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PROOF OF MORLEY'S THEOREMS

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Abstract. In this paper we will prove Morley's Theorems through four methods, two trigonometric, with complex numbers and using barycentric coordinates.

Key words. Morley's Theorems

MSC 2000. 51M04

This article is a methodical paper in which known results will be demonstrated through both old and new demonstration ideas.

In the following we will recall some known notations and results which we will use in the following.

For a triangle ABC, we will use the established notations: A, B, C the measures of the angles; a, b, c the lengths of the sides and $\mathcal{C}(O, R)$ the circumscribed circle, with center O and radius R. In the demonstration using complex numbers, by z_A we will understand the affix of the point A and $\varepsilon_\alpha = \cos\alpha + i\sin\alpha$, $\alpha \in \mathbb{R}$.

The following identities are well known

$$\sin x = \sin\frac{x}{3}\left(3 - 4\sin^2\frac{x}{3}\right) = 4\sin\frac{x}{3}\cdot\sin\frac{\pi + x}{3}\sin\frac{\pi - x}{3} \tag{1}$$

if $x \in \mathbb{R}$,

$$\sin^2 x + \sin^2 y - 2\sin x\sin y\cos z = \sin^2 z, \tag{2}$$

if $x, y, z \in \mathbb{R}$ such that $x + y + z = \pi$,

$$\sin^2 x + \sin^2 y + 2\sin x\sin y\cos z = \sin^2 z \tag{3}$$

if $x, y, z \in \mathbb{R}$ such that $x + y + z = 0$ and

$$\sin x\sin y\sin z = \tag{4}$$
$$= \frac{1}{4}\big(\sin(-x+y+z) + \sin(x-y+z) + \sin(x+y-z) - \sin(x+y+z)\big)$$

if $x, y, z \in \mathbb{R}$.

The following theorem was discovered in 1899 by Anglo-American mathematician Frank Morlay.

THEOREM 1 (Frank Morley, September 9, 1860-October 17, 1937). *In any triangle, the three points of intersection of the adjacent angle bisectors form an equilateral triangle.*

Demonstration 1. Applying the law of sine in triangle BO_1C (see Fig. 1), we have respectively

$$CO_1 = \frac{a\sin\dfrac{B}{3}}{\sin\left(\pi - \dfrac{B+C}{3}\right)} = \frac{2R\sin A\sin\dfrac{B}{3}}{\sin\left(\dfrac{\pi}{3} - \dfrac{A}{3}\right)} = \frac{4R\sin A\sin\dfrac{B}{3}}{\sqrt{3}\cos\dfrac{A}{3} - \sin\dfrac{A}{3}}.$$

Amplify with $\sqrt{3}\cos\dfrac{A}{3} + \sin\dfrac{A}{3} = 2\sin\dfrac{\pi+A}{3}$ and taking (1) into account

$$CO_1 = \frac{4R\sin\dfrac{A}{3}\left(3 - 4\sin^2\dfrac{A}{3}\right)\sin\dfrac{B}{3}2\sin\dfrac{\pi+A}{3}}{3\cos^2\dfrac{A}{3} - \sin^2\dfrac{A}{3}},$$

from where

(5) $$CO_1 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{\pi+A}{3}.$$

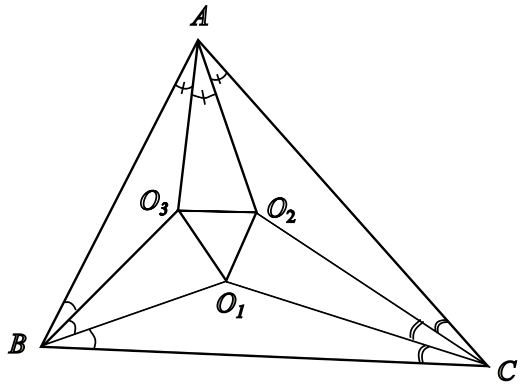

Figure 1

Similar in triangle AO_2C we have

(6) $$CO_2 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{\pi+B}{3}.$$

Applying the cosine theorem in triangle CO_1O_2 and taking (5) and (6) into account, we have

$$O_1O_2^2 = CO_1^2 + CO_2^2 - 2CO_1CO_2\cos\frac{C}{3} =$$
$$= 64R^2\sin^2\frac{A}{3}\sin^2\frac{B}{3}\left(\sin^2\frac{\pi+A}{3} + \sin^2\frac{\pi+B}{3} - 2\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\cos\frac{C}{3}\right).$$

Because $\dfrac{\pi+A}{3} + \dfrac{\pi+B}{3} + \dfrac{C}{3} = \dfrac{2\pi + (A+B+C)}{3} = \pi$ and taking relation (2) into account, we obtain

(7) $$O_1O_2 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}.$$

Similarly $O_2O_3 = O_3O_1 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}$, it follows that the triangle $O_1O_2O_3$ is equilateral triangle.

