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A CONVERGENCE THEOREM AND ITS APPLICATION CONCERNING RIEMANN'S ZETA FUNCTION

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Abstract. The present text presents the Euler-McLaurin integral formula along with its demonstration and a new convergence criterion for a certain type of complex number series. An asyptotic development for Riemann's Zeta function is derived, using the Euler-McLaurin integral formula and the new presented convergence criterion

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1. EULER-MACLAURIN FORMULA

1.1. Bernoulli polynomials.

Bernoulli polynomials $B_n(x)$ for $n \in \{0, 1, 2, ...\}$ are defined recurrently with $B_0(x) = 1$ and $B_n(x)$ satisfying

$$B_n'(x) = nB_{n-1}(x); \tag{1}$$

and

$$\int_0^1 B_n(x)dx = 0; \tag{2}$$

for all $n \in \{1, 2, 3, ...\}$.

Example 1. For $n = 1$ one can compute $B_1'(x) = 1 \Rightarrow B_1(x) = x + c$. From here $\int_0^1 (x + c)dx = \frac{1}{2} + c = 0 \Rightarrow c = -\frac{1}{2}$, so $B_1(x) = x - \frac{1}{2}$.

Lemma 1. For $n > 1$ the Bernoulli polynomials satisfy $B_n(1) = B_n(0)$.

Proof. Let n be a natural number $n > 1$. Then from (2) one has $\int_1^0 nB_{n-1}(x)dx = 0$ so $\int_1^0 B_n'(x)dx = 0$. Therefore $B_n(1) - B_n(0) = 0$. □

Definition 1 (Bernoulli number). For all $n \in \{0, 1, 2, ...\}$ the Bernoulli number B_n is defined as $B_n = B_n(1)$.

Definition 2 (Periodic Bernoulli polynomials). For all $n \in \{0, 1, 2, ...\}$ the periodic Bernoulli polynomial $P_n(x)$ is defined as $P_n(x) = B_n(x - [x])$, where $[x]$ denotes the integer part of x not greater then x.

1.2. The Euler-MacLaurin formula.

The following theorem is a particular case of Euler-McLaurin integral formula, for functions defined on positive real semiaxis and infinitly derivable.

Theorem 1 (Euler-McLaurin integral formula). Let $f \in C^{\infty}[0,\infty)$, then the following relation in true, for $n, p \in \{1, 2, ...\}$:

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n) - f(0)}{2} + \sum_{k=2}^{p} (-1)^k \frac{B_k}{k!} \left[f^{(k-1)}(n) - f^{(k-1)}(0) \right] + \frac{(-1)^{(p+1)}}{p!} \int_0^n f^{(p)}(x) P_p(x) dx \tag{3}$$

Proof. The proof follows somehow [1]. Let $k \in \mathbb{N}$. Then

$$\int_k^{k+1} f(x)dx = \int_k^{k+1} f(x) P_0(x) dx \tag{4}$$

and using equation (1) for $P_1(x)$, the following relation is true

$$\int_k^{k+1} f(x) P_0(x) dx = \frac{1}{1} \int_k^{k+1} f(x) P_1'(x) dx \tag{5}$$

Integrating by parts one obtains the following

$$\int_k^{k+1} f(x)dx = f(x) P_1(x)|_k^{k+1} - \int_k^{k+1} f'(x) P_1(x) dx \tag{6}$$

$$\int_k^{k+1} f(x)dx = f(k+1) P_1(k+1) - f(k) P_1(k) - - \int_k^{k+1} f'(x) P_1(x) dx \tag{7}$$

$$\int_k^{k+1} f(x)dx = f(k+1) B_1(1) - f(k) B_1(0) - - \int_k^{k+1} f'(x) P_1(x) dx \tag{8}$$

Using the above derivation, the integral from 0 to n can be expressed in the following way:

$$\int_0^n f(x)dx = \sum_{k=0}^{n-1} \int_k^{k+1} f(x)dx = \sum_{k=0}^{n-1} \left[f(k+1) B_1(1) - f(k) B_1(0) \right] - - \int_0^n f'(x) P_1(x) dx; \tag{9}$$

