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CANONICAL JORDAN FORM FOR QUADRATIC MATRICES (SECOND ORDER)

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1. PRELIMINARY NOTIONS

Let $A = (a_{ij})_{i,j\in 1,2} \in M_2(\mathbb{C})$. We denote by $\operatorname{Spec}(A)$ the set of eigenvalues of A, that is, the roots of the equation

$$\begin{vmatrix} a_{11}-x & a_{12} \\ a_{21} & a_{22}-x \end{vmatrix} = 0$$

The eigenvalues of the matrix are the complex numbers c for which the system

$$\begin{cases} (a_{11}-c)\,x + a_{12}y = 0 \\ a_{21}x + (a_{22}-c)\,y = 0 \end{cases}$$

are nonzero solutions, that is, complex numbers c for which there exists a nonzero vector $v \in M_{2,1}(\mathbb{C})$ such that $Av = cv$.

Proposition 1.1.

Let $A \in M_2(\mathbb{C}), A \neq O_2$ be a nilpotent matrix. Then there exists an invertible matrix $S \in M_2(\mathbb{C})$ such that $A = S^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S$.

Proof. Since A is a nilpotent matrix, we deduce that $\det A = \operatorname{Tr} A = 0$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$ and $ad - bc = 0$.
If $c = 0$, it follows that $a = d = 0$ and $b \neq 0$. We consider $S = \begin{pmatrix} 0 & 1 \\ \frac{1}{b} & 0 \end{pmatrix} \in GL_2(\mathbb{C})$. In this case, it is easily verified that $SAS^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, from which we obtain the required relation.

If $c \neq 0$, we consider $S = \begin{pmatrix} 1 & -\frac{a}{c} \\ 0 & \frac{1}{c} \end{pmatrix} \in GL_2(\mathbb{C})$. Again, it is verified that $SAS^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

□

Proposition 1.2.

Let $A \in M_2(\mathbb{C})$ with $\mathrm{Spec}(A) = \{\lambda\}, A \neq \lambda I_2$, where $\lambda \in \mathbb{C}$. Then there exists $S \in GL_2(\mathbb{C})$ such that $A = S^{-1} \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} S$.

Proof. Clearly, $A - \lambda I_2 \neq O_2$.
First, we show that $A - \lambda I_2$ is a nilpotent matrix. Let α be an eigenvalue of $A - \lambda I_2$. It follows that there exists $v \in M_{2,1}(\mathbb{C}), v \neq \binom{0}{0}$ such that $(A - \lambda I_2)v = \alpha v$. Then $Av = (\lambda + \alpha)v$. From this, we deduce that $\lambda + \alpha \in \mathrm{Spec}(A)$, so $\alpha = 0$. Since the eigenvalues of $A - \lambda I_2$ are all zero, it follows that $A - \lambda I_2$ is a nilpotent matrix.

□

Using Proposition 1.1, we find that there exists $S \in GL_2(\mathbb{C})$ such that $A - \lambda I_2 = S^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S$, which implies that $A = S^{-1} \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} S$.

Definition 1.1.

Under the assumptions of the proposition, the matrix $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ represents **the Jordan form of A**.

Proposition 1.3.

Let $A \in M_2(\mathbb{C})$ with $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2\}$, where $\lambda_1, \lambda_2 \in \mathbb{C}$ are distinct. Then there exists $S \in GL_2(\mathbb{C})$ such that $A = S^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S$.

Proof. Since $\lambda_i \in \mathbb{C}$ is an eigenvalue of A for each $i \in \{1, 2\}$, it follows that there exists $v_i \in M_{2,1}(\mathbb{C}), v_i \neq \binom{0}{0}$ such that $Av_i = \lambda_i v_i$ for each $i \in \{1, 2\}$. (1)

Let $v_1 = \binom{x_1}{x_2}$ and $v_2 = \binom{y_1}{y_2}$, and consider $T = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. We show that T is an invertible matrix.
Assume by contradiction that $\det T = 0$. Then the columns of T are proportional, so there exists $c \in \mathbb{C}$ such that $v_2 = cv_1$. Therefore, $Av_2 = cAv_1$. From this, it follows that $\lambda_2 v_2 = c\lambda_1 v_1$, so $\lambda_2 c v_1 = c\lambda_1 v_1$. Thus, we obtain $(\lambda_1 - \lambda_2)cv_1 = \binom{0}{0}$.
But $\lambda_1 \neq \lambda_2$ and $v_1 \neq \binom{0}{0}$. It follows that $c = 0$, so $v_2 = \binom{0}{0}$, a contradiction.
Let $S = T^{-1}$. We verify that $AT = T\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}$. Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we have

$$AT = \begin{pmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ cx_1 + dx_2 & cy_1 + dy_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 y_1 \\ \lambda_1 x_2 & \lambda_2 y_2 \end{pmatrix} = T\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(the penultimate equality follows from (1)). Thus, $A = S^{-1}\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} S$.
$\square$

Definition 1.2.

