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APPLICATIONS OF WEIERSTRASS' FACTORIZATION THEOREM

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Abstract. In this paper, I want to give a quick overview of Weierstrass' Factorization Theorem for complex functions, explain its importance, and then present some of the most important applications of the theorem: writing the trigonometric functions as infinite products, solving the Basel problem, proving Wallis' Formula and justifying one of the definitions of the Riemann Zeta Function.

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1. INTRODUCTION OF THE WEIERSTRASS' FACTORIZATION THEOREM

To understand Weierstrass' Factorization Theorem, we first have to touch on the definition and properties of functions of infinite products of function.

Definition 1. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions that have complex values, defined in an open set $D \subset \mathbb{C}$, then $f = \prod_{n=1}^{\infty} f_n$ is said to be **compactly convergent** in D if for all compact sets $K \subset D$ there exists an $N \in \mathbb{N}$ such that $\left\{\prod_{n=N}^{d} f_n\right\}_{d=1}^{\infty}$ converges uniformly on K to a non-vanishing function g as $d \to \infty$, so $f = f_1 \cdot f_2 \cdot \ldots \cdot f_N \cdot g$ on K.

This definition can be found in Zakeri [9, Definition 8.13, p. 233] and Kohr, Mocanu [5, Definition 7.2.2, p. 240]. Following the definition, they list the following properties as well:

Theorem 1. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions that have complex values, defined on D such that $\displaystyle\sum_{n=1}^{\infty} |f_n - 1|$ converges compactly in D, then the following are true

(1) The infinite product $f = \displaystyle\prod_{n=1}^{\infty} f_n$ converges compactly in D.

(2) $f(p) = 0 \iff \exists\, n \in \mathbb{N}$ such that $f_n(p) = 0$

(3) $f = \displaystyle\prod_{n=N}^{\infty} f_n \to 1$ compactly in D as $N \to \infty$

(4) If all f_n are not identically zero, then we also have the logarithmic differentiation as $\dfrac{f'}{f} = \displaystyle\sum_{n=1}^{\infty} \dfrac{f_n'}{f_n}$ in D.

According to the fundamental theorem of algebra, every non-constant polynomial P has a unique factorization of the form

$$P(z) = Cz^m \prod_{n=1}^{k} \left(1 - \frac{z}{z_n}\right)$$

where $C \neq 0$ is a constant, $m \geq 0$ is an integer, and $z_1, \ldots, z_k$ are the non-zero roots of P, repeated according to their multiplicity. Note that if two polynomials have the same roots of the same multiplicities they will agree up to a multiplicative constant.

Now, we wonder if it is possible to find such a factorization for **entire** functions. The answer is **Yes! due to Weierstrass**.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex values in a subset D with no accumulation point in D, with the possibility of a point being repeated a finite number of times.

Taking into account the properties of complex-valued infinite products, if it is possible to find analytic functions $g_n(z)$ on D, with no zeros in D such that $\sum_{n=1}^{\infty} |(z - a_n)g_n(z) - 1|$ converges compactly on subsets of D, then $f(z) = \prod_{n=1}^{\infty}(z - a_n)g_n(z)$ is analytic and has zeros only at $\{a_n\}_{n=1}^{\infty}$.

The easiest way to make sure that $g_n(z)$ is never zero is to use the exponential function in combination with an analytic function $h(z)$, to get $g_n(z) = e^{h_n(z)}$. The functions $g_n(z)$ we need were introduced by Weierstrass.

Definition 2. The entire functions E_d defined for $d \in \mathbb{N}$ as:

$$E_d(z) = \begin{cases} 1 - z & \text{if } d = 0 \\ (1 - z)\exp\left(\displaystyle\sum_{n=1}^{d} \frac{z^n}{n}\right) & \text{if } d > 0 \end{cases}, \text{ where } exp(x) = e^z$$

are called the **Weierstrass' elementary factors**

This definition appears in Zakery [9, p. 236], as well as in Conway [1, p. 168]. They are also part of Kohr, Mocanu [5, Lemma 7.2.14, p. 247]. All these sources immediately continue with the following remarks:

- The elementary factors have only a simple zero, that is $z = 1$ so $E_d(1) = 0$.
- The function $E_d(z/a)$ has a zero only at $z = a$ and no other zero.
- If $|z| \leq 1$ then, the larger d is, the closer $E_d(z)$ is to 1. Also note that $E_d(0) = 1$.