Therefore

$$\int_0^n f(x)dx = \sum_{k=0}^{n-1} \left[f(k+1) + f(k)\right] B_1(1) - \tag{10}$$
$$- \int_0^n f'(x)P_1(x)dx$$

The expresion (10) can be further modified:

$$\int_0^n f(x)dx = \left[f(0) + f(n)\right] B_1(1) + 2\sum_{k=1}^{n-1} f(k)B_1(1) - \tag{11}$$
$$- \int_0^n f'(x)P_1(x)dx$$

$$\int_0^n f(x)dx = \left[f(0) - f(n)\right] B_1(1) + 2\sum_{k=1}^{n} f(k)B_1(1) - \tag{12}$$
$$- \int_0^n f'(x)P_1(x)dx$$

$$2B_1(1)\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \left[f(n) - f(0)\right] B_1(1) + \tag{13}$$
$$+ \int_0^n f'(x)P_1(x)dx$$

Using the above Bernoulli number definition, Equation (13) is rewritten as

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{\left[f(n) - f(0)\right]}{2} + \tag{14}$$
$$+ \int_0^n f'(x)P_1(x)dx$$

Let us evaluate $\int_k^{k+1} f'(x)P_1(x)dx$:

$$\int_k^{k+1} f'(x)P_1(x)dx = \int_k^{k+1} f'(x)\frac{P_2'(x)}{2}dx \tag{15}$$

because $P_2'(x) = B_2'(x - [x]) = 2B_1(x - [x]) = 2P_1(x)$. Integrating again by parts one can obtain:

$$\int_k^{k+1} f'(x)P_1(x)dx = f'(x)\frac{P_2(x)}{2}\bigg|_k^{k+1} - \int_k^{k+1} f''(x)\frac{P_2(x)}{2}dx \tag{16}$$

$$\int_k^{k+1} f'(x)P_1(x)dx = \left[f'(k+1) - f'(k)\right]\frac{B_2(1)}{2} - \\ -\frac{1}{2}\int_k^{k+1} f''(x)P_2(x)dx \tag{17}$$

Hence again

$$\int_0^n f'(x)P_1(x)dx = \sum_{k=0}^{n-1}\int_k^{k+1} f'(x)P_1(x)dx \\ = \frac{B_1(1)}{2}\sum_{k=0}^{n-1}\left[f'(k+1) - f'(k)\right] - \\ -\frac{1}{2}\int_0^n f''(x)P_2(x)dx \tag{18}$$

$$\int_0^n f'(x)P_1(x)dx = \frac{B_2}{2}\left[f'(n) - f'(0)\right] - \frac{1}{2}\int_0^n f''(x)P_2(x)dx \tag{19}$$

Replacing Equation (19) in Equation (14) one obtains

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n) - f(0)}{2} + \\ + \frac{B_2}{2}\left[f'(n) - f'(0)\right] - \frac{1}{2}\int_0^n f''(x)P_2(x)dx \tag{20}$$

Let us now evaluate $\int_k^{k+1} f''(x)P_2(x)dx$:

$$\int_k^{k+1} f''(x)P_2(x)dx = \int_k^{k+1} f''(x)\frac{P_3'(x)}{3}dx \\ = f''(x)\frac{P_3(x)}{3}\Big|_k^{k+1} - \int_k^{k+1} f'''(x)\frac{P_3(x)}{3}dx \\ = \frac{B_3}{3}\left[f''(k+1) - f''(k)\right] - \\ -\frac{1}{3}\int_k^{k+1} f'''(x)P_3(x)dx \tag{21}$$

Therefore

$$\int_0^n f''(x)P_2(x)dx = \frac{B_3}{3}\left[f''(n) - f''(0)\right] - \frac{1}{3}\int_0^n f'''(x)P_3(x)dx \tag{22}$$

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n) - f(0)}{2} + \frac{B_2}{2}\left[f'(n) - f'(0)\right] - \\ -\frac{1}{2}\left[\frac{B_3}{3}\left[f''(n) - f''(0)\right] - \frac{1}{3}\int_0^n f'''(x)P_3(x)dx\right] \tag{23}$$