Under the assumptions of Proposition 1.3, the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}$ represents **the Jordan canonical form of $\boldsymbol{A}$**.

2. JORDAN CANONICAL FORM FOR SQUARE MATRICES (OF ORDER N)

Theorem 2.1. *(Schur's unitary triangularization theorem)*

Let $A \in M_n(\mathbb{C})$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that the matrix U^*AU is upper triangular, with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal.
If $A \in M_n(\mathbb{R})$ and all the eigenvalues of A are real, then there exists an orthogonal matrix $U \in M_n(\mathbb{R})$ such that the matrix U^tAU has the above property.

Theorem 2.2. *(simultaneous triangularization theorem)*

Let $A, B \in M_n(\mathbb{C})$ such that $AB = BA$. Then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that the matrices U^*AU and U^*BU are both upper triangular.

Moreover, if $F \subset M_n(\mathbb{C})$ is a commutative family (i.e., a family with the property that any two of its matrices commute), then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that U^*AU is upper triangular, for any $A \in F$.

Definition 2.1.

Let $m \in \mathbb{N}$. A Jordan block of order $\boldsymbol{m}$ is any upper triangular matrix in $M_m(\mathbb{C})$, of the form:

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

A Jordan matrix is called any block matrix $J \in M_n(\mathbb{C})$, of the form:

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{n_2}(\lambda_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{n_k}(\lambda_k) \end{pmatrix}$$

where $J_{n_i}(\lambda_i)$ are Jordan blocks, and $n_1 + n_2 + \ldots + n_k = n$. The orders n_i of the Jordan blocks, as well as the values λ_i, are not necessarily distinct.

Theorem 2.3. *(Jordan canonical form theorem)*

For any $A \in M_n\mathbb{C}$, there exists an invertible matrix $S \in M_n\mathbb{C}$ such that

$$S^{-1}AS = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{n_2}(\lambda_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{n_k}(\lambda_k) \end{pmatrix} =: J_A$$

The matrix J_A is called the Jordan matrix (or Jordan canonical form) of A. It is unique, up to a permutation of the diagonal blocks. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of A, and they are not necessarily distinct.

If $A \in M_n(\mathbb{R})$ and all eigenvalues of A are real, then the similarity matrix S can be chosen from $M_n(\mathbb{R})$.

Note: Let $A \in M_n(\mathbb{C})$ be a matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ with algebraic multiplicities $s_1, s_2, \ldots, s_m \in \mathbb{N}$. Then the characteristic polynomial of A has the form:

$$p_A(t) = (t - \lambda_1)^{s_1} \cdot \ldots \cdot (t - \lambda_m)^{s_m}.$$

According to Frobenius' theorem, the minimal polynomial of A has the form:

$$m_A(t) = (t - \lambda_1)^{r_1} \cdot \ldots \cdot (t - \lambda_m)^{r_m},$$

where $r_1, r_2, \ldots, r_m \in \mathbb{N}$ such that $1 \leq r_i \leq s_i$ for all $i \in \{1, 2, \ldots, m\}$. The relationship between the Jordan matrix J_A and the two polynomials is summarized below:

- The total number k of Jordan blocks (counting the repeated appearances of the same block) represents the number of linearly independent eigenvectors of A;
- The number of Jordan blocks corresponding to an eigenvalue λ_i represents the geometric multiplicity of the eigenvalue λ_i, i.e., the dimension of the eigenspace of eigenvectors associated with λ_i, $x \in \mathbb{C}^n \mid Ax = \lambda_i x$;
- The sum of the orders of the Jordan blocks corresponding to an eigenvalue λ_i represents the algebraic multiplicity s_i of the eigenvalue λ_i, i.e., the power of the factor $(t - \lambda_i)$ in the factorization of the characteristic polynomial p_A;
- The order of the largest Jordan block corresponding to an eigenvalue λ_i represents the power r_i of the factor $(t - \lambda_i)$ in the factorization of the minimal polynomial m_A.

An immediate, but important consequence in applications of the last remark is:

Theorem 2.4.

If $A \in M_n(\mathbb{C})$ and $f \in \mathbb{C}[X]$ is a polynomial with only simple roots such that $f(A) = O_n$, then A is diagonalizable.

Applications: Problem 2.1.

