

ROOTS OF A POLYNOMIAL IN A GIVEN INTERVAL

Cristina Blaga and Paul A. Blaga

Abstract. In this note, we present an application of Sturm's theorem to determine the number of positive roots of a third-degree polynomial arising in the study of a compact object. The coefficients of the polynomial depend on several quantities obtained from measurements; therefore, an analysis of the existence of its roots within an interval, for a given set of conditions satisfied by the parameters, is useful.

MSC 2000. 12D10.

Key words. Polynomials: real roots, Sturm theorem.

1. INTRODUCTION

Students learn in middle school to solve practical problems that lead to first-degree equations or, at most, second-degree equations. Later, in high school an important place in mathematics curriculum is occupied by the study of polynomials. Sometimes problems from astrophysics lead to solving third-degree or higher equations, with coefficients depending on parameters derived from measurements. In the sixteen century, formulae for the roots of a third and fourth degree equations were found. The search for formulae of solution for fifth and higher degrees equations in terms of their coefficients was unsuccessful, therefore in the nineteenth century, theories were developed that allow us to determine whether an equation has a solution in a given interval. For a brief history of the search of formulae for the roots of the third and higher degree equations see [2] or [6].

In this note we present the problem from astrophysics which leads to the third order equation analysed using the Sturm theorem. In the section 2 we describe the context in which the third degree equation arises. In the section 3 we obtain the terms of the Sturm sequence and discuss their sign for the given set of parameters. Some final remarks are given in section 4.

2. THE PROBLEM

The third order equation we analyse appears in the study of a fascinating object called *black hole*. The existence of such objects was mentioned by Pierre-Simon Laplace (1749-1827), in 1795, in his book *Exposition of*

the System of the World, where he predicted the existence of bright stars that we do not see, because the escape velocity from their surface is greater than the speed of light (see [4]). In Einstein's theory of general relativity, a black hole is defined as astrophysical objects so dense that even light cannot escape from it (see [1]). Around a black hole there is a boundary, where the light cannot escape, *the event horizon*. Today different metrics are used to describe the (spacetime) geometry around a dense object. The existence and the spatial location of horizons depends on the metric we use.

Next we consider the metric

$$ds^2 = B(r)dt^2 - B^{-1}(r)dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,$$

where

$$B(r) = 1 - 3\beta\gamma - \frac{\beta(2 - 3\beta\gamma)}{r} + \gamma r + kr^2 ,$$

with β , γ and k constants, obtained by Mannheim and Kazanas in [3]. The constants β , γ are positive, $\beta\gamma \ll 1$ and k is negative (see [3]).

3. HORIZONS

To determine whether an horizon is formed, we analyze the zeros of the lapse function,

$$B(r) = \frac{-kr^3 + \gamma r^2 + (-3\beta\gamma + 1)r + 3\beta^2\gamma - 2\beta}{r} .$$

We observe that

$$\lim_{r \rightarrow 0^+} B(r) = \text{sgn}[\beta(3\beta\gamma - 2)] \cdot \infty = -\infty ,$$

since $\beta\gamma \ll 1$. Moreover,

$$\lim_{r \rightarrow \infty} B(r) = -\text{sgn}(k) \cdot \infty = +\infty ,$$

because $k < 0$. It follows that there exists $r_0 \in (0, \infty)$ such that $B(r_0) = 0$. Hence, *an event horizon exists* (see [5]).

The number of positive roots of the numerator of the lapse function — a cubic polynomial — can be determined using the Sturm theorem [2], which provides a method to evaluate the number of distinct real roots of a polynomial within a given interval by constructing its Sturm sequence. The difference in the number of sign changes of this sequence, evaluated at the endpoints of the interval, gives the number of distinct real roots of the polynomial in that interval.

