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ALGEBRAIC METHODS FOR SOLVING GEOMETRY PROBLEMS: APPLICATIONS OF MATRIX DETERMINANTS AND QUATERNIONS

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Abstract. This article explores the application of quaternions in elementary three dimensional geometry, emphasizing their utility in geometric rotations. We begin with a review of determinant formulas for the areas of triangles and the volumes of parallelepipeds and tetrahedrons. Subsequently, we introduce quaternions and provide an algorithm that showcases their effectiveness in performing geometric rotations. The final section presents practical applications, including the proof of the volume formula for a regular tetrahedron using quaternion rotations and determinants, as well as a similar approach for an irregular tetrahedron. The article is framed within a didactical context, highlighting the potential of using quaternions as a pedagogical tool in geometric problem-solving.

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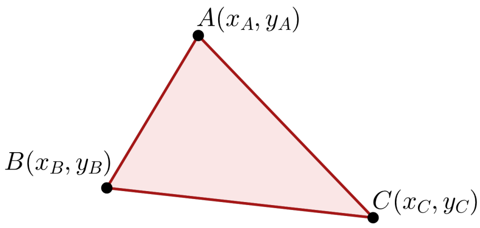
Key words. Quaternions, rotations with quaternions, applications of determinants in geometry, polyhedra volumes, tetrahedron volume.

1. INTRODUCTION

In the Romanian educational system, high school students are familiarized with using algebraic methods in order to compute certain geometric characteristics. As an example in two dimensional Euclidean geometry, they use determinants to compute the area of a triangle (see, for example [1, Elemente de calcul matriceal și sisteme de ecuații liniare III.2.3.]).

Proposition 1 (Area of a triangle in two dimensions). Consider a Cartesian system of coordinates in two dimensions Oxy. If we know the coordinates of the vertices of a triangle ABC: $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$,



then the area of $\triangle ABC$ is given by

$$\mathcal{A}_{\triangle ABC} = \frac{1}{2} \cdot \left| \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \right|.$$

Given that high school students have also studied three dimensional Euclidean geometry (in the last year of middle school) and two dimensional vector geometry (in the first year of high school) one can easily present as an extracurricular lesson similar formulas for three dimensional Euclidean geometry. Such a lesson is listed in Section 2.

In the final year of high school, Romanian students may encounter while studying examples of skew fields, the notion of quaternions (see, for example, [3, pp. 148–149]). While doing so, it is pointed out to them that quaternions can play a big role in three dimensional geometry problem-solving, yet no examples are given in this direction, even as an extracurricular lesson. The motivation behind this appears to be that quaternions are most useful in describing three dimensional rotations, yet in the Romanian high school geometry problem ecosystem there seems to be a lack of such problems. In order to address this issue, at least as an extracurricular lesson of geometrical applications of quaternions, in Section 3 we review the main tools needed to deal with such constructions, while in the last section (Section 4) we will present two geometrical applications which link together both the presented determinant-based formulas and the tools provided by quaternions discussed in this paper.

2. APPLICATIONS OF DETERMINANTS IN THREE DIMENSIONAL GEOMETRY

In order to present our selected applications, we need to introduce a few definitions, where we follow in our description [2, p. 23].

Consider a Cartesian system of coordinates in three dimensions $Oxyz$. A *location vector* is a vector whose initial point is anchored at the origin $O(0,0,0)$. If the terminal point is $P(x_P, y_P, z_P)$, then our location vector is $\overrightarrow{OP}$, whose coordinates are also (x_P, y_P, z_P). Moreover, given an additional point $P'(x_{P'}, y_{P'}, z_{P'})$, for a free vector $\overrightarrow{PP'}$, given that $\overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}$, we consider the coordinates of $\overrightarrow{PP'}$ to be

$$\overrightarrow{PP'}(x_{P'} - x_P, y_{P'} - y_P, z_{P'} - z_P).$$

Furthermore, note that its length is given by:

$$|\overrightarrow{PP'}| = \sqrt{(x_{P'} - x_P)^2 + (y_{P'} - y_P)^2 + (z_{P'} - z_P)^2}.$$