Let $A \in M_2(\mathbb{C})$ such that $\det(A^2 + A + I_2) = \det(A^2 - A + I_2) = 3$. Prove that:

$$A^2 \cdot (A^2 + I_2) = 2I_2.$$

(National Mathematics Olympiad, county phase, 2016)

Solution. If A is of the form αI_2, we get $(\alpha^2+\alpha+1)^2 = (\alpha^2-\alpha+1)^2 = 3$. Since $(\alpha^2 + \alpha + 1)^2 - (\alpha^2 - \alpha + 1)^2 = 0$, it follows that $\alpha(\alpha^2 + 1) = 0$, so $\alpha \in \{0, i, -i\}$. None of these values satisfy the equations.

If A has a single eigenvalue λ and $A \neq \lambda I_2$, then there exists S such that

$$A = S^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S.$$

From $\det(A^2 + A + I_2) = \det(A^2 - A + I_2) = 3$, it follows that

$$\det\left(S^{-1} \begin{pmatrix} \lambda^2+\lambda+1 & 0 \\ 2\lambda+1 & \lambda^2+\lambda+1 \end{pmatrix} S\right) =$$

$$= \det\left(S^{-1} \begin{pmatrix} \lambda^2-\lambda+1 & 0 \\ 2\lambda-1 & \lambda^2-\lambda+1 \end{pmatrix} S\right) = 3,$$

from which it follows that $(\lambda^2 + \lambda + 1)^2 = (\lambda^2 - \lambda + 1)^2 = 3$, for which we have seen that there are no solutions.

If A has two distinct eigenvalues λ_1, λ_2, then there exists $S \in GL_2(\mathbb{C})$ such that

$$A = S^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix} S.$$

From $\det(A^2 + A + I_2) = \det(A^2 - A + I_2) = 3$, it follows that

$$\det\left(S^{-1}\begin{pmatrix} \lambda_1^2+\lambda_1+1 & 0 \\ 0 & \lambda_2^2+\lambda_2+1 \end{pmatrix} S\right) =$$

$$= \det\left(S^{-1}\begin{pmatrix} \lambda_1^2-\lambda_1+1 & 0 \\ 0 & \lambda_2^2-\lambda_2+1 \end{pmatrix} S\right) = 3,$$

and we have that

$$(2) \qquad (\lambda_1^2+\lambda_1+1)(\lambda_2^2+\lambda_2+1) = (\lambda_1^2-\lambda_1+1)(\lambda_2^2-\lambda_2+1) = 3$$

By performing the multiplications and simplifying the like terms, we obtain

$$(\lambda_1+\lambda_2)(\lambda_1\lambda_2+1) = 0.$$

From this, it follows that $\lambda_2 = -\lambda_1$ or $\lambda_1\lambda_2 = -1$.
If $\lambda_2 = -\lambda_1$, then (2) becomes $\lambda_1^4+\lambda_1^2 = \lambda_2^4+\lambda_2^2 = 2$.
If $\lambda_1\lambda_2 = -1$, it follows that $\lambda_2 = -\frac{1}{\lambda_1}$. Substituting into (2) gives $\lambda_1^2 + \frac{1}{\lambda_1^2} = 2$.
Thus, $\left(\lambda_1 - \frac{1}{\lambda_1}\right)^2 = 0$, therefore $\lambda_1^2 = 1$ and we obtain $\lambda_1^4+\lambda_1^2 = \lambda_2^4+\lambda_2^2 = 2$.
Thus, $\lambda_1^4+\lambda_1^2 = \lambda_2^4+\lambda_2^2 = 2$ which leads to $A^2(A^2+I_2) =$
$= S^{-1}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} \lambda_1^2+1 & 0 \\ 0 & \lambda_2^2+1 \end{pmatrix} S = 2I_2.$

Problem 2.2.

Let $A \in M_2(\mathbb{C})$ be a matrix that satisfies the conditions $\det(A^{2014} - I_2) = \det(A^{2014}+I_2)$ and $\det(A^{2016}-I_2) = \det(A^{2016}+I_2)$.

Prove that $\det(A^n - I_2) = \det(A^n + I_2)$, for any non-zero natural number n.

(National Mathematics Olympiad, national phase, 2016)

Solution. If A is of the form αI_2, it immediately follows that $\alpha = 0$, so $A = O_2$. In this case, it is easy to verify that $\det(A^n - I_2) = \det(A^n + I_2)$, for any non-zero natural number n.