The reader can see now that the process can be repeated, so after p steps the following relation holds:

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n)-f(0)}{2} + \sum_{k=2}^{p}(-1)^k \frac{B_k}{k!}\left[f^{(k-1)}(n) - f^{(k-1)}(0)\right] + \frac{(-1)^{(p+1)}}{p!}\int_0^n f^{(p)}(x)P_p(x)dx \tag{24}$$

which ends the demostration. □

EXAMPLE 2. Let $f(x) = \frac{1}{(1+x)^s}$ with $x \in \mathbb{R}_+$ and $s > 1$. Then $\int_0^n f(x)dx = \frac{(n+1)^{1-s}}{1-s} - \frac{1}{1-s}$ and $f^{(k)}(x) = (-1)^k \prod_{p=0}^{k-1}(s+p)\frac{1}{(1+x)^{k+s}}$. Applying the Theorem 1 one obtains:

$$\sum_{k=1}^{n} \frac{1}{(1+k)^s} = \frac{(n+1)^{-s+1}}{-s+1} - \frac{1}{-s+1} + \frac{(1+n)^{-s}-1}{2} - \sum_{k=2}^{m}\frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p)\left[\frac{1}{(n+1)^{s+k-1}} - 1\right] - \frac{1}{m!}\int_0^n \prod_{p=0}^{m-1}(s+p)\frac{1}{(1+x)^{s+m}}P_m(x)dx \tag{25}$$

The last term in Equation(25) is the remainder term, it shall be denoted $R_{m,n}$, where m denotes how many derivatives are considered and n is the upper limit of the integral or the sum.

1.3. A convergence criterion. In this subsection a convergence criterion for some series is enounced and an original proof is given. This convergence criterion resembles an already known theorem due to Cauchy and MacLaurin, but the reader will notice differences in demonstration and in formulation of it. First the known theorem, see [2]

THEOREM 2 (Cauchy, MacLaurin). If f(x) is positive, continuous, and tends monotOTonically to 0, then an Euler constant γ_f, which is defined below, exists

$$\gamma_f = \lim_{n\to\infty}\left(\sum_{i=1}^{i=n} f(i) - \int_1^n f(x)dx\right) \tag{26}$$

Proof. The theorem and the proof follows closely the presentation from [2]. The continuity of f guarantees the existence of the integral $\int_1^n f(x)dx$ for $n \in 1, 2, ...$ Since f is decreasing, the maximum and minimum of f over a closed interval is known:

$$\inf_{x\in[k,k+1]} f(x) = f(k+1) \tag{27}$$

$$\sup_{x\in[k,k+1]} f(x) = f(k), \tag{28}$$

therefore the following inequatity holds:

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k) \tag{29}$$

and summing form $k = 1$ to $n-1$, one obtains

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx \leq \sum_{k=1}^{n-1} f(k) \tag{30}$$

Substracting $\sum_{k=1}^{n} f(k)$ from both sides in Equation (30)

$$-f(1) \leq \int_1^n f(x)dx - \sum_{k=1}^{n} f(k) \leq -f(n) \tag{31}$$

and after multiplying with -1

$$f(1) \geq \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx \geq f(n) \geq 0 \tag{32}$$

The sequence $s_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx$ is therefore bounded and

$$s_{n+1} - s_n = f(n+1) - \int_n^{n+1} f(x)dx \leq 0 \tag{33}$$

monotonically decreasing, so it has a limit. □

The almost novel convergence criterion this paper presents is enounced below:

THEOREM 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function at least two times derivable with $|f''|$ monotonically decreasing with $\sum_{k=1}^{n} |f''(k)|$ convergent for $n \to \infty$. Then the sequence $u_n = (\sum_{k=1}^{n} f'(k) - f(n))$ is also convergent.