Demonstration 2.

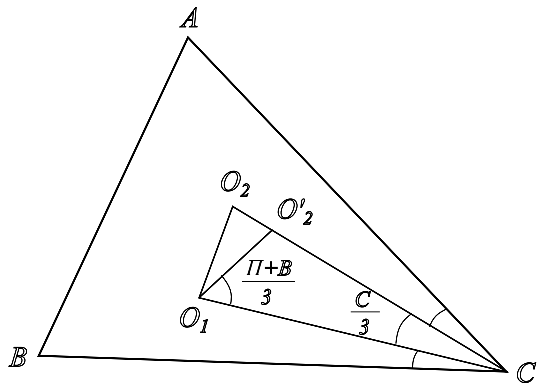

Figure 2

Let the point $O_2' \in (CO_2$ such that $\widehat{CO_1O_2'} = \frac{\pi+B}{3}$ (see Fig. 2) and then

$$\widehat{CO_2'O_1} = \pi - \frac{\pi+B}{3} - \frac{C}{3} = \frac{2\pi - B - C}{3} = \frac{\pi+A}{3}.$$

From the theorem of sine in the triangle $O_1O_2'C$, we have that

$$\frac{CO_2'}{\sin\frac{\pi+B}{3}} = \frac{CO_1}{\sin\frac{\pi+A}{3}}$$

and taking (5) into account, we obtain that

$$CO_2' = \frac{CO_1 \cdot \sin\frac{\pi+B}{3}}{\sin\frac{\pi+A}{3}} = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{\pi+B}{3}.$$

Then, according to (6) it follows that $CO_2' = CO_2$ and how $O_2' \in (CO_2$ it results that O_2' coincides with O_2. At this point we are in the situation in Demonstration 1.

Demonstration 3. We will give a proof with complex numbers (see Fig. 1). We have immediately that

$$Z_{O_1} - Z_C = \frac{CO_1}{b}(Z_A - Z_C)\varepsilon_{\frac{2C}{3}},\ Z_{O_2} - Z_C = \frac{CO_2}{b}(Z_A - Z_C)\varepsilon_{\frac{C}{3}},$$

from which by subtraction, we obtain that

$$Z_{O_1} - Z_{O_2} = \frac{Z_A - Z_C}{b}\varepsilon_{\frac{C}{3}}\left(CO_1 \cdot \varepsilon_{\frac{C}{3}} - CO_2\right).$$

Taking (5) and (6) into account, we have

$$Z_{O_1} - Z_{O_2} = \frac{Z_A - Z_C}{b}\varepsilon_{\frac{C}{3}} 8R\sin\frac{A}{3}\sin\frac{B}{3}\left(\sin\frac{\pi+A}{3}\cdot\varepsilon_{\frac{C}{3}} - \sin\frac{\pi+B}{3}\right)$$
$$= \frac{Z_A - Z_C}{b}\cdot\varepsilon_{\frac{C}{3}}\cdot 8R\sin\frac{A}{3}\sin\frac{B}{3}\cdot$$
$$\cdot\left(\left(\sin\frac{\pi+A}{3}\cos\frac{C}{2} - \sin\frac{\pi+B}{3}\right) + i\sin\frac{\pi+A}{3}\ \sin\frac{C}{3}\right).$$

Moving to the modulus, we have that

$$O_1O_2 = |Z_{O_1} - Z_{O_2}| = \frac{Z_A - Z_C}{b}\left|\varepsilon_{\frac{C}{3}}\right| 8R\sin\frac{A}{3}\sin\frac{B}{3}\cdot$$
$$\cdot\sqrt{\left(\sin\frac{\pi+A}{3}\cos\frac{C}{3} - \sin\frac{\pi+B}{3}\right)^2 + \left(\sin\frac{\pi+A}{3}\sin\frac{C}{3}\right)}$$
$$= 8R\sin\frac{A}{3}\sin\frac{B}{3}\cdot$$
$$\cdot\sqrt{\sin^2\frac{\pi+A}{3} + \sin^2\frac{\pi+B}{3} - 2\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\cos\frac{C}{3}}.$$

Because $\frac{\pi+A}{3} + \frac{\pi+B}{3} + \frac{C}{3} = \pi$ and taking (2) into account, from above we obtain that $O_1O_2 = 8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}$, so $O_1O_2O_3$ is equilateral triangle.