If A has a single eigenvalue λ and $A \neq \lambda I_2$, then there exists S such that

$$A = S^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S.$$

From $\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$, it follows that

$$\det\left(S^{-1} \begin{pmatrix} \lambda^{2014} - 1 & 0 \\ 2014\lambda^{2013} & \lambda^{2014} - 1 \end{pmatrix} S\right) =$$
$$\det\left(S^{-1} \begin{pmatrix} \lambda^{2014} + 1 & 0 \\ 2014\lambda^{2013} & \lambda^{2014} + 1 \end{pmatrix} S\right),$$

from which it follows that $(\lambda^{2014} - 1)^2 = (\lambda^{2014} + 1)^2$. Thus, we deduce that $\lambda = 0$.
Therefore, $(\lambda^n - 1)^2 = (\lambda^n + 1)^2$, for any non-zero natural number n.
Thus, $\det(A^n - I_2) = \det(A^n + I_2)$, for any non-zero natural number n.
If A has two distinct eigenvalues λ_1, λ_2, then there exists $S \in GL_2(\mathbb{C})$ such that

$$A = S^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix} S.$$

From $\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$, it follows that

$$\det\left(S^{-1} \begin{pmatrix} \lambda_1^{2014} - 1 & 0 \\ 0 & \lambda_2^{2014} - 1 \end{pmatrix} S\right) =$$
$$\det\left(S^{-1} \begin{pmatrix} \lambda_1^{2014} + 1 & 0 \\ 0 & \lambda_2^{2014} + 1 \end{pmatrix} S\right),$$

and then we have

$$(\lambda_1^{2014} - 1)(\lambda_2^{2014} - 1) = (\lambda_1^{2014} + 1)(\lambda_2^{2014} + 1).$$

We deduce that $\lambda_1^{2014} + \lambda_2^{2014} = 0$.
Similarly, we obtain that $\lambda_1^{2016} + \lambda_2^{2016} = 0$. If λ_1, λ_2 are non-zero, then $\lambda_1^2 = \lambda_2^2$, from which it follows that $\lambda_1 = \lambda_2 = 0$, which contradicts $\lambda_1 \neq \lambda_2$.

Problem 2.3.

Let $A, B \in M_2(\mathbb{R})$ be nilpotent matrices, different from the zero matrix, such that $AB = BA$. Prove that:
a) $AB = O_2$;
b) There exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $A = \alpha B$.

(D. Miheț, L. Duican Competition, 2011/2)

Solution. a) Since the matrices A and B are nilpotent, all their eigenvalues are zero. Applying the simultaneous triangularization theorem (Theorem 2.2), we deduce that there exists an orthogonal matrix $U \in M_2\mathbb{R}$ such that the matrices $T_1 := U^tAU$ and $T_2 := U^tBU$ are upper triangular. Furthermore, the earlier observation about the eigenvalues shows that T_1 and T_2 are of the form $T_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, and $T_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R} \setminus \{0\}$ (since A and B are non-zero).
Since $T_1T_2 = O_2$, it follows that $AB = UT_1T_2U^t = O_2$.

b) We have $A = UT_1U^t$ and $B = UT_2U^t$. Let $\alpha = \frac{a}{b}$. Then, $T_1 = \alpha T_2$, so $A = \alpha B$.

Problem 2.4.

Let $A, B \in M_n(\mathbb{C})$ be matrices with the property that $AB = BA$. Prove that if there exists a natural number $k \geq 1$ such that $B^k = O_n$, then $\det(A + B) = \det(A)$.

Solution. Since $AB = BA$, by the simultaneous triangularization theorem (Theorem 2.2), there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that the matrices $T_1 := U^*AU$ and $T_2 := U^*BU$ are both upper triangular. Since A and T_1 are similar, it follows that T_1 has the same eigenvalues as A (in some order). Similarly, T_2 has the eigenvalues of B on its diagonal. But since B is nilpotent, it has all its eigenvalues equal to zero, so T_2 has zeros on its diagonal. We have

$$A + B = UT_1 + T_2U^*,$$

so $\det(A + B) = \det(T_1 + T_2)$. Since the matrix $T_1 + T_2$ is upper triangular, $\det(T_1+T_2)$ is the product of the diagonal elements of T_1+T_2, which is exactly the product of the eigenvalues of A, i.e., $\det(A)$.
Therefore, $\det(A + B) = \det(A)$.

Problem 2.5.

Let $A \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ be an eigenvalue of the matrix A^n, and let v be a corresponding eigenvector. Prove that if the vectors $v, Av, A^2v, \ldots, A^{n-1}v$ are linearly independent, then $A^n = \lambda I_n$.

(T. Lalescu Competition, National Phase, 2013)

Solution. Let $p_A(t) := \det(tI_n - A) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of A. By the Cayley-Hamilton theorem, we have

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = O_n \tag{3}$$

Multiplying this relation by v on the right and using the fact that $A^n v = \lambda v$, we obtain

$$a_{n-1}A^{n-1}v + \cdots + a_1 Av + a_0 v + \lambda v = O_n.$$

Since $v, Av, A^2v, \ldots, A^{n-1}v$ are linearly independent, it follows that $a_{n-1} = \cdots = a_1 = 0$, and $a_0 = -\lambda$. Substituting into relation (3), we deduce that

$$A^n = \lambda I_n.$$

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