THEOREM 2. **Weierstrass' factorization theorem from 1876**
If the function f is an entire function, non-identically zero whose zeros we can arrange in a sequence of the form $\{z_n\}_{n=1}^{\infty}$ so that each zero appears as many times as its multiplicity $m \geq 0$, and if $\{d_n\}_{n=1}^{\infty}$ is any sequence of non-negative integers such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{d_n+1} \quad \textit{is convergent } \forall\, r \geq 0 \text{ , } \textit{then}$$

we can give the function f the following factorization:

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_{d_n}\left(\frac{z}{z_n}\right) \quad \textit{where } \ g \in H(\mathbb{C}).$$

This theorem appears along with its full proof in Zakery [9, Theorem 8.24, p. 239], as well as in Conway [1, Theorem 5.14, p. 170] and in Kohr, Mocanu [5, Theorem 7.2.19, p. 251]. In all these sources, the necessary partial results are presented together with their proof, building up to a shorter overall proof of the factorization theorem itself.

REMARK 1. From the theorem, we observe that:

- To create a zero of order m at $z = 0$, all we have to do is multiply the infinite product by z^m
- The necessary convergence of the sum can be achieved if $d_n \geq n - 1$, but choosing d_n to be as small as possible gives us an advantage since the elementary factors will be smaller.

2. FACTORIZATION OF THE COMPLEX TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Using the consequence of Weierstrass's Factorization Theorem 2 we can rewrite the sine function as an infinite product. This is a powerful tool that allows us to give an infinite product form to all the other trigonometric functions, and hyperbolic functions as well. It is also an important part of proving Wallis's formula and solving the Basel problem.

Being such an important proof for further results, the proof of the infinite form of the sinus is given in a detailed form in Conway [1, p. 174–175], and in a blueprint form in Kohr, Mocanu [4, Example 7.2.21, p. 251]. Following their approach, I wanted to provide an extended proof that elaborates more on each step of the process. The rest of the results are given as exercises for the reader in Conway [1, p. 176] and in Kohr, Mocanu [4, p. 253].

(1) $\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

First, we must identify the zeros of $\sin(\pi z)$ using its complex form :

$$\sin(\pi z) = 0 \Leftrightarrow \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0 \Leftrightarrow e^{i\pi z} - \frac{1}{e^{i\pi z}} = 0 \Leftrightarrow e^{2i\pi z} = 1$$

Now by applying Euler's Formula we have $\cos(2\pi z) + i\sin(2\pi z) = 1$. This equality holds for all points in $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, so the zeros are all the integers, and their multiplicity, m, is 1.

We notice that $\sin(\pi z) \in H(\mathbb{C})$, and is not identically zero. Also, we can arrange its zeros in the sequence $z_n = n$ and $z_n \neq 0$ for $n \in \mathbb{Z}^*$, with $\lim_{n\to\infty} |z_n| = \infty$. By picking $d_n = 1$, a sequence of non-negative integers, we have that

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{r}{z_n}\right)^{d_n+1} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{r}{n}\right)^2 = \sum_{n=-1}^{-\infty} \left(\frac{r}{n}\right)^2 + \sum_{n=1}^{\infty} \left(\frac{r}{n}\right)^2$$
$$= 2 \cdot \sum_{n=1}^{\infty} \left(\frac{r}{n}\right)^2 < \infty$$

thus, this series is convergent for all $r > 0$.