Demonstration 4. In this proof we will use barycentric coordinates. We consider $\alpha_1, \beta_1, \gamma_1$ the barycentric coordinates of O_1, $\alpha_2, \beta_2, \gamma_2$ barycentric coordinates of O_2, F be the area of triangle ABC and let

$$p = \sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}.$$

Using (5), (6) and their analogues, we have

$$\alpha_1 = \frac{\alpha(O_1, BC)}{\alpha(A, BC)} = \frac{CO_1\sin\frac{C}{3}a}{2F} = \frac{8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{\pi+A}{3}\sin\frac{C}{3}a}{2F}$$

from where

$$\alpha_1 = \frac{4apR}{F}\sin\frac{\pi+A}{3}, \tag{8}$$

$$\beta_1 = \frac{d(O_1, CA)}{d(B, CA)} = \frac{CO_1\sin\frac{2C}{3}b}{2F} = \frac{8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{\pi+A}{3}\sin\frac{2C}{3}b}{2F}$$

and because $\sin\frac{2C}{3} = 2\sin\frac{C}{3}\cos\frac{C}{3}$, we obtain

$$\beta_1 = \frac{8bpR}{F}\sin\frac{\pi+A}{3}\cos\frac{C}{3}, \tag{9}$$

$$\gamma_1 = \frac{d(O_1, AB)}{d(C, AB)} = \frac{BO_1\sin\frac{2B}{3}c}{2F} = \frac{8R\sin\frac{A}{3}\sin\frac{C}{3}\sin\frac{\pi+A}{3}\sin\frac{2B}{3}}{2F}$$

from where

$$\gamma_1 = \frac{8cpR}{F} \sin\frac{\pi + A}{3} \cos\frac{B}{3} . \tag{10}$$

Similarly, taking into account that $\alpha_2 = \dfrac{d(O_2, BC)}{d(A, BC)}$, $\beta_2 = \dfrac{d(O_2, CA)}{d(B, CA)}$ and $\gamma_2 = \dfrac{d(O_2, AB)}{d(C, AB)}$, we have that

$$\alpha_2 = \frac{8apR}{F} \sin\frac{\pi + B}{3} \cos\frac{C}{3} , \tag{11}$$

$$\beta_2 = \frac{4bpR}{F} \sin\frac{\pi + B}{3} \tag{12}$$

and

$$\gamma_2 = \frac{8cpR}{F} \sin\frac{\pi + B}{3} \cos\frac{A}{3} . \tag{13}$$

Using (8) and (11), we have $\alpha_1 - \alpha_2 = \dfrac{4apR}{F}\left(\sin\dfrac{\pi + A}{3} - 2\sin\dfrac{\pi + B}{3}\cos\dfrac{C}{3}\right)$ and because

$$2\sin\frac{\pi + B}{3}\cos\frac{C}{3} = \sin\frac{\pi + B + C}{3} + \sin\frac{\pi + B - C}{3} =$$
$$= \sin\frac{2\pi - A}{3} + \sin\frac{\pi + B - C}{3} = \sin\left(\pi - \frac{\pi + A}{3}\right) + \sin\frac{\pi + B - C}{3} =$$
$$= \sin\frac{\pi + A}{3} + \sin\left(\pi + \frac{B - C - 2\pi}{3}\right),$$

so

$$\alpha_1 - \alpha_2 = \frac{4apR}{F} \sin\frac{B - C - 2\pi}{3} . \tag{14}$$

Similarly, using (9) and (12), we have

$$\beta_1 - \beta_2 = \frac{4bpR}{F}\left(2\sin\frac{\pi + A}{3}\cos\frac{C}{3} - \sin\frac{\pi + B}{2}\right) = \frac{4bpR}{F}\sin\frac{\pi + A - C}{3} ,$$

so

$$\beta_1 - \beta_2 = \frac{4bpR}{F} \sin\frac{2\pi - B - 2C}{3} . \tag{15}$$

From (10) and (13) we have

$$\gamma_1 - \gamma_2 = \frac{8cpR}{F}\left(\sin\frac{\pi + A}{3}\cos\frac{B}{3} - \sin\frac{\pi + B}{3}\cos\frac{A}{3}\right) =$$
$$= \frac{8cpR}{F}\cdot\frac{1}{2}\left(\sin\frac{\pi{+}A{+}B}{3} + \sin\frac{\pi{+}A{-}B}{3} - \sin\frac{\pi{+}B{+}A}{3} - \sin\frac{\pi{+}B{-}A}{3}\right) =$$
$$= \frac{8cpR}{F}\sin\frac{A - B}{3}\cos\frac{\pi}{3} , \quad \text{so}$$

$$\gamma_1 - \gamma_2 = \frac{4cpR}{F} \sin\frac{\pi - 2B - C}{3} . \tag{16}$$

From (14)-(16) and distance formula between two points, we have that

$O_1O_2^2 = -(\beta_1-\beta_2)(\gamma_1-\gamma_2)a^2-(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)b^2-(\alpha_1-\alpha_2)(\beta_1-\beta_2)c^2$ and $a = 2R\sin A = 2R\sin(\pi - B - C)$ and analogues, we have