Proof. Let $\epsilon > 0$, we shall prove $\exists n_\epsilon \in \mathbb{N}$ such that $\forall n, m > n_\epsilon$ one has $|u_n - u_m| < \epsilon$ meaning that (u_n) is a Cauchy sequence. Because $\mathbb{R}$ is a complete space, that will make (u_n) convergent. Let us evaluate $|u_n - u_m|$, presuming $n > m$:

$$\begin{aligned} |u_n - u_m| &= \left| \sum_{k=1}^{n} f'(k) - f(n) - \sum_{k=1}^{m} f'(k) + f(m) \right| \\ &= \left| \sum_{k=m+1}^{n} f'(k) - \sum_{k=m+1}^{n} (f(k) - f(k-1)) \right| \end{aligned} \tag{34}$$

Using the Lagrange's mean value theorem $\exists c_k$ such that $f(k) - f(k-1) = f'(c_k)(k-(k-1)) = f'(c_k)$, hence

$$|u_n - u_m| = \left|\sum_{k=m+1}^{n} \left(f'(k) - f'(c_k)\right)\right| \tag{35}$$

where $c_k \in (k-1, k)$. Using again the Lagrange mean value theorem $\exists d_k \in (c_k, k)$ such that $f'(k) - f'(c_k) = f''(d_k)(k - c_k)$.

$$\begin{aligned} |u_n - u_m| &\le \sum_{k=m+1}^{n} \left|f'(k) - f'(c_k)\right| \\ &\le \sum_{k=m+1}^{n} \left|f''(d_k)\right| \le \sum_{k=m+1}^{n} \left|f''(k-1)\right| \end{aligned} \tag{36}$$

since $|f''|$ is monotonically decreasing. The above sum is the rest of a convergent series. This ends the demonstration. □

Remark 1. The Theorem 2 give similar results with Theorem3, if instead of f one considers f'.

Remark 2. Theorem 2 asks for the function to be positive, monotonically decresing to zero, whereas Theorem 3 asks for the second derivative to be monotonically decreasing and $\sum_{k=1}^{n} |f''(k)|$ to be convergent which implies it's convergence to zero. Note that it not necesary to be positive.

The Theorem 3 can be genreralised for the following case:

Theorem 4. Let $f : \mathbb{R} \to \mathbb{C}$ be a double differentiable function with $|f''|$ monotonically decreasing and $\sum_{k=1}^{n} |f''(k)|$ is a convergent real series. Then $u_n = \left(\sum_{k=1}^{n} f'(k) - f(n)\right)$ is a convergent sequence.

Proof. Let $\epsilon > 0$, we shall prove $\exists n_\epsilon \in \mathbb{N}$ such that $\forall n, m > n_\epsilon$ one has $|u_n - u_m| < \epsilon$ meaning that (u_n) is a Cauchy sequence. Because $\mathbb{C}$ is a complete space, that will make (u_n) convergent. Let us evaluate $|u_n - u_m|$, presuming $n > m$:

$$\begin{aligned} |u_n - u_m| &= \left|\sum_{k=1}^{n} f'(k) - f(n) - \sum_{k=1}^{m} f'(k) + f(m)\right| \\ &= \left|\sum_{k=m+1}^{n} f'(k) - \sum_{k=m+1}^{n} (f(k) - f(k-1))\right| \\ &\le \left|\Re\left(\sum_{k=m+1}^{n} f'(k) - \sum_{k=m+1}^{n} (f(k) - f(k-1))\right)\right| + \\ &+ \left|\Im\left(\sum_{k=m+1}^{n} f'(k) - \sum_{k=m+1}^{n} (f(k) - f(k-1))\right)\right| \end{aligned} \tag{37}$$

Using again Lagrange's mean value theorem for real and imaginary parts, independently, will result in the existence of r_k and i_k such that $\Re\left(f(k)-f(k-1)\right)=\Re\left(f'(r_k)\right)$ and $\Im\left(f(k)-f(k-1)\right)=\Im\left(f'(i_k)\right)$, therefore

$$|u_n-u_m| \leq \sum_{k=m+1}^{n} \Re\left(f'(k)-f'(r_k)\right)+\sum_{k=m+1}^{n} \Im\left(f'(k)-f'(i_k)\right)$$
$$\leq \sum_{k=m+1}^{n} \left|f'(k)-f'(r_k)\right|+\sum_{k=m+1}^{n} \left|f'(k)-f'(i_k)\right| \tag{38}$$

where r_k and i_k are in the interval $(k-1,k)$. Using again the mean value theorem

$$|u_n-u_m| \leq \sum_{k=m+1}^{n} \left|f''(c_k)\right|+\sum_{k=m+1}^{n} \left|f''(d_k)\right|$$
$$\leq \sum_{k=m+1}^{n} \left|f''(k-1)\right|+\sum_{k=m+1}^{n} \left|f''(k-1)\right| \tag{39}$$