So, we have checked all the requirements needed to apply the Weierstrass Factorization Theorem 2. It says that there exists a factorization of $\sin(\pi z)$ with the following form, for a function $g(z) \in H(\mathbb{C})$:

$$\sin(\pi z) = e^{g(z)} \cdot z^m \cdot \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} E_{d_n}\left(\frac{z}{z_n}\right)$$

$$= e^{g(z)} \cdot z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} E_1\left(\frac{z}{n}\right) = e^{g(z)} \cdot z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) \cdot e^{\frac{z}{n}}$$

then we can rearrange and regroup the terms like this:

$$= e^{g(z)} z \dots \left(1 + \frac{z}{2}\right) e^{-\frac{z}{2}} \left(1 + \frac{z}{1}\right) e^{-\frac{z}{1}} \left(1 - \frac{z}{1}\right) e^{\frac{z}{1}} \left(1 - \frac{z}{2}\right) e^{\frac{z}{2}} \dots$$
$$= e^{g(z)} z \left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{2}\right)\left(1 - \frac{z}{2}\right) \dots$$
$$= e^{g(z)} z \left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right) \dots$$
$$= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Now the problem is identifying the function $g(z)$ or rather $e^{g(z)}$. Comparing two forms of the same differentiation will help us. First, we replace this new form in logarithmic differentiation and apply the rule for product differentiation in the upper half by doing $\frac{\sin(\pi z)'}{\sin(\pi z)} =$

$$= \frac{\left(e^{g(z)}\right)' z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}{e^{g(z)} z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} + \frac{e^{g(z)} z' \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}{e^{g(z)} z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} + \frac{e^{g(z)} z \left(\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)'}{e^{g(z)} z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}$$

$$= \frac{e^{g(z)}\left(g(z)\right)'}{e^{g(z)}} + \frac{1}{z} + \frac{\left(\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)'}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = g(z)' + \frac{1}{z} + \frac{\left(\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)'}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}$$

We are going to work separately on the last fraction, using a technique similar one would use to prove the Logarithmic Differentiation for an infinite product:

$$\frac{\left(\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)'}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = \frac{\left(1-\frac{z^2}{1^2}\right)' \cdot \prod_{n=2}^{\infty}\left(1-\frac{z^2}{n^2}\right)}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} + \frac{\left(1-\frac{z^2}{2^2}\right)' \cdot \prod_{\substack{n=1\\ n\neq 2}}^{\infty}\left(1-\frac{z^2}{n^2}\right)}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} + \dots$$

$$= \frac{\left(1-\frac{z^2}{1^2}\right)'}{\left(1-\frac{z^2}{1^2}\right)} + \frac{\left(1-\frac{z^2}{2^2}\right)'}{\left(1-\frac{z^2}{2^2}\right)} + \dots = \sum_{n=1}^{\infty} \frac{\left(1-\frac{z^2}{n^2}\right)'}{\left(1-\frac{z^2}{n^2}\right)}$$

$$= \sum_{n=1}^{\infty} \frac{\left(n^2-z^2\right)'}{\left(n^2-z^2\right)} = \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

Replacing in the relation before we have:

$$\frac{\sin(\pi z)'}{\sin(\pi z)} = g(z)' + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

On the other hand, using the identity given in the Theorem from [6, page 327] by Remmert, we can determine the nature of $g(z)$ like so:

$$\frac{\sin(\pi z)'}{\sin(\pi z)} = \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \pi\,\mathrm{ctg}(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

By matching coefficients, we see that $g(z)'$ is equal to 0, so $g(z)$ is constant. Let's set $g(z) = c$ for a constant c, and replace it in the factorization.

$$\sin(\pi z) = e^c z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

next we are going to divide both sides by πz, and then apply the limit for $z \to 0$ in both sides:

$$\frac{\sin(\pi z)}{\pi z} = \frac{e^c}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} = \lim_{z\to 0} \frac{e^c}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

the limit on the left-hand side tends to 1 as $z \to 0$ and all z from the right hand side vanish leaving behind the following:

$$1 = \frac{e^c}{\pi} \cdot 1 \Rightarrow e^c = \pi$$

Finally, we replace $e^c = \pi$ in the factorization and thus, obtain a very useful relation for the results that follow:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

(2) $\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$

Using the trigonometric formula for $\sin(2x)$ we determine a form of $\cos(x)$ that only depends on $\sin(x)$ like:

$$\sin(2x) = 2\sin(x)\cos(x) \Rightarrow \cos(x) = \frac{\sin(2x)}{2\sin(x)}$$

Since the **Identity Theorem** states that given $D \subset \mathbb{C}$ a domain and $f, g \in H(D)$, if the set $\{z \in D : f(z) = g(z)\}$ **has at least one limit point in** D**, then** $f = g$ **everywhere in** D, the above formula holds for complex numbers.

