DIDACTICA MATHEMATICA, Vol. 43 (2025) , pp. 01–10

THE PREY-PREDATOR MODEL HIDDEN IN A PROBLEM OF ELEMENTARY GEOMETRY

Medeea-Drusila Apăscăriţei[1] and Eduard Ştefan Grigoriciuc[1,2]

Abstract. We consider in this paper a problem of elementary geometry that can be interpreted in a completely unexpected context, as a very simple prey-predator mathematical model. The aim of the paper is to offer students a chance to see a real application of what they learn in the geometry class.

MSC 2000. 11C20, 15A06

Key words. elementary geometry, applied mathematics, mathematical modeling, prey-predator model

1. INTRODUCTION

What are the applications of mathematics in real life? This is probably the most common question among students in the first years of study. Beyond formulas, theorems, and unsolved problems, many of them want to see applications of what they learn in everyday life. But how do you explain to an young child that, in fact, everything around us can be modeled with the help of mathematics? What is “mathematical modeling”? And what does this involve?

In this paper we will look at a problem of elementary geometry that can be interpreted in a completely unexpected context. A simple problem of similarity of triangles can be interpreted as a prey-predator mathematical model. Obviously, this model will be extremely simple and we will not take into account all the elements that appear in such a context. We just want to give students who solve this problem the chance to see (as much as possible) a real application of what they learn in the geometry class.

2. A PROBLEM OF ELEMENTARY GEOMETRY

In order to illustrate the discussion presented above, let us consider a problem of elementary geometry. First of all, we solve this problem using simple results in geometry, such as the similarity of triangles and congruence criteria (for details, one may consult [2], [3], [4] or [6]).

Problem 1. Let $ABCD$ be a parallelogram and let d be a line passing through A such that $d \cap CD = \{M\}$ and $d \cap BC = \{N\}$. Prove that the product $DM \cdot BN$ is constant.

Proof. We prove that $DM \cdot BN = const.$ using the similarity of triangles. Depending on the position of the line d, the points M and N may lie either on the sides of the parallelogram or on their extensions.

I. We shall first consider the case when d passes through the interior of the parallelogram (so that both M and N lie on the sides CD and BC). In this case, we distinguish three subcases according to the position of M on CD.

- **Case 1**: Let $DM < DC$ (see Figure 2.1). First, because $ABCD$ is a parallelogram, we have that $AB \parallel CD$ and $BC \parallel AD$. If we consider d a transversal line that intersects these pairs of parallels, then

$$\widehat{MAD} \equiv \widehat{ANB}$$

and

$$\widehat{DMA} \equiv \widehat{NAB}.$$

Hence, according to the Angle-Angle criteria (i.e. if two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar, because the third angle in both triangles must be equal, see e.g. [2] or [4]), we deduce that $ADM \sim NBA$. Moreover,

(1) $$\frac{AD}{NB} = \frac{DM}{BA} = \frac{AM}{NA}.$$

From relation (1) we obtain that

$$DM \cdot BN = AB \cdot AD = const.$$

because the dimensions of the $ABCD$ parallelogram are constant.

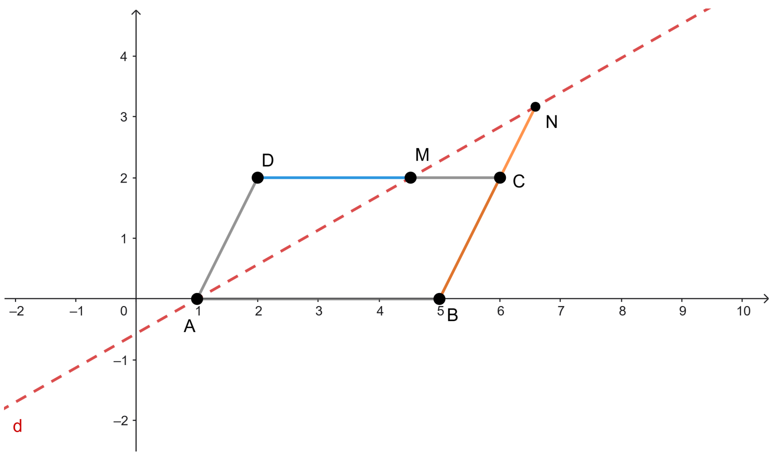

Fig. 2.1 – The case when $DM < DC$.

- **Case 2**: Let $DM = DC$ (see Figure 2.2). In this case, the point M coincides with C. Consequently, N coincides also with C. Then we have

$$DM \cdot BN = DC \cdot BC,$$

which is exactly the same constant as before (since $ABCD$ is a parallelogram, $BC = AD$ and $AB = DC$).

