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ON A CONVERGENCE PROBLEM FOR SERIES

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Abstract. This article presents a complete solution for a difficult problem involving series of real numbers. In order to finish its study, several theorems concerning the convergence theory of series are employed. Students attending mathematical competitions and those interesting in deepening their understanding of the subjects are primary beneficiaries.

We start by pointing out the fact that throughout this article the set of natural numbers, in accordance to the inductive theory associated to it, starts at 1, namely 0 is not a natural number. Therefore,

$$\mathbb{N} = \{1, 1+1, 1+1+1, ...\}.$$

For more details on this approach see [1]

1. SERIES OF REAL NUMBERS

Let us recall some basic notions and results on series of real numbers. Having a sequence of real numbers $(x_n)_{n\geq k}$, where $k \in \mathbb{N} \cup \{0\}$, one can generate its so-called **sequence of partial sums**, by summing up all the terms of the initial sequence up to the current index. Thus, for a random $n > k \in \mathbb{N}$ one has

$$s_n = x_k + x_{k+1} + ... + x_n.$$

An ordered pair of two sequences $((x_n), (s_n))_{n\geq k}$ is said to be series of real numbers, and is denoted by

$$\sum_{n\geq k} x_n \text{ (or simplier } \sum x_n).$$

If there exists the limit of the sequence of the partial sums, it is said to be **the sum of the series** and is denoted by

$$\sum_{k=1}^{\infty} x_n = \lim_{n\to\infty} s_n.$$

A series is said to be convergent or divergent, if its sequence of partial sums is convergent or divergent, respectively. One of the mostly used series in comparison, is the so-called **harmonic series**,

$$\sum \frac{1}{n^\alpha}$$

which is convergent for each $\alpha > 1$, and divergent for each $\alpha \leq 1$. There are several convergence criteria concerning series of real numbers (for details see [1]) we list here those needed to understand the problem solved in this article.

Theorem 1: *Consider* $\sum x_n$ *a series of real numbers. If the series is convergent, then* $\lim_{n\to\infty} x_n = 0$.

Notice that this theorem is mostly used in practice through contraposition, thus

Corollary 2: *Consider* $\sum x_n$ *a series of real numbers. If* $\lim_{n\to\infty} x_n \neq 0$ *then the series is divergent.*

Theorem 3: (First comparison criterion for series with positive terms) *Consider* $\sum x_n$ *and* $\sum y_n$ *two series with positive terms, such that for each* $n \geq k$, $x_n \leq y_n$. *If the series* $\sum y_n$ *is convergent then the series* $\sum x_n$ *is convergent and, if the series* $\sum x_n$ *is divergent then the series* $\sum y_n$ *is divergent.*

Theorem 4: (The consequence of the second comparison criterion for series with positive terms) *Consider* $\sum x_n$ *and* $\sum y_n$ *two series with positive terms, such that there exist the limit*

$$\lim_{n\to\infty} \frac{x_n}{y_n} \in (0, \infty).$$

Then the two series have the same nature.

A series of real numbers is said to be **absolutely convergent** if the series generated by the absolute values of its general terms, namely $\sum |x_n|$, is convergent.

Theorem 5: (the connection between absolutely convergent and convergent series) *Consider the serie* $\sum x_n$ *such that it is absolutely convergent. Then it is also convergent.*

Notice that the converse statement of the theorem above does not hold, namely there are convergent series which are not necessarily absolutely convergent.

Theorem 6: (the Abel-Dirichlet theorem on series with random terms) *Consider* (a_n) *and* (u_n) *two sequences of real numbers. If the sequence* (a_n) *is decreasing and has the limit* 0, *and if the sequence* (u_n) *has its sequence of partial sums bounded, that the series*

$$\sum u_n a_n$$

is convergent.