Let $P_0(r)$ denote the numerator of $B(r)$, and seek its positive roots. Its Sturm sequence [2] can be derived from $P_0(r)$ as follows: set $P_1(r) =$

$P'_0(r)$. Then divide $P_0(r)$ by $P_1(r)$ and take the remainder, with reversed sign, as $P_2(r)$. In general, once $P_{k-1}(r)$ and $P_k(r)$ have been obtained, the next term $P_{k+1}(r)$ is defined as the remainder (with reversed sign) of the division of $P_{k-1}(r)$ by $P_k(r)$,

$$(1) \quad P_{k-1}(r) = P_k(r)q_k(r) - P_{k+1}(r),$$

where $q_k(r)$ is the quotient of the division. Hence, the Sturm sequence of the numerator of $B(r)$ is:

$$(2) \quad P_0(r) = -kr^3 + \gamma r^2 + (1 - 3\beta\gamma)r - \beta(2 - 3\beta\gamma),$$

$$(3) \quad P_1(r) = -3kr^2 + 2\gamma r + 1 - 3\beta\gamma,$$

$$(4) \quad P_2(r) = \frac{-6k(1 - 3\beta\gamma) - 2\gamma^2}{9k}r + \frac{9\beta k(2 - 3\beta\gamma) - \gamma^2(1 - 3\beta\gamma)}{9k},$$

$$(5) \quad P_3 = \frac{9k(27\beta^2 k - 3\beta\gamma - 1)[k(2 - 3\beta\gamma)^2 + \gamma^2(1 - \beta\gamma)]}{4(9\beta\gamma - \gamma^2 - 3k)^2}.$$

We determine the sign changes of the polynomials of the Sturm sequence for $r = 0$ and $r = +\infty$ using the following results.

THEOREM 1. If $\beta > 0$, $\gamma > 0$, $\beta\gamma \ll 1$, and $k < 0$, then the signs of the polynomials in the Sturm sequence at $r = 0$ are:

- (1) $\text{sgn}[P_0(0)]$ is negative,
- (2) $\text{sgn}[P_1(0)]$ is positive,
- (3) $\text{sgn}[P_2(0)]$ is positive, and
- (4) $\text{sgn} P_3 = \text{sgn}[k(2 - 3\beta\gamma)^2 + \gamma^2(1 - \beta\gamma)]$.

Proof. Evaluating the first three terms of the Sturm sequence at $r = 0$ gives

$$(6) \quad P_0(0) = -\beta(2 - 3\beta\gamma),$$

$$(7) \quad P_1(0) = 1 - 3\beta\gamma,$$

$$(8) \quad P_2(0) = \frac{9\beta k(2 - 3\beta\gamma) - \gamma^2(1 - 3\beta\gamma)}{9k}.$$

From the parameter constraints, it follows that $P_0(0) < 0$ and $P_1(0) > 0$. For $P_2(0)$, note that its denominator has the sign of k (negative), while its numerator is also negative (a sum of negative terms), implying $P_2(0) > 0$. The sign of P_3 is determined by the third factor in its numerator, as the first two factors are negative and the denominator is positive. \square

THEOREM 2. If $\beta > 0$, $\gamma > 0$, $\beta\gamma \ll 1$, and $k < 0$, then the signs of the polynomials in the Sturm sequence at $r = +\infty$ are:

- (1) $\text{sgn}[P_0(+\infty)]$ is positive,
- (2) $\text{sgn}[P_1(+\infty)]$ is positive,
- (3) $\text{sgn}[P_2(+\infty)] = \text{sgn}\left(\frac{3k(1-3\beta\gamma)+\gamma^2}{k}\right)\infty$,
- (4) $\text{sgn} P_3 = \text{sgn}[k(2-3\beta\gamma)^2 + \gamma^2(1-\beta\gamma)]$.

Proof. The signs follow from taking limits of the polynomials as $r \rightarrow \infty$:

$$(9) \quad \lim_{r \rightarrow \infty} P_0(r) = -\text{sgn}(k) \infty = +\infty,$$

$$(10) \quad \lim_{r \rightarrow \infty} P_1(r) = -\text{sgn}(k) \infty = +\infty,$$

$$(11) \quad \lim_{r \rightarrow \infty} P_2(r) = \text{sgn}\left(\frac{3k(1-3\beta\gamma)+\gamma^2}{k}\right)\infty.$$

Again, the sign of P_3 is determined by the third factor in its numerator. \square

The sign changes of the Sturm sequence at $r = 0$ and $r = \infty$ depend on the signs of $P_2(+\infty)$ and P_3 .