For vectors $\vec{v}_1(x_1, y_1, z_1) = x_1 \cdot \vec{i} + y_1 \cdot \vec{j} + z_1 \cdot \vec{k}$, $\vec{v}_2(x_2, y_2, z_2) = x_2 \cdot \vec{i} + y_2 \cdot \vec{j} + z_2 \cdot \vec{k}$ and $\vec{v}_3(x_3, y_3, z_3) = x_3 \cdot \vec{i} + y_3 \cdot \vec{j} + z_3 \cdot \vec{k}$, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the versors of the Cartesian system; we consider their dot product:

$$\vec{v}_1 \cdot \vec{v}_2 = x_1x_2 + y_1y_2 + z_1z_2,$$

their cross product:

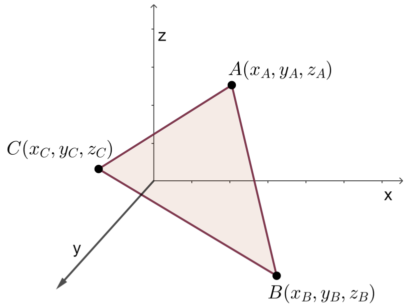
$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

and their box product:

$$[\vec{v}_1, \vec{v}_2, \vec{v}_3] = (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Given these notions, we can introduce a formula, similarly to Proposition 1, that gives the area of a triangle in three dimensions (see, for example, [2, p. 62]).

Proposition 2 (Area of a triangle in three dimensions). If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$ are the vertices of $\triangle ABC$,

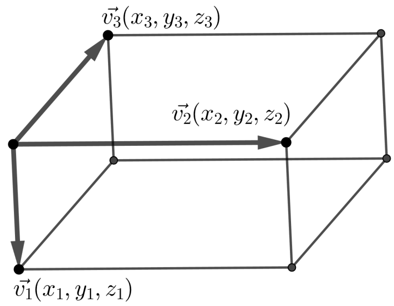


then the area of $\triangle ABC$ is given by

$$\mathcal{A}_{\triangle ABC} = \frac{1}{2} \cdot |\overrightarrow{CA} \times \overrightarrow{CB}| = \frac{1}{2} \cdot \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A - x_C & y_A - y_C & z_A - z_C \\ x_B - x_C & y_B - y_C & z_B - z_C \end{vmatrix} \right|.$$

Furthermore, we can move to formulas for volume, where we will start with the formula for the volume of a parallelepiped.

Proposition 3 (Volume of a parallelepiped). The volume of a parallelepiped spanned by vectors $\vec{v}_1(x_1, y_1, z_1)$, $\vec{v}_2(x_2, y_2, z_2)$ and $\vec{v}_3(x_3, y_3, z_3)$

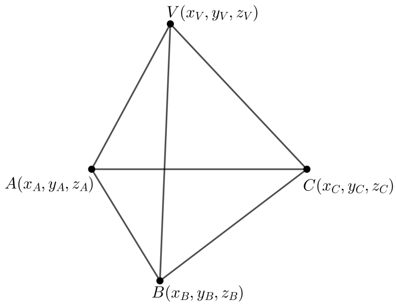


is given by

$$\mathcal{V} = |[\vec{v}_1, \vec{v}_2, \vec{v}_3]| = \left| \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \right|.$$

From this formula, one can easily deduce (see, for example, [2, pp. 65–68]) the formula for the volume of a tetrahedron.

COROLLARY 1 (Volume of a tetrahedron). If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$ and $V(x_V, y_V, z_V)$ are the vertices of a tetrahedron $VABC$,



then the volume of the tetrahedron $VABC$ is given by

$$\mathcal{V}_{VABC} = \frac{1}{6} \cdot |[\overrightarrow{VA}, \overrightarrow{VB}, \overrightarrow{VC}]| = \frac{1}{6} \cdot | \begin{vmatrix} x_A - x_V & y_A - y_V & z_A - z_V \\ x_B - x_V & y_B - y_V & z_B - z_V \\ x_C - x_V & y_C - y_V & z_C - z_V \end{vmatrix} |,$$

or equivalently

$$\mathcal{V}_{VABC} = \frac{1}{6} \cdot | \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_V & y_V & z_V & 1 \end{vmatrix} |.$$

3. THE GEOMETRY OF QUATERNIONS

We know that complex numbers can be identified with points in a two dimensional plane (pairs of real numbers), or, even better, they can be identified with the location vector with the terminal point in that respective point; and that the operations we defined between said numbers have thus clear geometric interpretations: complex number addition corresponds to vector translation, and complex number multiplication corresponds to vector rotation in a two dimensional plane. From this idea, another question immediately arises: is it possible to describe three dimensional space and its isometries with triplets of real numbers?