2. PRELIMINARY NOTIONS

2.1. Trigonometric Identities. Let us recall some basic trigonometrical identities

$$\sin a \cdot \sin b = -\frac{1}{2}\Big(\cos(a+b) - \cos(a-b)\Big), \tag{1}$$

$$\sin a \cdot \cos b = \frac{1}{2}\Big(\sin(a+b) + \sin(a-b)\Big), \tag{2}$$

$$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right), \tag{3}$$

$$1 - \cos a = 2\sin^2 \frac{a}{2} \tag{4}$$

Consider k a random natural number, and $x \in \mathbb{R}\backslash\{2t\pi : t \in \mathbb{Z}\}$, from (1)

$$\sin\frac{x}{2} \cdot \sin(kx) = -\frac{1}{2}\left(\cos\left(\frac{(2k+1)x}{2}\right) - \cos\left(\frac{(2k-1)x}{2}\right)\right)$$

therefore

$$\sin(kx) = \frac{\cos\left(\frac{(2k+1)x}{2}\right) - \cos\left(\frac{(2k-1)x}{2}\right)}{-2\sin\frac{x}{2}}. \tag{5}$$

In order to sum up from 1 to $n \in \mathbb{N}$ we consider

$$\sin(x) = \frac{\cos\left(\frac{3x}{2}\right) - \cos\left(\frac{x}{2}\right)}{-2\sin\frac{x}{2}},$$

$$\sin(2x) = \frac{\cos\left(\frac{5x}{2}\right) - \cos\left(\frac{3x}{2}\right)}{-2\sin\frac{x}{2}},$$

$$\dots$$

$$\sin(nx) = \frac{\cos\left(\frac{(2n+1)x}{2}\right) - \cos\left(\frac{(2n-1)x}{2}\right)}{-2\sin\frac{x}{2}}.$$

Thus

$$\sum_{k=1}^{n}\sin(kx) = \frac{\cos\left(\frac{(2n+1)x}{2}\right) - \cos\left(\frac{x}{2}\right)}{-2\sin\frac{x}{2}} = \frac{\cos\left(nx+\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)}{-2\sin\frac{x}{2}} =$$

$$\frac{\cos(nx)\cos\frac{x}{2} - \sin(nx)\sin\frac{x}{2} - \cos\left(\frac{x}{2}\right)}{-2\sin\frac{x}{2}} = \frac{\cos\frac{x}{2}\left(\cos(nx)-1\right) - \sin(nx)\sin\frac{x}{2}}{-2\sin\frac{x}{2}}$$

By applying (4) we get

$$\sum_{k=1}^{n}\sin(kx) = \frac{-2\cos\frac{x}{2}\sin^2\frac{nx}{2} - \sin(nx)\sin\frac{x}{2}}{-2\sin\frac{x}{2}} =$$

$$= \frac{-2\cos\frac{x}{2}\sin^2\frac{nx}{2} - 2\sin\frac{nx}{2}\cos\frac{nx}{2}\sin\frac{x}{2}}{-2\sin\frac{x}{2}} = \frac{\sin\frac{nx}{2}\left(\cos\frac{x}{2}\sin\frac{nx}{2} + \cos\frac{nx}{2}\sin\frac{x}{2}\right)}{\sin\frac{x}{2}}$$

In conclusion

$$\sum_{k=1}^{n}\sin(kx) = \frac{\sin\frac{nx}{2}\sin\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}. \tag{6}$$

By applying a similar procedure, we cand deduce that

$$\sum_{k=1}^{n}\cos(kx) = \frac{\sin\frac{nx}{2}\cos\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}. \tag{7}$$

Due to the fact that all the functions above are differentiable, we notice that

$$\left(\sum_{k=1}^{n}\cos(kx)\right)' = \left(\frac{\sin\frac{nx}{2}\cos\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}\right)',$$

thus

$$\sum_{k=1}^{n} k\sin(kx) = -\frac{\left(\sin\frac{nx}{2}\cos\left(\frac{(n+1)x}{2}\right)\right)' \cdot \sin\frac{x}{2} - \left(\sin\frac{nx}{2}\cos\left(\frac{(n+1)x}{2}\right)\right)\sin'\frac{x}{2}}{\sin^2\frac{x}{2}}.$$