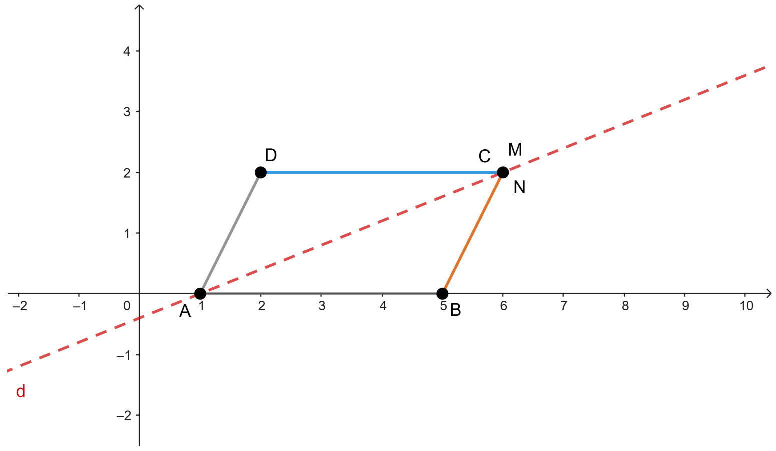

Fig. 2.2 – The case when $DM = DC$.

- **Case 3**: Let $DM > DC$ (see Figure 2.3). Here, the point N lies on the extension of BC beyond C. The triangles $\triangle ADM$ and $\triangle NBA$ remain similar. Hence, the same proportionality holds:

$$\frac{DM}{AB} = \frac{AD}{BN} \implies DM \cdot BN = AB \cdot AD, \tag{2}$$

and the product remains constant.

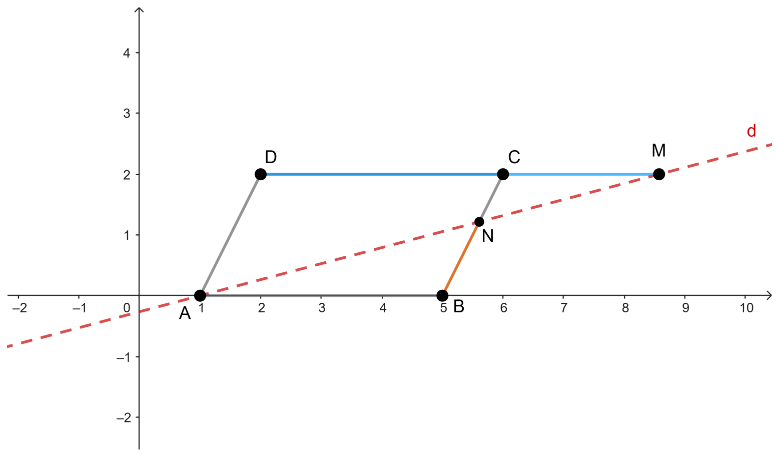

Fig. 2.3 – The case when $DM > DC$.

II. Finally, we consider the situation when the line d does not pass through the interior of the parallelogram, so that M lies on the extension of

CD beyond D and N lies on the extension of BC beyond B (see Figure 2.4).

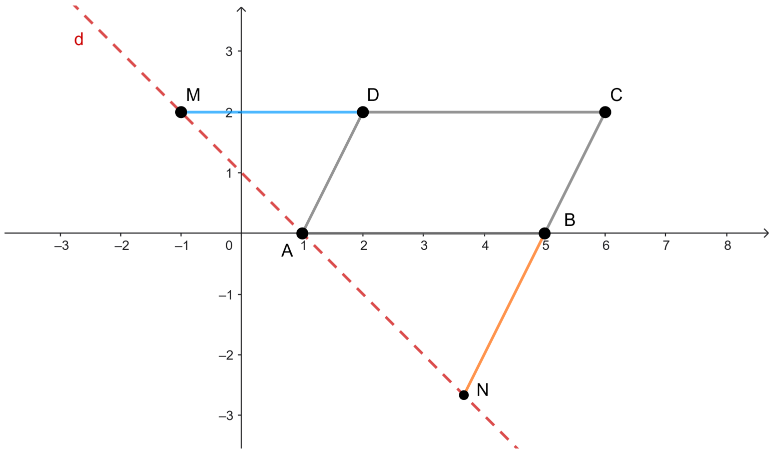

Fig. 2.4 – The line d does not pass through the interior of the parallelogram

The triangles $\triangle ADM$ and $\triangle NBA$ remain similar by the same angle argument as above, and the proportionality

$$\frac{DM}{AB} = \frac{AD}{BN} \implies DM \cdot BN = AB \cdot AD, \tag{3}$$

still holds. Therefore, the product $DM \cdot BN$ is constant also in this configuration and this completes the proof.