This question has intrigued Irish mathematician William Rowan Hamilton for many years. As he initially tried to extend complex numbers from having one real part and one imaginary part to having one real part and two imaginary parts, he was unable to answer this question positively. In fact, later on, Frobenius even proved that this was not mathematically possible. This did not stop Hamilton in his mission to algebraically describe three dimensional space; it only just forced him to move from using triplets of real numbers to quadruplets, and so in 1843, quaternions were first described.

Today, the set of quaternions, denoted as $\mathbb{H}$ in honor of mathematician William Rowan Hamilton, contains numbers with one real part and three imaginary parts and has the form:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

Quaternions take their name from the Latin word *quaternio*, which means "fourfold". In addition to the fact that

$$i^2 = j^2 = k^2 = ijk = -1, \tag{1}$$

another defining property is that quaternion multiplication is not commutative, but rather exhibits a different property called anticommutativity, i.e. from (1) one can easily prove the following:

$$ij = -ji = k, \qquad ik = -ki = j, \qquad jk = -kj = i.$$

This property was one of the obstacles that Hamilton had to accept. The operation of multiplication between quaternions can be summarized in the following table.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 1 – The multiplication table for quaternions

Additionally, this table can be easily visualized in the following figure (Figure 3.1), where the product of two quaternions in the order indicated by the arrows will result in the third quaternion with a positive sign, and in the reverse direction of the arrows, it will result in the same quaternion but with a negative sign.

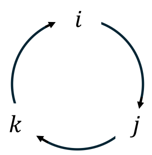


Fig. 3.1 – The multiplication of quaternions

3.1. Operations with quaternions. From an algebraic perspective (see for example [4, Chapter 5]), the operations between quaternions are naturally defined: addition is performed component-wise, while multiplication is carried out by expanding brackets, using Table 1 and then collecting the terms.

Therefore, if $q_1, q_2 \in \mathbb{H}$, where $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$, then we have

$$
\begin{aligned}
\bullet\ \text{addition: } q_1 + q_2 &= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \\
\bullet\ \text{multiplication: } q_1 \cdot q_2 &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
&\quad + i(a_1b_2 + b_1a_2) + j(a_1c_2 + c_1a_2) \\
&\quad + k(a_1d_2 + d_1a_2) + ij(b_1c_2 - c_1b_2) \\
&\quad + ik(b_1d_2 - d_1b_2) + jk(c_1d_2 - d_1c_2) \\
&= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
&\quad + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\
&\quad + j(a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2) \\
&\quad + k(a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2).
\end{aligned}
$$

Moreover, if $q = a + bi + cj + dk \in \mathbb{H}$ and $\lambda \in \mathbb{R}$, we have

- scalar multiplication: $\lambda(a + bi + cj + dk) = \lambda a + (\lambda b)i + (\lambda c)j + (\lambda d)k$
- the conjugate of a quaternion: $\overline{q} = a - bi - cj - dk$
- the modulus of a quaternion: $|q| = \sqrt{q \cdot \overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$
- the inverse of a nonzero quaternion: $q^{-1} = \dfrac{\overline{q}}{|q|^2} = \dfrac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$

Definition 1. A quaternion is called a *unit quaternion* if and only if its modulus is equal to 1.

Remark 1. If q is a unit quaternion, then $|q| = 1$ and $q^{-1} = \dfrac{\overline{q}}{|1|^2} = \overline{q}$.