Now we replace each sinus with its corresponding form according to the proven formula for $\sin(\pi z)$

$$\cos(\pi z) = \frac{\sin(2\pi z)}{2\sin(\pi z)} = \frac{2\pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{(2z)^2}{n^2}\right)}{2 \cdot \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)} = \frac{\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}$$

Now we are going to split the numerator by even and odd n:

$$\cos(\pi z) = \frac{\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)} = \frac{\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n)^2}\right) \cdot \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)}$$

$$= \frac{\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{4n^2}\right) \cdot \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)}$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$$

(3) $\sinh(z) = z \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)$

For this and the next proof, we are going to use the connection between $\sinh(z)$ and $\sin(z)$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = i \cdot \frac{e^{-i^2 z} - e^{-(-i^2 z)}}{2i} = -i \cdot \frac{e^{i\cdot(iz)} - e^{-i\cdot(iz)}}{2i}$$

$$= -i \cdot \sin(iz) = -i \cdot \sin\left(\pi \cdot \frac{iz}{\pi}\right)$$

and just apply the formula provided at $\sin(\pi z)$:

$$= -i \cdot \pi \cdot \frac{iz}{\pi} \cdot \prod_{n=1}^{\infty}\left(1 - \frac{\left(\frac{iz}{\pi}\right)^2}{n^2}\right) = z \cdot \prod_{n=1}^{\infty}\left(1 - \frac{i^2 z^2}{\pi^2 n^2}\right)$$

$$= z \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)$$

(4) $\cosh(z) = z \cdot \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right)$

Just like in the previous example, we establish the connection $\cosh(z)$ and $\cos(z)$:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \frac{e^{-i^2 z} + e^{-(-i^2 z)}}{2} = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \cos(iz) = \cos\left(\pi \cdot \frac{iz}{\pi}\right)$$

and just apply the product formula for $\cos(\pi z)$:

$$= \prod_{n=1}^{\infty}\left(1-\frac{4\left(\frac{iz}{\pi}\right)^2}{(2n-1)^2}\right) = \prod_{n=1}^{\infty}\left(1-\frac{4i^2z^2}{(2n-1)^2\pi^2}\right)$$

$$= \prod_{n=1}^{\infty}\left(1+\frac{4z^2}{(2n-1)^2\pi^2}\right)$$

3. WALLIS' FORMULA

In 1656, John Wallis published this product formula for π:

$$\sqrt{\frac{\pi}{2}} = \prod_{n=1}^{\infty}\frac{2n}{\sqrt{(2n-1)(2n+1)}} = \lim_{n\to\infty}\frac{2\cdot 4\cdot 6\ldots 2n}{1\cdot 3\cdot 5\ldots(2n-1)}\cdot\frac{1}{\sqrt{2n+1}}$$

Proof. Using the formula for $\sin(\pi z)$ we have:

$$\sin(z) = \sin(\pi\cdot\frac{z}{\pi}) = \pi\cdot\frac{z}{\pi}\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{z}{\pi}\right)^2}{n^2}\right) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

Now we replace z with $\frac{\pi}{2}$ and since $\sin\left(\frac{\pi}{2}\right) = 1$:

$$\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{\pi}{2}\right)^2}{n^2\pi^2}\right) = \frac{\pi}{2}\cdot\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = 1 \Rightarrow$$

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = \frac{2}{\pi} \Rightarrow \prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right) = \frac{2}{\pi}$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$$

