N}^*$. Applying the l'Hospital rule, $\lim\limits_{n\to\infty} \frac{\ln x_n}{x_n^{p-1}} = 0$. So,

$$\lim_{n\to\infty} G(x_n) \geq \lim_{n\to\infty} x_n^{p-1} \left(f(x_n)^{p-1} - q \frac{\ln x_n}{x_n^{p-1}} \right) = \infty,$$

in contradiction with G being decreasing.

3. REMARKS

Remark 1. In the case $p = 1$, we can easily give examples of continuous functions f that satisfy the inequality of the problem, but don't satisfy $\lim\limits_{x\to\infty} f(x) = 0$, using the following observation: if f is decreasing, then f satisfies the inequality of the problem for $p = 1$. Example: $f(x) = e^{-x} + 1$, $x \geq 0$.

Remark 2. In the case $p \in (0, 1)$, according to the above solutions, the only function f that satisfies the conditions of the problem is the zero function. In the case $p > 1$, there are functions f that satisfy the conditions of the problem and are not identically zero. Example: $(xe^{-x})^p \leq xe^{-x} \leq \int_0^x e^{-t} \mathrm{d}t, \forall x \geq 0$, for $p > 1$.

REFERENCES

[1] Hardy, G.H., Littlewood, J.E., and Pólya, G., *Inequalities*, Cambridge University Press, 1952.

Faculty of Mathematics and Computer Science, Babeș-Bolyai University, Cluj-Napoca
e-mail: mihai.iancu@ubbcluj.ro