3.2. Rotations with quaternions. We present below an algorithm to rotate a point $P(a, b, c)$ around a vector $v = (v_x, v_y, v_z)$ by a given angle θ, thus obtaining the point

$$P' = \mathrm{rot}_{\theta,v}\, P.$$

For an overview of the proof of this algorithm, we refer the reader to [4, 5.15].

Step 1: If the vector v is not a unit vector, we normalize it.

$$v \leftarrow \frac{v}{|v|}.$$

Step 2: We calculate a so-called *rotation quaternion* using the formula

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \cdot (iv_x + jv_y + kv_z).$$

It is worth noting that q is a unit quaternion. Indeed

$$\begin{aligned}
|q| &= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cdot v_x^2 + \sin^2\left(\frac{\theta}{2}\right)\cdot v_y^2 + \sin^2\left(\frac{\theta}{2}\right)\cdot v_z^2 \\
&= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cdot \left(v_x^2 + v_y^2 + v_z^2\right) \\
&= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cdot 1 \\
&= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) \\
&= 1.
\end{aligned}$$

As a consequence

$$q^{-1} = \overline{q} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cdot (iv_x + jv_y + kv_z).$$

Step 3: We convert the point we want to rotate into a quaternion by adding an additional component which will represent the real part of the quaternion. We choose it to be 0, and the three coordinates of the point will become the imaginary components of the quaternion. Here, the point $P(a, b, c)$ will have the corresponding quaternion:

$$p = (0, a, b, c).$$

This is a so-called *pure quaternion* (see [4, p. 114]).

Step 4: We perform the rotation via two multiplications, either one of the following ones:

- for an active rotation (when the point is rotated relative to the coordinate system):
$$p' = q^{-1}\cdot p\cdot q;$$
- for a passive rotation (when the coordinate system is rotated relative to the point):
$$p' = q\cdot p\cdot q^{-1}.$$

Step 5: We extract the new coordinates, obtained after the rotation, from the quaternion p', i.e. the quaternion p' obtained after the rotation will contain four coordinates, like any other quaternion, where the first will be 0 (i.e. a pure quaternion), and the other three will be the coordinates of the point P' obtained from P by rotating it by an angle θ around the vector v:

$$p' = (0, a', b', c') \Rightarrow P'(a', b', c').$$

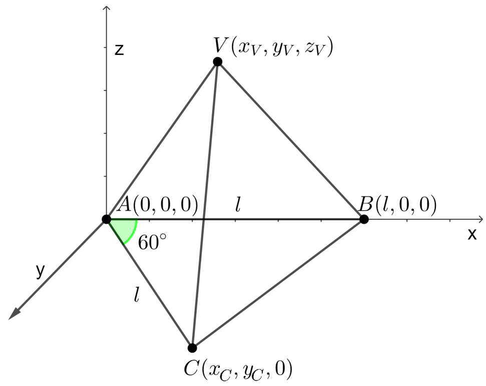
4. APPLICATIONS

As a first application for the described methods, we will prove the formula for the volume of a regular tetrahedron.

THEOREM 1 (Volume of a regular tetrahedron). If $VABC$ is a regular tetrahedron with edge length l, then its volume is given by the following formula:

$$\mathcal{V}_{VABC} = \frac{l^3\sqrt{2}}{12}.$$

Proof. We consider the Cartesian coordinate system $Oxyz$ and place the vertex A at the origin of the system, so $A(0,0,0)$. We place the vertex B on the Ox-axis, so $B(l,0,0)$; and the point C is chosen to be in the xOy-plane.

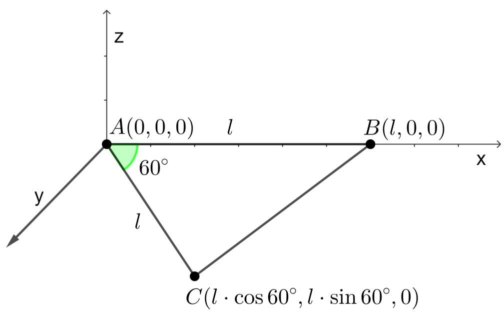


Since $AB = l$ and $m(\measuredangle BAC) = 60°$, using the representation of points with complex numbers, we obtain the coordinates of vertex C:

$$C(l \cdot \cos 60°, l \cdot \sin 60°, 0),$$

which simplifies to

$$C\left(\frac{l}{2}, \frac{l\sqrt{3}}{2}, 0\right).$$



To represent the apex of the tetrahedron, we perform a rotation of the point C by an angle of $\arccos(\frac{1}{3})$ around the edge AB. Thus, the apex is given by

$$V = \mathrm{rot}_{\arccos(\frac{1}{3}),Ox} C.$$

Applying the described algorithm for rotations using quaternions, we have:

Step 1: We choose as the unit vector (around which the rotation will be performed) the versor of the Ox-axis, namely:

$$v = (1, 0, 0).$$

Step 2: We calculate the rotation quaternion using the formula

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \cdot (iv_x + jv_y + kv_z),$$

where

$$\theta = \arccos\left(\frac{1}{3}\right).$$

We know that $\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cdot \cos^2(x) - 1$, which gives $\cos(x) = \pm\sqrt{\frac{\cos(2x)+1}{2}}$. Therefore

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{\cos(\theta)+1}{2}} = \sqrt{\frac{\cos(\arccos\left(\frac{1}{3}\right))+1}{2}} = \sqrt{\frac{\frac{1}{3}+1}{2}} = \sqrt{\frac{\frac{4}{3}}{2}} = \sqrt{\frac{2}{3}}.$$

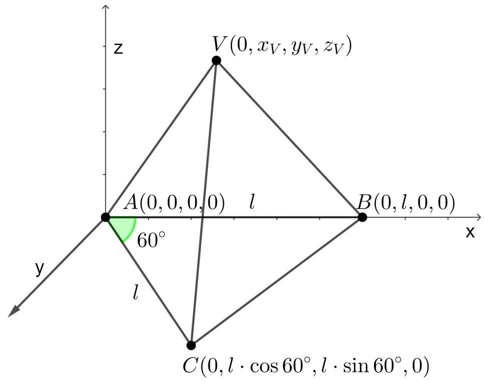
Thus, from the fact that $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$, we obtain that $\sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{3}}$.

Since $v = (1, 0, 0)$ we obtain

$$q = \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}} \cdot i.$$

Step 3: We will use quaternions to write the coordinates of the points, and since a quaternion has four coordinates, we will add an additional coordinate, 0, at the beginning of each point to represent the real part of the quaternion. Thus, we have the four dimensional coordinates:

$$A(0, 0, 0, 0), \qquad B(0, l, 0, 0) \quad \text{and} \quad C(0, \frac{l}{2}, \frac{l\sqrt{3}}{2}, 0).$$



Step 4: We will perform an active rotation. We compute:

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} \cdot i}{1^2} = \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} \cdot i.$$

Furthermore, we have

$$v = q^{-1} \cdot c \cdot q = \left(\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} \cdot i\right)\left(0 + \frac{l}{2} \cdot i + \frac{l\sqrt{3}}{2} \cdot j + 0 \cdot k\right)\left(\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}} \cdot i\right).$$

Performing the quaternion multiplications, we obtain:

$$v = 0 + \frac{l}{2} \cdot i + \frac{l\sqrt{3}}{6} \cdot j - \frac{l\sqrt{6}}{3} \cdot k.$$

Step 5: We have obtained the coordinates of the apex V of the tetrahedron $VABC$, namely

$$V\left(\frac{l}{2}, \frac{l\sqrt{3}}{6}, -\frac{l\sqrt{6}}{3}\right).$$

Given Corollary 1, the volume of the tetrahedron $VABC$ is:

$$\mathcal{V}_{VABC} = \frac{1}{6} \cdot \left| \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_V & y_V & z_V & 1 \end{vmatrix} \right| = \frac{1}{6} \cdot \left| \begin{vmatrix} 0 & 0 & 0 & 1 \\ l & 0 & 0 & 1 \\ \frac{l}{2} & \frac{l\sqrt{3}}{2} & 0 & 1 \\ \frac{l}{2} & \frac{l\sqrt{3}}{6} & -\frac{l\sqrt{6}}{3} & 1 \end{vmatrix} \right|.$$