Now we apply the square root on both sides:

$$\sqrt{\frac{\pi}{2}} = \prod_{n=1}^{\infty}\frac{2n}{\sqrt{(2n-1)(2n+1)}}$$

$$= \lim_{n\to\infty}\frac{2\cdot 4\cdot 6\ldots 2n}{\sqrt{2-1}\cdot\sqrt{2+1}\cdot\sqrt{2\cdot 2-1}\cdot\sqrt{2\cdot 2+1}\ldots\sqrt{(2n-1)(2n+1)}}$$

$$= \lim_{n\to\infty}\frac{2\cdot 4\cdot 6\ldots 2n}{\sqrt{1}\cdot\sqrt{3}\cdot\sqrt{3}\cdot\sqrt{5}\ldots\sqrt{2n-1}\cdot\sqrt{2n+1}}$$

$$= \lim_{n\to\infty}\frac{2\cdot 4\cdot 6\ldots 2n}{1\cdot 3\cdot 5\ldots(2n-1)}\cdot\frac{1}{\sqrt{2n+1}}$$

□

4. THE BASEL PROBLEM

Proposed by Pietro Mengoli in the year 1650, the Basel Problem was solved by Euler more than 80 years later, in 1734. The problem is named after the Swiss city of Basel, where Euler was born. Heavily debated in the beginning, it withstood the test of time, gathering a plethora of equivalent proofs.

I decided to present here the proof that made Euler famous. It is important to mention, that his version of the proof was not considered rigorous at the time because it involved proving the product form of $\sin(\pi z)$, which was made rigorous only do to Weierstrass's factorization Theorem 2, almost 150 years later.

The problem posed was determining the sum of the inverses of the squares of all natural numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Proof. First, we take the infinite product form of $\sin(z)$ and try to open the parenthesis, one by one, and then we observe an interesting pattern

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = z\left(1 - \frac{z^3}{\pi^2}\right) \prod_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$= \left(z - \frac{z^3}{\pi^2}\right) \prod_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = \left(z - \frac{z^3}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right) \prod_{n=3}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$= \left(z - \left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2}\right) z^3 + \frac{z^5}{\pi^2(2\pi)^2}\right) \prod_{n=3}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$= \left(z - \left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2}\right) z^3 + \frac{z^5}{\pi^2(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right) \prod_{n=4}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$= \left(z - \left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2}\right) z^3 + \left(\left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2}\right)\frac{1}{(3\pi)^2} + \frac{1}{\pi^2(2\pi)^2}\right) z^5 - \frac{z^7}{\pi^2(2\pi)^2(3\pi)^2}\right)$$
$$\cdot \prod_{n=4}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \dots$$

If we continue to do this process we can see that the first 2 terms would look like this, and we could also factor out both z^3 and π^2:

$$= z - \left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \dots\right) z^3 + \dots = z - z^3 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + \dots$$

$$= z - \frac{z^3}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} + \dots$$

The coefficient of z^3 contains exactly the sum that we need. Now, Euler thought of another form of $\sin(z)$ that involves a z^3, its Taylor expansion

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

Since the 2 forms of $\sin(z)$ are equivalent, we can match coefficients, and by doing the following computations we obtain our answer:

$$-\frac{z^3}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{z^3}{3!} \Rightarrow \frac{1}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

Euler used this method in his paper [2, "Various Observations about Infinite Series"] to calculate even more values for what will be presented in the next section as the Riemann Zeta function.

5. RIEMANN ZETA FUNCTION CONNECTION TO PRIME NUMBERS

I would like to start this section by touching, albeit briefly on the monumental paper [8, **On the Number of Primes Less Than a Given Magnitude**] by Bernard Riemann, published in the November 1859 edition of the Monthly reports of the Royal Prussian Academy of Sciences in Berlin.