$$O_1O_2^2 = -\frac{16R^2p^2a^2bc}{F^2}\sin\frac{2\pi-2C-B}{3}\sin\frac{\pi-2B-C}{3}-$$
$$-16\frac{R^2p^2ab^2c}{F^2}\sin\frac{B-C-2\pi}{3}\sin\frac{\pi-2B-C}{3}-16\frac{R^2p^2abc^2}{F^2}\sin\frac{B-C-2\pi}{3}\cdot$$
$$\cdot\sin\frac{2\pi-B-2C}{3}=\frac{-16R^2p^22Rabc}{F^2}\left(\sin\frac{2\pi-B-2C}{3}\sin\frac{\pi-2B-C}{3}\cdot\right.$$

$$\cdot\sin(\pi-B-C)+\sin\frac{\pi-2B-C}{3}\sin\frac{B-C-2\pi}{3}\sin B+$$
$$\left.+\sin\frac{B-C-2\pi}{3}\sin\frac{2\pi-B-2C}{3}\sin C\right).$$

Using the identity (4) and as $A+B+C=\pi$, we have

$$\sin\frac{2\pi-B-2C}{3}\sin\frac{\pi-2B-C}{3}\sin(\pi-B-C)+\sin\frac{\pi-2B-C}{3}\cdot$$
$$\cdot\sin\frac{B-C-2\pi}{3}\sin B+\sin\frac{B-C-2\pi}{3}\sin\frac{2\pi-B-2C}{3}\sin C=$$
$$=\frac{1}{4}\left(\sin\frac{2\pi+4B-2C}{3}+\sin\frac{4\pi-2B-4C}{3}-\sin(2\pi-2B-2C)-\right.$$
$$-\sin(\pi-2B)-\sin\frac{\pi+4B+2C}{3}+\sin\frac{\pi-2B+2C}{3}+$$

$$\left.+\sin\frac{4\pi-2B+2C}{3}+\sin(-2C)-\sin\frac{4\pi-2B-4C}{3}\right)=$$
$$=-\frac{1}{4}(\sin 2A+\sin 2B+\sin 2C)+$$
$$+\frac{1}{4}\left(\sin\frac{2\pi-4B-2C}{3}-\sin\frac{\pi+4B+2C}{3}\right)=-\sin A\sin B\sin C.$$

Substituting into the above identity, we have that

$$O_1O_2^2=\frac{32abcp^2R^3}{F^2}\sin A\sin B\sin C.$$

But $F=\dfrac{abc}{4R}$, $\sin A=\dfrac{a}{2R}$ and analogues, then $O_1O_2^2=64p^2R^2$, so $O_1O_2 = 8R\sin\dfrac{A}{3}\sin\dfrac{B}{3}\sin\dfrac{C}{3}$.

In the same way $O_2O_3 = O_3O_1 = 8pR$, so $O_1O_2O_3$ is an equilateral triangle.

Theorem 2 (The second Morley's triangle). *In any triangle, the three points of intersection of the adjacent exterior angle bisectors, form an equilateral triangle.*

Demonstration 1.

Similarly Demonstration 1 from Theorem 1 (see Fig. 3), we have in triangle BO_1C that

$$CO_1 = \frac{a \sin \dfrac{\pi - B}{3}}{\sin \dfrac{\pi + A}{3}} = \frac{2R \sin A \sin \dfrac{\pi - B}{3}}{\sin \dfrac{\pi + A}{3}} = \frac{-2R \sin(\pi + A) \sin \dfrac{\pi - B}{3}}{\sin \dfrac{\pi + A}{3}}.$$

Taking (1) into account, we have

$$CO_1 = \frac{-2R \cdot 4 \sin \dfrac{\pi + A}{3} \cdot \sin \dfrac{2\pi + A}{3} \sin\left(-\dfrac{A}{3}\right) \sin \dfrac{\pi - B}{3}}{\sin \dfrac{\pi + A}{3}},$$

from where

(17) $$CO_1 = 8R \sin \frac{A}{3} \sin \frac{\pi - A}{3} \sin \frac{\pi - B}{3}$$

and similarly

(18) $$CO_2 = 8R \sin \frac{B}{3} \sin \frac{\pi - B}{3} \sin \frac{\pi - A}{3}.$$