We will tackle the numerator first:

$$\begin{aligned} A &= \left(\tfrac{n}{2}\cos\tfrac{nx}{2}\cos\tfrac{(n+1)x}{2} - \tfrac{(n+1)}{2}\sin\tfrac{nx}{2}\sin\tfrac{(n+1)x}{2}\right)\cdot\sin\tfrac{x}{2} - \\ &\quad -\tfrac{1}{2}\cos\tfrac{x}{2}\sin\tfrac{nx}{2}\cos\tfrac{(n+1)x}{2} = \\ &= \tfrac{n}{2}\left(\cos\tfrac{nx}{2}\cos\tfrac{(n+1)x}{2} - \sin\tfrac{nx}{2}\sin\tfrac{(n+1)x}{2}\right)\cdot\sin\tfrac{x}{2} - \\ &\quad -\tfrac{1}{2}\left(\sin\tfrac{x}{2}\sin\tfrac{(n+1)x}{2} + \cos\tfrac{x}{2}\cos\tfrac{(n+1)x}{2}\right)\cdot\sin\tfrac{nx}{2} = \\ &= \tfrac{n}{2}\cos\tfrac{(2n+1)x}{2}\sin\tfrac{x}{2} - \tfrac{1}{2}\sin\tfrac{nx}{2}\cos\tfrac{nx}{2} \end{aligned}$$

by using (2)

$$\begin{aligned} A &= \tfrac{n}{4}\left[\sin\left(\tfrac{x}{2} + \tfrac{(2n+1)x}{2}\right) - \sin\left(\tfrac{x}{2} - \tfrac{(2n+1)x}{2}\right)\right] - \tfrac{1}{2}\sin\tfrac{nx}{2}\cos\tfrac{nx}{2} \\ &= \tfrac{n}{4}\left(\sin((n+1)x) - \sin(nx)\right) - \tfrac{2}{4}\sin\tfrac{nx}{2}\cos\tfrac{nx}{2} \\ &= \tfrac{1}{4}\left[n\sin((n+1)x) - n\sin(nx) - \sin(nx)\right] \\ &= \tfrac{1}{4}\left[n\sin((n+1)x) - (n+1)\sin(nx)\right] \end{aligned}$$

Thus we may conclude that

$$\sum_{k=1}^{n} k\sin(kx) = \frac{n\sin((n+1)x) - (n+1)\sin(nx)}{4\sin^2\frac{x}{2}}. \tag{8}$$

3. PROBLEM STATEMENT

The main idea being this article is the following problem: **Study with discussion on the real parameter $a > 1$, the nature of the series**

$$\sum_{n\geq 1} \frac{\sin 1 + 2\sin 2 + 3\sin 3 + ... + n\sin n}{n^a}.$$

To begin with, we use the following notations, for each $n \in \mathbb{N}$

$$x_n = \frac{\sin 1 + 2\sin 2 + 3\sin 3 + ... + n\sin n}{n^a}$$

$$y_n := \sin 1 + 2\sin 2 + 3\sin 3 + ... + n\sin n$$

Whit the notation above, we see that the goal is to investigate the nature of the series

$$\sum_{n\geq 1} x_n.$$

By applying basic abstract value properties and the boundaries of the sinus function we conclude that, for a random $n \in \mathbb{N}$ and a random $a > 1$

$$|x_n| = \frac{|y_n|}{n^a} \leq \frac{1}{n^a}\cdot(1|\sin a| + 2|\sin s| + ... + n|\sin n|) \leq \frac{1}{n^a}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2n^{a-1}}.$$

For each $n \in \mathbb{N}$, we denote

$$z_n := \frac{n+1}{2n^{a-1}}.$$

Thus, we can conclude that

$$|x_n| \leq z_n. \tag{9}$$

By applying Theorem 4 for the seires $\sum z_n$ and the generalized harmonic series, we conclude that

$$\sum z_n \sim \sum \frac{1}{n^{a-2}}. \tag{10}$$

Case 1: $a > 3$. For $a > 3$ we have $a - 2 > 1$, thus the harmonic series is convergent for such a choice therefore, $\sum z_n$ is convergent. By using (9) and Theorem 3, we conclude that $\sum |x_n|$ is absolutely convergent, therefore, by further going to Theorem 5, we are led to the conclusion that the series $\sum x_n$ is also convergent.
Case 2: $a \in (1, 3]$ In this case, by using (10) we conclude that $\sum z_n$ is divergent, which does not deliver any conclusion on our original $\sum x_n$. We will employ Theorem 6. From (6) applied for $x = 1$ we obtain that

$$y_n = \frac{n \sin(n+1) - (n+1)\sin(n)}{4\sin^2 \frac{1}{2}}$$

therefore

$$x_n = \frac{n \sin(n+1) - (n+1)\sin(n)}{4n^a \sin^2 \frac{1}{2}} = \frac{1}{4\sin^2 \frac{1}{2}} \cdot \frac{n \sin(n+1) - (n+1)\sin(n)}{n^a}$$