□

3. OTHER PROOFS OF PROBLEM 1

In this section we present alternative solutions to Problem 1, illustrating the variety of methods that can be employed in the study of an elementary geometry problem (for details, one may consult [2], [3], [4] or [6]).

3.1. The second solution to Problem 1. The first alternative proof uses simple elements of vector geometry. Without going into too much detail, we can prove that the product $DM \cdot BN$ is constant.

Proof. Since points D, M, C are collinear, the vector $\overrightarrow{DM}$ must be proportional to $\overrightarrow{DC}$. Hence, there exists $k_1 \in \mathbb{R}$ such that

$$\overrightarrow{DM} = k_1 \, \overrightarrow{DC}.$$

Similarly, since the points B, N, C are collinear, the vector $\overrightarrow{BN}$ is proportional to $\overrightarrow{BC}$. Thus, there exists $k_2 \in \mathbb{R}$ such that

$$\overrightarrow{BN} = k_2 \, \overrightarrow{BC}.$$

Multiplying the two relations, we obtain

$$\overrightarrow{DM} \cdot \overrightarrow{BN} = (k_1\overrightarrow{DC}) \cdot (k_2\overrightarrow{BC}) = k_1k_2\,(\overrightarrow{DC} \cdot \overrightarrow{BC}).$$

If we are interested in the product of the segment lengths rather than the scalar product of vectors, the same proportionality gives

$$DM \cdot BN = |k_1|\,|k_2|\,DC \cdot BC. \tag{4}$$

Since DC and BC are fixed sides of the parallelogram, the quantity $|k_1||k_2|$ depends only on how the line d intersects the two sides, but the geometric configuration forces the product k_1k_2 (or $|k_1||k_2|$ for lengths) to remain constant. Therefore, the product $DM \cdot BN$ is equal to a constant value that depends only on the parallelogram, not on the choice of the line d, and this completes the proof. □

3.2. The third solution to Problem 1. The next proof is based on the idea of placing the parallelogram in the complex plane and identifying the points with their complex affixes.

Proof. Let

$$A = a, \qquad B = a + b, \qquad D = a + d,$$

with $a \in \mathbb{C}$ and $b, c, d \in \mathbb{C}^*$. Moreover, assume that $d \neq xb$, for all $x \in \mathbb{R}$. In this case,

$$C = a + b + d.$$

The line d through $A = a$ meets CD and BC at points M and N, so there exist $s, t \in \mathbb{R}$such that

$$M = D + s \cdot (B - A) = a + s \cdot b + d$$

and

$$N = B + s \cdot (D - A) = a + b + t \cdot d,$$

with $t, s \in \mathbb{R}$. The collinearity of A, M, N means the complex numbers M and N are real multiples of one another, hence

$$\frac{M - A}{N - A} = \frac{s \cdot b + d}{b + t \cdot d} \in \mathbb{R}. \tag{5}$$

Therefore, there exists $k \in \mathbb{R}$ such that

$$s \cdot b + d = k \cdot (b + t \cdot d).$$

Rearranging the terms, we obtain that

$$b \cdot (s - k) + d \cdot (1 - k \cdot t) = 0.$$

Since b and d are linearly independent (because $d \neq x \cdot b$, for all $x \in \mathbb{R}$), the two real coefficients must vanish, i.e.

$$s - k = 0, \quad \text{and} \quad 1 - k \cdot t = 0,$$

where $k \neq 0$. Then $s = k$ and $t = \frac{1}{k}$. Thus

$$|s \cdot t| = |s| \cdot |t| = k \cdot \frac{1}{k} = 1.$$

Now DM is the length of the segment from D to M, $DM = |M - D|$. In our normalization $AB = |b|$ and $AD = |d|$, and because $M = a + s \cdot b + d$ and $N = a + b + td$, with $s, t \in \mathbb{R}$, it follows that

$$DM = |s| \cdot |AB| = |s| \cdot |b|$$

and

$$BN = |t| \cdot |AD| = |s| \cdot |d|.$$

Therefore

$$DM \cdot BN = |t| \cdot |s| \cdot |b| \cdot |d| = |st| \cdot |b| \cdot |d| = 1 \cdot |b| \cdot |d| = AB \cdot AD$$

and this completes the proof. □

3.3. The fourth solution to Problem 1. The last proof returns to elements of elementary geometry, more precisely, the study of the properties of the angles of the quadrilateral $ABCD$.