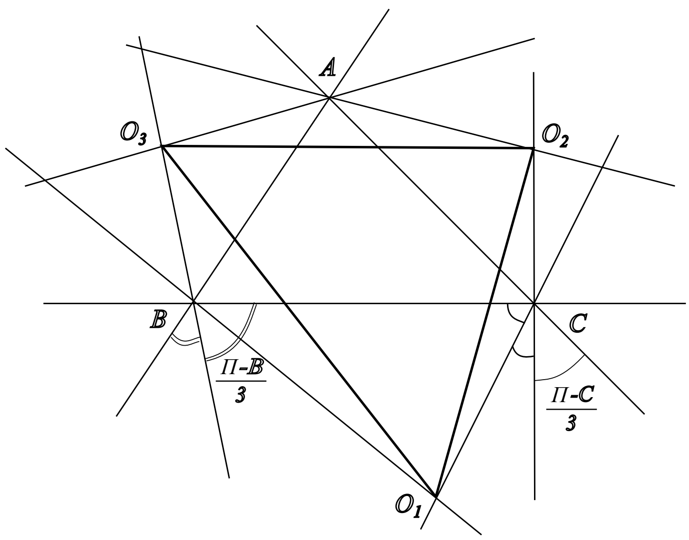

Figure 3

Applying the cosine theorem in triangle CO_1O_2 and taking (17) and (18) into account, we have

$$\begin{aligned} O_1O_2^2 &= CO_1^2 + CO_2^2 - 2CO_1 \cdot CO_2 \cdot \cos \frac{2\pi + C}{3} \\ &= 64R^2 \sin^2 \frac{\pi - A}{3} \sin^2 \frac{\pi - B}{3} \cdot \\ &\cdot \left(\sin^2 \frac{A}{2} + \sin^2 \frac{B}{3} - 2 \sin \frac{A}{3} \sin \frac{B}{3} \cos \frac{2\pi + C}{3} \right). \end{aligned}$$

Because $\dfrac{A}{3} + \dfrac{B}{3} + \dfrac{2\pi + C}{3} = \pi$, and using relation (2), we obtain that

$O_1O_2^2 = 64R^2 \sin^2 \dfrac{\pi - A}{3} \sin^2 \dfrac{\pi - B}{3} \sin^2 \dfrac{2\pi + C}{3}$ and how

$\sin \dfrac{2\pi + C}{3} = \sin\left(\pi - \dfrac{2\pi + C}{3}\right) = \sin \dfrac{\pi - C}{3}$, then

$O_1O_2^2 = 64R^2 \sin^2 \dfrac{\pi - A}{3} \sin^2 \dfrac{\pi - B}{3} \sin^2 \dfrac{\pi - C}{3}$, so

$$O_1O_2 = 8R \sin \frac{\pi - A}{3} \sin \frac{\pi - B}{3} \sin \frac{\pi - C}{3}. \tag{19}$$

Similarly $O_2O_3 = O_3O_1 = \sin \dfrac{\pi - A}{3} \sin \dfrac{\pi - B}{3} \sin \dfrac{\pi - C}{3}$, so triangle $O_1O_2O_3$ is equilateral triangle.

Demonstration 2. Let $O_2' \in (CO_2$ such that $\widehat{O_1O_2'C} = \dfrac{A}{3}$ and then

$\widehat{O_2'O_1C} = \pi - \dfrac{A}{3} - \dfrac{2\pi + C}{3} = \dfrac{\pi - A - C}{3} = \dfrac{B}{3}$ (see Fig. 4).

From the theorem of sine in the triangle $O_1O_2'C$, we have $\dfrac{CO_1}{\sin \frac{A}{3}} = \dfrac{CO_2'}{\sin \frac{B}{3}}$ and taking (17) into account, we obtain

$$CO_2' = \frac{CO_1 \sin \frac{B}{3}}{\sin \frac{A}{3}} = 8R \sin \frac{B}{3} \sin \frac{\pi - A}{3} \sin \frac{\pi - B}{3}.$$

Then according to (18), it follows that $CO_2' = CO_2$ and how $O_2' \in (CO_2$, it results that O_2' coincides with O_2. At this point we are in the situation in Demonstration 1.