Expanding along the first row, we obtain:

$$\mathcal{V}_{VABC} = \frac{1}{6} \cdot \left| \begin{vmatrix} l & 0 & 0 \\ \frac{l}{2} & \frac{l\sqrt{3}}{2} & 0 \\ \frac{l}{2} & \frac{l\sqrt{3}}{6} & -\frac{l\sqrt{6}}{3} \end{vmatrix} \right|.$$

Expanding again along the first row, we obtain:

$$\mathcal{V}_{VABC} = \frac{l}{6} \cdot \left| \begin{vmatrix} \frac{l\sqrt{3}}{2} & 0 \\ \frac{l\sqrt{3}}{6} & -\frac{l\sqrt{6}}{3} \end{vmatrix} \right| = \frac{l}{6} \cdot \left| -\frac{l^2\sqrt{18}}{6} \right| = \frac{l}{6} \cdot \frac{l^2 3\sqrt{2}}{6}.$$

Therefore

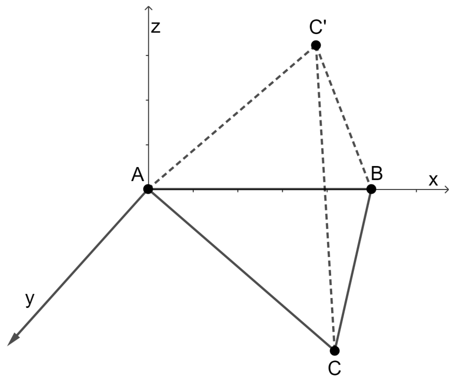
$$\mathcal{V}_{VABC} = \frac{l^3\sqrt{2}}{12}. \qquad \square$$

Finally, we provide an exercise which requests the volume of an irregular tetrahedron.

PROBLEM 1. Let the triangle ABC have side lengths: $AB = 5$, $BC = 7$ and $CA = 10$. By rotating point C in space around the line AB by $\frac{\pi}{3}$ radians, we obtain a point C'. Determine the volume of the tetrahedron $C'ABC$.

Solution:

We consider the same coordinate system as in the previous problem: A is the origin of the system, hence we have $A(0, 0, 0)$. Given that $AB = 5$ and because we want B to be on the Ox-axis, we have $B(5, 0, 0)$. The point C is chosen to be in the plane Oxy, therefore $C(x_C, y_C, 0)$.



We know the lengths of all the sides of triangle ABC, therefore, we can calculate the area of the triangle using Heron's formula:

$$\mathcal{A}_{\triangle ABC} = \sqrt{p \cdot (p-a) \cdot (p-b) \cdot (p-c)}, \text{where } p = \frac{a+b+c}{2}.$$

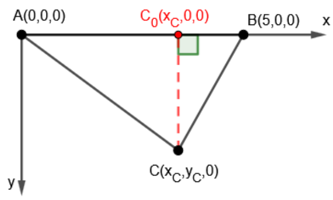
We have $p = \frac{BC+AC+AB}{2} = \frac{7+10+5}{2} = \frac{22}{2} = 11$, hence

$$\mathcal{A}_{\triangle ABC} = \sqrt{11 \cdot (11-7) \cdot (11-10) \cdot (11-5)} = \sqrt{11 \cdot 4 \cdot 1 \cdot 6} = \sqrt{264}.$$

Therefore,

$$\mathcal{A}_{\triangle ABC} = 2\sqrt{66}.$$

Let $CC_0 \perp AB$, with $C_0 \in AB$, as in the following figure.