Proof. From the geometric configuration presented above, we have the angle equalities

$$\widehat{DAM} = \widehat{ANB} \qquad \text{and} \qquad \widehat{DMA} = \widehat{NAB}.$$

Therefore,

$$\sin \widehat{DAM} = \sin \widehat{ANB} \quad \text{and} \quad \sin \widehat{DMA} = \sin \widehat{NAB}. \tag{6}$$

Consider triangle ADM and let DE be the altitude from D onto AM. Then:

$$\sin \widehat{DMA} = \frac{DE}{DM} \quad \Longrightarrow \quad DM = \frac{DE}{\sin \widehat{DMA}}.$$

Also,

$$\sin \widehat{DAM} = \frac{DE}{AD} \quad \Longrightarrow \quad DE = AD \cdot \sin \widehat{DAM}.$$

Substituting DE into the expression for DM, we obtain

$$DM = \frac{AD \cdot \sin \widehat{DAM}}{\sin \widehat{DMA}}. \tag{7}$$

Next, consider triangle ABN and let BF be the altitude from B onto AN. Then:

$$\sin \widehat{NAB} = \frac{BF}{AB} \quad \Longrightarrow \quad BF = AB \cdot \sin \widehat{NAB},$$

and

$$\sin \widehat{ANB} = \frac{BF}{BN} \quad \Longrightarrow \quad BN = \frac{BF}{\sin \widehat{ANB}}.$$

Now, substituting BF, it follows that

$$BN = \frac{AB \cdot \sin \widehat{NAB}}{\sin \widehat{ANB}}. \tag{8}$$

Using the previously established equalities

$$\sin \widehat{DAM} = \sin \widehat{ANB}, \qquad \sin \widehat{DMA} = \sin \widehat{NAB},$$

and multyplying the formulas for DM and BN, we deduce that

$$DM \cdot BN = \frac{AD \cdot \sin \widehat{DAM}}{\sin \widehat{DMA}} \cdot \frac{AB \cdot \sin \widehat{NAB}}{\sin \widehat{ANB}}.$$

Replacing each sine term with its corresponding equal one gives

$$DM \cdot BN = \frac{AD \cdot \sin \widehat{ANB}}{\sin \widehat{NAB}} \cdot \frac{AB \cdot \sin \widehat{NAB}}{\sin \widehat{ANB}}. \tag{9}$$

All sine terms cancel out, leaving the constant product:

$$DM \cdot BN = AB \cdot AD$$

and this completes the proof. □

4. MORE THAN A PROBLEM OF ELEMENTARY GEOMETRY

Let us denote $AB = x_0$ and $BC = y_0$ the dimensions of the $ABCD$ parallelogram. Then,

$$CD = AB = x_0 > 0, \quad AD = BC = y_0 > 0$$

and

$$AB \cdot AD = x_0 \cdot y_0 = c, \tag{10}$$

where $c > 0$ is a positive constant.

Problem 2. Let us consider that

- $x_0 = AB > 0$ is a fixed positive number of carrots,
- $y_0 = AD > 0$ is a fixed positive number of rabbits,

which are together in a rabbit pen. In view of relation (10), we know that between them there holds a relation of the form

$$y_0 = \frac{c}{x_0},$$

where $c > 0$ is a constant. Hence, x_0 and y_0 are inversely proportional.

Remark 1. In this context, we say that there is an equilibrium between rabbits and carrots if the product

$$x_0 y_0 = c$$

is constant. In other words, there is enough food in the pen for all the rabbits, but not too much so that it is wasted. The equilibrium value $c > 0$ can be established from the beginning of the problem, considering the geometric interpretation of the two dimensions x_0 and y_0 (see Figure 2.2).