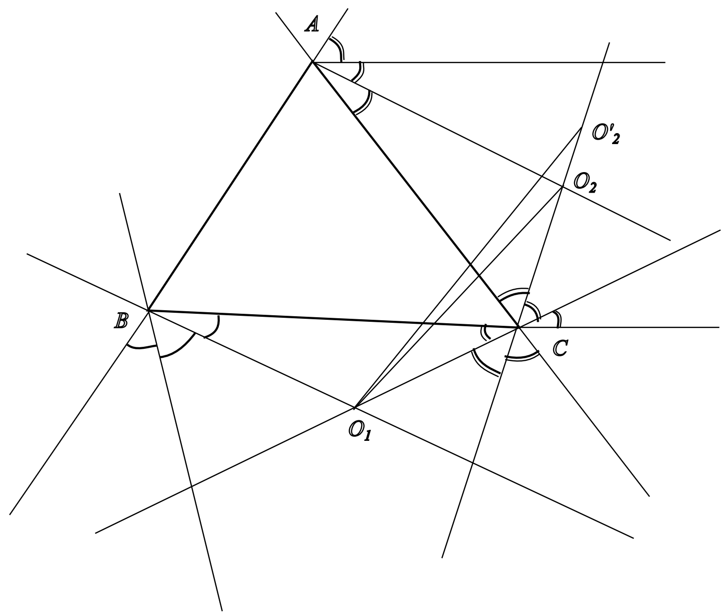

Figure 4

Demonstration 3. We have $ZO_1 - Z_C = \dfrac{CO_1}{b}(Z_A - Z_C)\varepsilon_{\frac{\pi+2C}{3}}$,
$Z_{O_2} - Z_C = \dfrac{CO_2}{b}(Z_A - Z_C)\varepsilon_{-\frac{\pi-C}{3}}$ and because $\varepsilon\dfrac{\pi+2C}{3} = -\varepsilon_{-\frac{2\pi-2C}{3}}$, it results that $Z_{O_1} - Z_C = -\dfrac{CO_2}{b}(Z_A - Z_C)\varepsilon_{-\frac{2\pi-C}{3}}$ (see Fig.3).
Then $ZO_1 - ZO_2 = -\dfrac{Z_A - Z_C}{b}\varepsilon_{\frac{\pi-C}{3}}\left(CO_1 \cdot \varepsilon_{-\frac{\pi-C}{3}} + CO_2\right)$ and taking (16) and (17) into account, we have

$$Z_{O_1} - Z_{O_2} =$$
$$= -\frac{Z_A - Z_C}{b}\varepsilon_{-\frac{\pi-C}{3}}\, 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\left(\sin\frac{A}{3}\,\varepsilon_{-\frac{\pi-C}{3}} + \sin\frac{B}{3}\right) =$$
$$= -\frac{Z_A - Z_C}{b}\,\varepsilon_{-\frac{\pi-C}{3}} 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\cdot$$
$$\cdot\left(\left(\sin\frac{A}{3}\cos\frac{\pi-C}{3} + \sin\frac{B}{3}\right) - \sin\frac{A}{3}\sin\frac{\pi-C}{3}\,i\right).$$

Moving to the modulus, we have that

$$O_1O_2 = |Z_{O_1} - Z_{O_2}| = \frac{|Z_A - Z_C|}{b}\left|\varepsilon_{-\frac{\pi-C}{3}}\right| 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\cdot$$
$$\cdot\sqrt{\left(\sin\frac{A}{3}\cos\frac{\pi-C}{3} + \sin\frac{B}{3}\right)^2 + \left(-\sin\frac{A}{3}\sin\frac{\pi-C}{3}\right)^2}$$
$$= 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\cdot\sqrt{\sin^2\frac{A}{3} + \sin^2\frac{B}{3} + 2\sin\frac{A}{3}\sin\frac{B}{3}\cos\frac{\pi-C}{3}} =$$
$$= 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{2}\sqrt{\sin^2\left(-\frac{A}{3}\right) + \sin^2\left(-\frac{B}{3}\right) + 2\sin\left(-\frac{A}{3}\right)\sin\left(-\frac{B}{3}\right)\cos\frac{\pi-C}{3}}\,.$$

Because $\left(-\dfrac{A}{3}\right) + \left(-\dfrac{B}{3}\right) + \dfrac{\pi-C}{3} = 0$ and taking (3) into account, from the above we obtain that $O_1O_2 = 8R\sin\dfrac{\pi-A}{3}\sin\dfrac{\pi-B}{3}\sin\dfrac{\pi-C}{3}$, so $O_1O_2O_3$ is an equilateral triangle.

Demonstration 4. We consider $\alpha_1, \beta_1\gamma_1$ the barycentric coordinates of O_1 and $\alpha_2, \beta_2, \gamma_2$ the barycentric coordinates of O_2. Using (17), (18) and their analogues, we have

$$\alpha_1 = -\frac{d(O_1, BC)}{d(A, BC)} = -\frac{CO_1 \cdot \sin\dfrac{2(\pi-C)}{3}a}{2F}$$
$$= -\frac{8R\sin\dfrac{A}{3}\sin\dfrac{\pi-A}{3}\sin\dfrac{\pi-B_2}{3}\sin\dfrac{\pi-C}{3}\cos\dfrac{\pi-C}{3}}{2F}$$

from where

(20) $$\alpha_1 = -\frac{4ap'R}{F}\sin\frac{A}{3}\,,$$

where $p' = \sin\dfrac{\pi - A}{3}\sin\dfrac{\pi - B}{3}\sin\dfrac{\pi - C}{3}$,

$$\beta_1 = \frac{d(O_1, CA)}{d(B, CA)} = \frac{CO_1 \sin\dfrac{2(\pi - C)}{3} b}{2F}$$
$$= \frac{8R\sin\dfrac{A}{3}\sin\dfrac{\pi - A}{3}\sin\dfrac{\pi - B}{3}\cdot 2\sin\dfrac{\pi - C}{3}\cos\dfrac{\pi - C}{3}\cdot b}{2F},$$