As the name of the paper suggests, Riemann was focused on analytical methods to count primes. He started his paper by observing, a connection to the prime numbers in the form given at Theorem 4 This was possible only due to the following relation given to us by Euler 1737:

Theorem 3.
$$\sum_{n=1}^{\infty} \frac{1}{n^a} = \prod_{p\ prime} \left(\frac{1}{1 - \frac{1}{p^a}} \right) \quad \text{for } a \in \mathbb{N} \setminus \{0, 1\}$$

Proof. Euler proved this affirmation using an algorithm called the Sieve of Eratosthenes. It is a pretty intuitive algorithm that allows us to filter the prime numbers. The process requires iterating through all the numbers up to any given limit, and upon encountering a prime number, eliminating all its multiples.

By repeating this procedure, we are left with only prime numbers, because if a number were composed, it would have already been eliminated from the set. Let:

$$\sum_{n=1}^{\infty} \frac{1}{n^a} = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} + \dots$$

Now we multiply both sides by $\frac{1}{2^a}$ and because this series is convergent we can rearrange the terms, then do a subtraction:

$$\frac{1}{2^a} \sum_{n=1}^{\infty} \frac{1}{n^a} = \frac{1}{2^a} + \frac{1}{4^a} + \frac{1}{8^a} + \dots$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^a} - \frac{1}{2^a} \sum_{n=1}^{\infty} \frac{1}{n^a} = 1 + \frac{1}{3^a} + \frac{1}{5^a} + \frac{1}{7^a} + \dots$$

We observe that all terms that had a multiple of 2 at the denominator disappeared. Let's group the terms on the left-hand side and repeat the procedure for $\frac{1}{3^a}$:

$$\left(1 - \frac{1}{2^a}\right) \sum_{n=1}^{\infty} \frac{1}{n^a} = 1 + \frac{1}{3^a} + \frac{1}{5^a} + \frac{1}{7^a} + \dots$$

$$\Rightarrow \frac{1}{3^a} \left(1 - \frac{1}{2^a}\right) \sum_{n=1}^{\infty} \frac{1}{n^a} = \frac{1}{3^a} + \frac{1}{9^a} + \frac{1}{15^a} + \dots$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^a} - \frac{1}{3^a} \left(1 - \frac{1}{2^a}\right) \sum_{n=1}^{\infty} \frac{1}{n^a} = 1 + \frac{1}{5^a} + \frac{1}{7^a} + \dots$$

Again, all the terms with a multiple of 3 at the denominator vanished. If we continue this process an infinite amount of times we get to the result:

$$\dots \left(1 - \frac{1}{7^a}\right)\left(1 - \frac{1}{5^a}\right)\left(1 - \frac{1}{3^a}\right)\left(1 - \frac{1}{2^a}\right) \sum_{n=1}^{\infty} \frac{1}{n^a} = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^a} = \frac{1}{\left(1 - \frac{1}{2^a}\right)\left(1 - \frac{1}{3^a}\right)\left(1 - \frac{1}{5^a}\right)\left(1 - \frac{1}{7^a}\right)\dots}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^a} = \prod_{p \ prime} \left(\frac{1}{1 - \frac{1}{p^a}}\right) \quad \text{for } a \in \mathbb{N} \setminus \{0, 1\}$$

□

We see that if we choose to replace a, with a complex number z whose real part is strictly greater than 1, in the left-hand side we retrieve the definition of the Zeta function, in its analytic form. Thus, Riemann said:

THEOREM 4.

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \ prime} \left(\frac{1}{1 - p^{-z}}\right) \quad \text{for } z \in D = \{z : Re(z) > 1\}$$

This infinite product form of the Riemann Zeta function would still not be possible without the Theorem 2 this is why I decided to include it in this paper.

The main takeaway from this paper is that the most general case of Weierstrass' Factorization Theorem is a powerful statement, as it allows us to craft specific functions that have zeros along a chosen sequence of numbers. Still, it comes with some restrictions we have to fulfill. One such requirement imposes our chosen sequence to have no limit points in the domain. Otherwise, we risk crafting a null function, which is a trivial case instead of a meaningful one.

Since its apparition in 1876, the Weierstrass' Factorization Theorem and its consequence gathered a tremendous amount of applications one of which is linked to one of the unsolved problems of the millennium, the Riemann Hypothesis.

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