THEOREM 3. If $\beta > 0$, $\gamma > 0$, $\beta\gamma \ll 1$, and $k < 0$, then:

- (1) if $k < \frac{-\gamma^2}{3(1-3\beta\gamma)}$, then $\lim_{r \rightarrow \infty} P_2(r) = -\infty$;
- (2) if $k < \frac{-\gamma^2(1-\beta\gamma)}{(2-3\beta\gamma)^2}$, then $P_3 < 0$.

Proof. These results follow directly from the parameter constraints, relation (11), and

$$(12) \quad \text{sgn} P_3 = \text{sgn}[k(2-3\beta\gamma)^2 + \gamma^2(1-\beta\gamma)].$$

\square

THEOREM 4. Let $f_1, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f_1(\beta, \gamma) = \frac{-\gamma^2}{3(1-3\beta\gamma)}, \quad f_2(\beta, \gamma) = \frac{-\gamma^2(1-\beta\gamma)}{(2-3\beta\gamma)^2}.$$

If $\beta\gamma < \frac{1}{3}$, then

$$f_1(\beta, \gamma) < 0, \quad f_2(\beta, \gamma) < 0, \quad \text{and} \quad f_1(\beta, \gamma) < f_2(\beta, \gamma).$$

Proof. The first two inequalities follow directly. Furthermore,

$$\frac{f_1(\beta, \gamma)}{f_2(\beta, \gamma)} = \frac{9\beta^2\gamma^2 - 12\beta\gamma + 4}{9\beta^2\gamma^2 - 12\beta\gamma + 3} > 1.$$

Since $f_2(\beta, \gamma) < 0$, it follows that $f_1(\beta, \gamma) < f_2(\beta, \gamma)$. \square

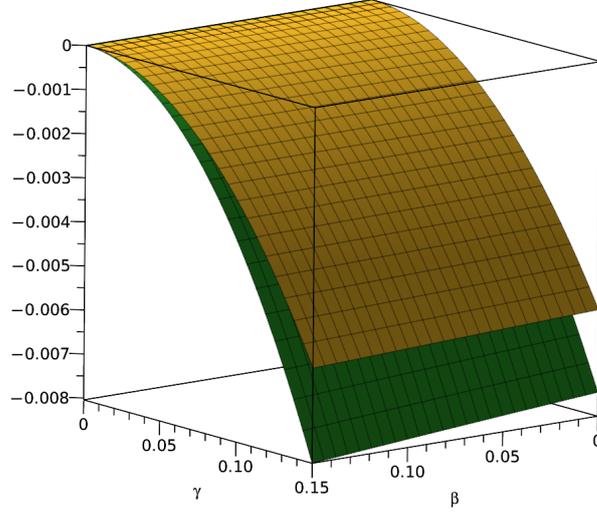


Fig. 3.1 – Plots of $f_1(\beta, \gamma)$ (green) and $f_2(\beta, \gamma)$ (gold).

We are now ready to discuss the number of positive roots of the equation $B(r) = 0$. From the parameter constraints $\beta > 0$, $\gamma > 0$, $\beta\gamma \ll 1$, and $k < 0$, there are three possible cases:

- (1) If $k < f_1(\beta, \gamma)$, the signs of the Sturm sequence at $r = 0$ and $r \rightarrow \infty$ are given in Table 1.

r	$P_0(r)$	$P_1(r)$	$P_2(r)$	P_3	Sign changes
0	–	+	+	–	2
$+\infty$	+	+	–	–	1

Table 1 – Sign changes in Case 1.

- (2) If $f_1(\beta, \gamma) < k < f_2(\beta, \gamma)$, the signs are shown in Table 2.

r	$P_0(r)$	$P_1(r)$	$P_2(r)$	P_3	Sign changes
0	–	+	+	–	2
$+\infty$	+	+	+	–	1

Table 2 – Sign changes in Case 2.