from where

$$\beta_1 = \frac{8bp'R}{F}\sin\frac{A}{3}\cos\frac{\pi - C}{3}, \tag{21}$$

$$\gamma_1 = \frac{d(O_1, AB)}{d(C, AB)} = \frac{BO_1 \cdot \sin\dfrac{2(\pi - B)}{3}\cdot c}{2F}$$
$$= \frac{8R\sin\dfrac{A}{3}\sin\dfrac{\pi - A}{3}\sin\dfrac{\pi - C}{3} 2\sin\dfrac{\pi - B}{3}\cos\dfrac{\pi - B}{3}}{2F},$$

from where

$$\gamma_1 = \frac{8cp'R}{F}\sin\frac{A}{3}\cos\frac{\pi - B}{3}. \tag{22}$$

Similarly, taking into account that $\alpha_2 = \dfrac{d(O_2, BC)}{d(A, BC)}$, $\beta_2 = -\dfrac{d(O_2, CA)}{d(B, CA)}$ and $\gamma_2 = \dfrac{d(O_2, AB)}{d(C, AB)}$, we have that

$$\alpha_2 = \frac{8ap'R}{F}\sin\frac{B}{3}\cos\frac{\pi - C}{3}, \tag{23}$$

$$\beta_2 = -\frac{4bp'R}{F}\sin\frac{B}{3} \tag{24}$$

and

$$\gamma_2 = \frac{8cp'R}{F}\sin\frac{B}{3}\cos\frac{\pi - A}{3}. \tag{25}$$

Using (20) and (23) we have $\alpha_1 - \alpha_2 = -\dfrac{4ap'R}{F}\left(\sin\dfrac{A}{3} + 2\sin\dfrac{B}{3}\cos\dfrac{\pi - C}{3}\right)$ and because $2\sin\dfrac{B}{3}\cos\dfrac{\pi - C}{3} = \sin\dfrac{\pi - C + B}{3} + \sin\dfrac{B - \pi + C}{3} =$
$= -\sin\left(\dfrac{\pi - C + B}{3} - \pi\right) + \sin\left(-\dfrac{A}{3}\right)$, so

$$\alpha_1 - \alpha_2 = -\frac{4ap'R}{F}\sin\frac{B - C - 2\pi}{3}. \tag{26}$$

Similarly, using (21) and (24), we have

$$\beta_1-\beta_2=\frac{4bp'R}{F}\Big(2\sin\frac{A}{3}\cos\frac{\pi-C}{3}+\sin\frac{B}{3}\Big)=\frac{4bp'R}{F}\sin\frac{A+\pi-C}{3},\text{ so}$$

(27)
$$\beta_1-\beta_2=\frac{4bp'R}{F}\sin\frac{\pi+B+2C}{3}\,.$$

From (22) and (25) we have

$$\gamma_1-\gamma_2=\frac{8cp'R}{F}\left(\sin\frac{A}{3}\cos\frac{\pi-B}{3}-\sin\frac{B}{3}\cos\frac{\pi-A}{3}\right)=$$
$$=\frac{8cp'R}{F}\,\frac{1}{2}\left(\sin\frac{A+\pi-B}{3}+\sin\frac{A-\pi+B}{3}-\frac{B+\pi-A}{3}-\sin\frac{B-\pi+A}{3}\right)=$$
$$=\frac{8cp'R}{F}\sin\frac{A-B}{3}\cos\frac{\pi}{3}\,,$$

so

(28)
$$\gamma_1-\gamma_2=\frac{4cp'R}{F}\sin\frac{\pi-2B-C}{3}\,.$$

Taking (26)-(28), $a=2R\sin A=2R\sin(\pi-B-C)$ and analogues into account, we have

$$O_1O_2^2=-(\beta_1-\beta_2)(\gamma_1-\gamma_2)a^2-(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)b^2-(\alpha_1-\alpha_2)(\beta_1-\beta_2)c^2=$$
$$=-\frac{32abcp'^2R^3}{F^2}\Big(\sin\frac{\pi+B+2C}{3}\sin\frac{\pi-2B-C}{3}\sin(\pi-B-C)-$$
$$-\sin\frac{\pi-2B-C}{3}\sin\frac{B-C-2\pi}{3}\sin B-\sin\frac{B-C-2\pi}{3}\sin\frac{\pi+B+2C}{3}\sin C\Big).$$

Using the identity (4) and as $A+B+C=\pi$, after calculus similarly Demonstration 4 from Theorem 1, we obtain

$O_1O_2=8p'R=8R\sin\dfrac{\pi-A}{3}\sin\dfrac{\pi-B}{3}\sin\dfrac{\pi-C}{3}$, so the triangle $O_1O_2O_3$ is an equilateral triangle.