An interesting case of the two populations (quantities) is when we vary the number of rabbits in the pen. For this, let x_1 be the number of carrots and y_1 be the number of rabbits at a given (arbitrary) time t. Using again Figure 1, we have $x_1 = DM$ (blue line) and $y_1 = BN$ (orange line). In this context, we distinguish two cases

- **Case 1:** $y_1 = AN < AD = y_0$, i.e. the number of rabbits is reduced (i.e. less than the number of rabbits that would be needed to reach the equilibrium state). In view of relation (10) we have that
$$x_1 \cdot y_1 = x_0 \cdot y_0 = c.$$
On the other hand,
$$\frac{y_1}{y_0} < 1$$
and hence
$$\frac{x_0}{x_1} = \frac{y_1}{y_0} < 1$$
which means that
$$x_0 < x_1,$$
i.e. the number of carrots is bigger than the number of carrots that would give the equilibrium in the pen (in this case, it is clear that the food will be wasted). Obviously, if the number of rabbits is smaller, then the number of carrots will be larger since fewer rabbits will eat fewer carrots.
- **Case 2:** $y_1 = BN > AD = y_0$, i.e. the number of rabbits is greater than the number of rabbits that would give the equilibrium in the pen. In view of relation (10) we have that
$$x_1 \cdot y_1 = x_0 \cdot y_0 = c.$$
On the other hand,
$$\frac{y_1}{y_0} > 1$$
and hence
$$\frac{x_0}{x_1} = \frac{y_1}{y_0} > 1$$
which means that
$$x_0 > x_1,$$
i.e. the number of carrots is smaller than the number of carrots that would give the equilibrium state in the pen. Again, it is clear that if the number of rabbits is greater, then the number of carrots will decrease (i.e. more rabbits will eat more carrots, so the food will not be wasted, but it may not be enough for each rabbit).

5. CONCLUSION

The geometric problem gives us a static relation between populations (an equilibrium curve). However, real predator–prey interactions evolve in time and are modeled by differential equations systems (for details, one may consult [1], [5] or [7]). The geometric invariant corresponds to one particular level set of such a dynamic system. The geometric identity

$$DM \cdot BN = AB \cdot AD$$

shows that the two quantities represented by the segments DM and BN are inversely proportional. When interpreted biologically, this expresses a static equilibrium between two interacting populations (for example, prey and predators; here, carrots and rabbits): if one quantity decreases, the other must increase so that their product remains constant. In this sense, the geometric configuration provides a simple and intuitive image of balance within an ecological system.

Of course, real biological populations do not remain in a fixed equilibrium. They evolve in time, influenced by birth rates, death rates, food availability, and many external factors. Classical predator-prey models, such as the Lotka-Volterra system (see e.g. [1] or [7]), describe this evolution using differential equations. A remarkable fact is that the trajectories of these dynamic systems often lie on level curves of a conserved quantity. In simplified situations, these level curves can resemble hyperbolas of the form

$$xy = c > 0,$$

where $c > 0$ is constant, precisely the type of invariant relation that appears in our problem. Thus, the geometric identity we studied can be viewed as the static analogue of an invariant curve in a genuine unsteady predator-prey model.

This parallel illustrates an important message: even simple geometric problems can contain the seeds of real-world mathematical modeling. The constancy of the product $DM \cdot BN$, obtained using elementary geometry, mirrors the idea of conserved quantities and equilibrium relations found in biological systems. By making this connection explicit, we offer an accessible pathway from the familiar world of triangles and parallelograms to the more advanced concepts of mathematical modeling. In this way, the study of geometry becomes not only a logical exercise, but also an invitation to explore how mathematics describes the living world.

REFERENCES

[1] E.S. Allman, J.A. Rhodes: *Mathematical Models in Biology - An Introduction*, Cambridge University Press, 2003

[2] D. Andrica: *GEOMETRIE. Teme pentru perfecţionarea profesorilor de matematică* (vol. 4), Casa Cărţii de Ştiinţă, 2017

[3] D. Andrica, C. Varga, D. Văcăreţu: *Teme şi probleme alese de geometrie*, Editura Plus, Bucureşti, 2002
[4] D. Andrica, D.I. Duca, I. Purdea, I. Pop: *Matematică de bază* (ediţia a V-a), Ed. Studium, Cuj-Napoca, 2004
[5] F. Hoppensteadt, *Predator-prey model*, Scholarpedia, 1(10):1563, 2006
[6] C. Pintea: *Geometrie. Elemente de geometrie analitica. Elemente de Geometrie diferentiala a curbelor si suprafetelor*, Presa Universitara Clujeana, 2001
[7] C.H. Taubes: *Modeling Differential Equations in Biology* (2nd Edition), Cambridge University Press, 2008

[1] *Faculty of Mathematics and Computer Science*
Babeş-Bolyai University
M. Kogălniceanu 1, 400084
Cluj-Napoca, Romania

[2] *Tiberiu Popoviciu Institute of Numerical Analysis*
Romanian Academy
P.O. Box 68-1, 400110
Cluj-Napoca, Romania
e-mail: `medeea.apascaritei@stud.ubbcluj.ro`
e-mail: `eduard.grigoriciuc@ubbcluj.ro`