- (3) If $f_2(\beta, \gamma) < k < 0$, the signs are listed in Table 3.

The graphs of the functions P_0 , P_1 , P_2 , and P_3 for three representative choices of the parameters (β, γ, k) , illustrating the three cases discussed above, are shown in Fig. 3.5. In all these three cases, the difference

r	$P_0(r)$	$P_1(r)$	$P_2(r)$	P_3	Sign changes
0	-	+	+	+	1
$+\infty$	+	+	+	+	0

Table 3 – Sign changes in Case 3.

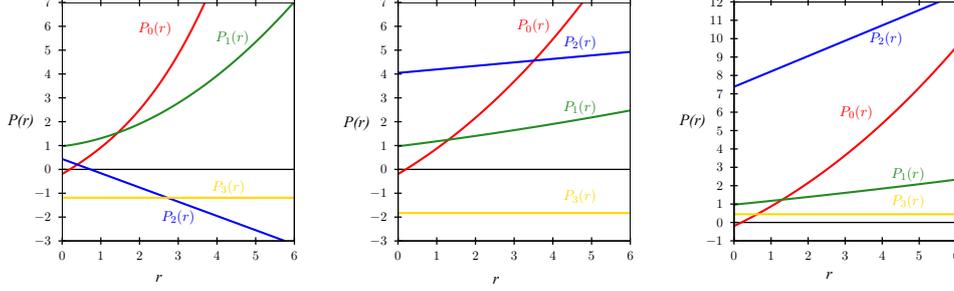


Fig. 3.2 – Case 1: $k < f_1(\beta, \gamma)$ Fig. 3.3 – 2: $f_1(\beta, \gamma) < k < f_2(\beta, \gamma)$ Fig. 3.4 – Case 3: $f_2(\beta, \gamma) < k$

Fig. 3.5 – Sturm sequences for the numerator of $B(r)$ in Weyl conformal gravity, for $(\beta, \gamma) = (0.1, 0.1)$: (a) Case 1: $k = -0.045 < f_1(0.1, 0.1) = -0.0034$; (b) Case 2: $f_1(0.1, 0.1) < k = -0.0028 < f_2(0.1, 0.1) = -0.0026$; (c) Case 3: $f_2(0.1, 0.1) < k = -0.0015$.

between the number of sign changes at 0 and ∞ equals 1, indicating a *single real positive root* of $B(r) = 0$. Hence, if $\beta > 0$, $\gamma > 0$, $\beta\gamma \ll 1$, and $k < 0$, then *an unique horizon exists*.

4. CONCLUDING REMARKS

In this note, we studied the existence of positive roots of a third-degree polynomial using Sturm's theorem. The equation arises in the analysis of the existence of the black hole horizon in Mannheim–Kazanas spacetime. The metric describing this spacetime depends on three parameters determined from measurements. Using Sturm's theorem, we proved that if β and γ are positive and their product is less than $1/3$, and if k is negative, then the cubic equation has a unique positive solution. This article is intended for those who wish to enrich their classes with practical examples and to encourage students to make connections with other disciplines, such as physics.

REFERENCES

- [1] CAMENZIND, M. *Compact Objects in Astrophysics*, Springer Verlag, 2007.

-
- [2] KUROSH, A. *Higher Algebra*, Mir Publishers, 1988.
 - [3] MANNHEIM P.D. and KAZANAS D. *Exact vacuum solution to conformal Weyl gravity and galactic rotation curves*, Ap. J., **342**, (1989), 635–638.
 - [4] NOVIKOV, I. *Black Holes and the Universe*, Cambridge University Press, 1990.
 - [5] TURNER G.E. and HORNE K. *Null geodesics in conformal gravity*. Class. Quantum Grav., *37*, (2020) 095012.
 - [6] VODĂ, V. GH. *Miraculous Equations*, Albatros Publisher, 1987 (in Romanian).

Faculty of Mathematics and Computer Sciences, Babes-Bolyai University
1 Kogălniceanu Street, 400083, Cluj-Napoca, Romania
e-mail: `cristina.blaga@ubbcluj.ro`
`aurel.blaga@ubbcluj.ro`