Theorem 3 (The third Morley's triangle). *In any triangle, the bisector of two exterior angles and one interior angle of the given triangle, form an equilateral triangle.*

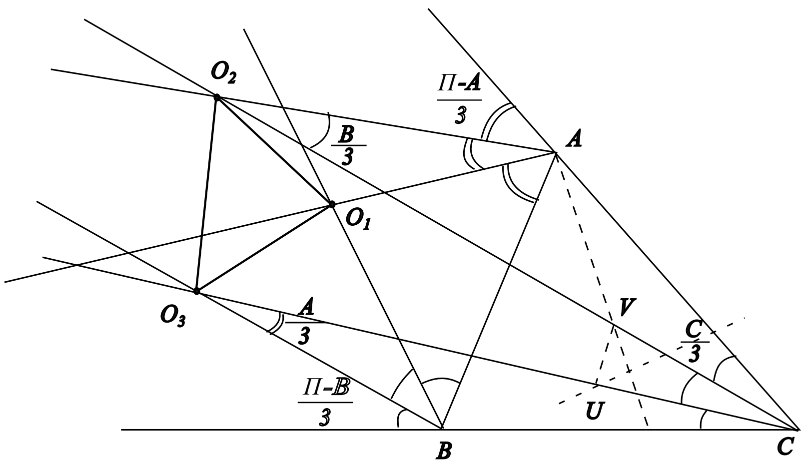

Figure 5

Demonstration. We consider the bisectors of the exterior angles A and B and the bisector of the interior angle C (see Fig. 5).
The four methods of proof this theorem are identical to those in Theorem 1 and Theorem 2, so we will not give the full proofs. For each proof, we will give the important results that appear in the demonstration.
Demonstration 1. We have

$$AO_1 = 8R\sin\frac{C}{3}\sin\frac{\pi-C}{3}\sin\frac{\pi-B}{3}, \tag{29}$$

$$AO_2 = 8R\sin\frac{C}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi-B}{3}, \tag{30}$$

$$CO_2 = 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\sin\frac{\pi+B}{3}, \tag{31}$$

$$CO_3 = 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\sin\frac{\pi+A}{3} \tag{32}$$

and

$$O_1O_2 = O_2O_3 = O_3O_1 = 8R\sin\frac{\pi-A}{3}\sin\frac{\pi-B}{3}\sin\frac{C}{3}. \tag{33}$$

Demonstration 3. We have

$$Z_{O_1} - Z_A = \frac{AO_1}{c}(Z_B - Z_A)\varepsilon_{-\frac{\pi-A}{3}} \tag{34}$$

and

$$Z_{O_2} - Z_A = \frac{AO_2}{c}(Z_B - Z_A)\varepsilon_{-\frac{2(\pi-A)}{3}}. \tag{35}$$

Demonstration 4. We have

$$\alpha_1 = \frac{8ap'R}{F}\sin\frac{\pi-C}{3}\cos\frac{\pi-B}{3}, \tag{36}$$

$$\beta_1 = \frac{8bp'R}{F}\sin\frac{\pi-C}{3}\cos\frac{\pi-A}{3}, \tag{37}$$

$$\gamma_1 = -\frac{4cp'R}{F}\sin\frac{\pi-C}{3}, \tag{38}$$

$$\alpha_2 = \frac{8ap'R}{F}\sin\frac{\pi+B}{3}\cos\frac{C}{3}, \tag{39}$$

$$\beta_2 = \frac{4bp'R}{F}\sin\frac{\pi+B}{3}, \tag{40}$$

$$\gamma_2 = -\frac{8cp'R}{F}\sin\frac{\pi+B}{3}\cos\frac{\pi-A}{3}, \tag{41}$$

$$\alpha_3 = \frac{4ap'R}{F}\sin\frac{\pi+A}{3}, \tag{42}$$

$$\beta_3 = \frac{4bp'R}{F}\sin\frac{\pi+A}{3}\cos\frac{C}{3}, \tag{43}$$

(44) $$\gamma_3 = -\frac{8cp'R}{F}\sin\frac{\pi + A}{3}\cos\frac{\pi - B}{3},$$

(45) $$\alpha_1 - \alpha_2 = \frac{4ap'R}{F}\sin\frac{B - C - \pi}{3},$$

(46) $$\beta_1 - \beta_2 = \frac{4bp'R}{F}\sin\frac{\pi - B - 2C}{3},$$

(47) $$\gamma_1 - \gamma_2 = \frac{4cp'R}{F}\cos\frac{\pi - 2B - C}{3},$$

(48) $$\alpha_2 - \alpha_3 = \frac{4ap'R}{F}\sin\frac{\pi + B - C}{3},$$

(49) $$\beta_2 - \beta_3 = -\frac{4ap'R}{F}\sin\frac{\pi + B + 2C}{3},$$

and

(50) $$\gamma_2 - \gamma_3 = 0.$$

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