We have

$$\mathcal{A}_{\triangle ABC} = \frac{CC_0 \cdot AB}{2} = 2\sqrt{66} \Rightarrow CC_0 = \frac{2 \cdot \mathcal{A}_{\triangle ABC}}{AB} = \frac{2 \cdot 2\sqrt{66}}{5} = \frac{4\sqrt{66}}{5},$$

then

$$y_C = \frac{4\sqrt{66}}{5}. \tag{2}$$

In the right triangle ACC_0, by the Pythagorean theorem, we have:

$$AC_0 = \sqrt{AC^2 - CC_0^2} = \sqrt{100 - \frac{16 \cdot 66}{25}} = \sqrt{\frac{1444}{25}} = \frac{38}{5},$$

therefore

$$x_C = \frac{38}{5}. \tag{3}$$

From (2) and (3), it follows that

$$C\left(\frac{38}{5}, \frac{4\sqrt{66}}{5}, 0\right).$$

To find the coordinates of the apex of the tetrahedron, we will perform a rotation of the point C by an angle of $\frac{\pi}{3}$ radians around the edge AB. Thus

$$C' = \operatorname{rot}_{\frac{\pi}{3}, Ox} C.$$

Applying the described rotation algorithm using quaternions, we obtain the following.

Step 1: We choose as the unit vector (around which the rotation will be performed) the versor of the Ox-axis, namely

$$v = (1, 0, 0).$$

Step 2: We compute the rotation quaternion using the formula:

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \cdot (iv_x + jv_y + kv_z),$$

where

$$\theta = \frac{\pi}{3}.$$

We obtain

$$q = \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot (i \cdot 1 + j \cdot 0 + k \cdot 0) = \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i.$$

Step 3: We use quaternions to write the coordinates of the points, and since a quaternion has four coordinates, we will add an additional coordinate, 0, at the beginning of each point to represent the real part of the quaternion. Thus we obtain the following four dimensional coordinates:

$$A(0,0,0,0), \qquad B(0,5,0,0) \quad \text{and} \quad C\left(0, \frac{38}{5}, \frac{4\sqrt{66}}{5}, 0\right).$$

Step 4: We will perform an active rotation. We calculate

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{\frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i}{1^2} = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i.$$

Next, we have

$$v = q^{-1} \cdot c \cdot q = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \cdot i\right) \left(0 + \frac{38}{5} \cdot i + \frac{4\sqrt{66}}{5} \cdot j + 0 \cdot k\right) \left(\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot i\right).$$

Performing the quaternion multiplications, we obtain:

$$v = 0 + \frac{38}{5} \cdot i + \frac{2\sqrt{66}}{5} \cdot j - \frac{6\sqrt{22}}{5} \cdot k.$$

Step 5: We have obtained the coordinates of the apex $C'$of the tetrahedron $C'ABC$, namely:

$$C'\left(\frac{38}{5}, \frac{2\sqrt{66}}{5}, -\frac{6\sqrt{22}}{5}\right).$$

Again, given Corollary 1, the volume of the tetrahedron $C'ABC$ is

$$\mathcal{V}_{C'ABC} = \frac{1}{6} \cdot \left| \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x'_C & y'_C & z'_C & 1 \end{vmatrix} \right|$$

$$= \frac{1}{6} \cdot \left| \begin{vmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 \\ \frac{38}{5} & \frac{4\sqrt{66}}{5} & 0 & 1 \\ \frac{38}{5} & \frac{2\sqrt{66}}{5} & -\frac{6\sqrt{22}}{5} & 1 \end{vmatrix} \right|.$$

Expanding along the first row, we obtain:

$$\mathcal{V}_{C'ABC} = \frac{1}{6} \cdot \left| \begin{vmatrix} 5 & 0 & 0 \\ \frac{38}{5} & \frac{4\sqrt{66}}{5} & 0 \\ \frac{38}{5} & \frac{2\sqrt{66}}{5} & -\frac{6\sqrt{22}}{5} \end{vmatrix} \right|.$$

Expanding again along the first row, we obtain:

$$\begin{aligned}\mathcal{V}_{C'ABC} &= \frac{5}{6}\cdot\left|\begin{vmatrix}\frac{4\sqrt{66}}{5} & 0\\ \frac{2\sqrt{66}}{5} & -\frac{6\sqrt{22}}{5}\end{vmatrix}\right|\\ &= \frac{5}{6}\cdot\left|\frac{4\sqrt{66}}{5}\cdot\frac{6\sqrt{22}}{5}\right|.\end{aligned}$$

Therefore, we obtain

$$\mathcal{V}_{C'ABC} = \frac{88\sqrt{3}}{5}.$$

□

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