We study separately the nature of the series generated by

$$\alpha_n := \frac{n}{n^a} \cdot \sin(n+1) = \frac{1}{n^{a-1}} \cdot \sin(n+1) \quad \textit{and} \quad \beta_n : \frac{(n+1)}{n^a} \cdot \sin n, \quad \forall n \in \mathbb{N},$$

for which we may apply the Abel Dirichlet theorem. To begin with, in the hypothesis of Theorem 6 consider

$$a_n := \frac{1}{n^{a-1}} \quad \textit{and} \quad u_n := \sin(n+1) \quad \forall n \in \mathbb{N}. \tag{11}$$

Due to the fact that $a > 1$, we conclude that

$$\forall n \in \mathbb{N},\ a_n < a_{n+1} \quad \textit{and} \quad \lim_{n\to\infty} a_n = 0.$$

Thus, the sequence (a_n) is both decreasing and has the limit 0. Moreover, the relation (6), we get

$$\sum_{k=1}^{n} u_n = \sum_{k=1}^{n} \sin(n+1) = \frac{\sin \frac{(n+1)}{2} \sin \frac{(n+2)}{2}}{\sin^2 \frac{1}{2}} - \sin 1.$$

Thus

$$|a_n| = \left| \frac{\sin \frac{(n+1)}{2} \sin \frac{(n+2)}{2}}{\sin^2 \frac{1}{2}} - \sin 1 \right| \le \left| \frac{\sin \frac{(n+1)}{2} \sin \frac{(n+2)}{2}}{\sin^2 \frac{1}{2}} \right| + |\sin 1| \le \frac{1}{\sin^2 \frac{1}{2}} + 1,$$

therefore the sequnce (a_n) is bounded. So, from Theorem 6

$$\sum \alpha_n \text{ is a convergent series.} \tag{12}$$

A similar reasoning to the one above may be applied for the series $\sum \beta_n$. To begin with, in the hypothesis of Theorem 6 consider

$$a'_n := \frac{n+1}{n^a} \quad \textit{and} \quad u'_n := \sin(n+1) \quad \forall n \in \mathbb{N}. \tag{13}$$

Due to the fact that $a > 1$, we conclude that

$$\forall n \in \mathbb{N},\ a'_n < a'_{n+1} \quad \textit{and} \quad \lim_{n\to\infty} a'_n = 0.$$

Thus, the sequence (a'_n) is both decreasing and has the limit 0. Moreover, the relation (6), we get

$$\sum_{k=1}^{n} u'_n = \sum_{k=1}^{n} \sin(n) = \frac{\sin\frac{n}{2}\sin\frac{(n+1)}{2}}{\sin^2\frac{1}{2}}.$$

Thus

$$|a'_n| = \left|\frac{\sin\frac{n}{2}\sin\frac{(n+1)}{2}}{\sin^2\frac{1}{2}}\right| \leq \frac{1}{\sin^2\frac{1}{2}},$$

therefore the sequnce (a'_n) is bounded. So, from Theorem 6

$$\sum \beta_n \text{ is a convergent series.} \tag{14}$$

From (12) and (14) we conclude that the sequence $\sum x_n$, may be written as a difference of two convergent sequences, namely

$$\sum \frac{1}{4\sin^2\frac{1}{2}} \cdot \alpha_n \quad \textit{and} \quad \sum \frac{1}{4\sin^2\frac{1}{2}} \cdot \beta_n$$

becoming thus itself convergent.

Conlusion: The series turns out to be convergent for all $a > 1$.

This problem was deeply analyzed with the firs year students in Calculus 1, during the university year 2022-2023. Their interesting in delivering a complete solution and its use of several convergence results for series was the inspiration in writing this article. Among the student, Gut Dan-Andrei had the most valuable input on handling the geometrical identities.

Further studies: As a follow up of the results above, it would be interesting to study what hapens when $a \leq 1$, and also the case when the general function is generated by the consinus.

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