The above sums are converging to zero, being the rests of convergent series. □

EXAMPLE 3. Let us apply Theorem 4 for the function of real variable x and complex values, $f(x)=\frac{1}{1-s}(1+x)^{1-s}$, for $x\in\mathbb{R}_+$ and $s\in\mathbb{C}$ with $\Re(s)>0$. The reader can verify that $f'(x)=\frac{1}{(1+x)^s}$ and $f''(x)=\frac{-s}{(1+x)^{s+1}}$, therefore f satisfies the condition in Theorem4, hence the sequence $Z_n(s)=\left(\sum_{k=1}^{n}\frac{1}{(1+k)^s}-\frac{1}{1-s}(1+n)^{1-s}\right)$ is convergent. For $s\in\mathbb{C}$ with $\Re(s)>1$ one has $\lim_{n\to\infty}\frac{1}{1-s}(1+n)^{1-s}=0$, hence $\lim_{n\to\infty}Z_n(s)=\zeta(s)-1$ punctually. It is important to mention that the convergence is uniform if $s\in K$ with K being a compact subset of the right complex semiplane. Indeed from Equation (39)

$$|Z_n(s)-Z_m(s)| \leq \sum_{k=m+1}^{n} \left|f''(k-1)\right|+\sum_{k=m+1}^{n} \left|f''(k-1)\right| \tag{40}$$

with $f''(x)=\frac{-s}{(1+x)^{s+1}}$ so

$$|Z_n(s)-Z_m(s)| \leq 2\,|s| \sum_{k=m+1}^{n} \left|\frac{1}{k^{s+1}}\right| \tag{41}$$

Because $s\in K$and K is compact, $\exists M\in\mathbb{R}_+$ with $|s|<M\forall s\in K$, and $\exists s_0\in K$ such that $\left|\frac{1}{k^{s+1}}\right|\leq\left|\frac{1}{k^{s_0+1}}\right|\forall s\in K$, therefore $\forall s\in K$

$$|Z_n(s)-Z_m(s)| \leq 2M \sum_{k=m+1}^{n} \left|\frac{1}{k^{s_0+1}}\right| \tag{42}$$

hence Z_n converges uniformly in K. Moreover $Z_n(s)$ is an holomorfic function which converges unifromly on every compact subset of the right complex plane, so using Weierstrass's theorem, it's limit is a holomorfic function, which has the same values with $\zeta(s) - 1$ on the interval $(1, \infty)$. Using the holomofic functions zeros theorem one obtains that the two function are identical for $s \in \mathbb{C}$ with $\Re(s) > 1$. But $\exists \lim_{n\to\infty} Z_n(s) = Z(s)$ for $s \in (C)$ with $\Re(s) > 0$ and $s \neq 1$, so $1 + Z(s) \equiv \zeta(s)$, being it's analytical continuation on $s \in \mathbb{C}$ with $\Re(s) > 0$ and $s \neq 1$. Therefore

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{(k+1)^s} - \frac{1}{1-s}(1+n)^{1-s} \right) = \zeta(s) - 1 \tag{43}$$

$$\lim_{n\to\infty} \left(\sum_{k=2}^{n+1} \frac{1}{(k)^s} - \frac{1}{1-s}(1+n)^{1-s} \right) = \zeta(s) - 1 \tag{44}$$

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n+1} \frac{1}{k^s} - \frac{1}{1-s}(1+n)^{1-s} \right) = \zeta(s) \tag{45}$$

EXAMPLE 4. Let us apply Theorem 4 for $f(x) = ln(x)$. Because $|ln''(x)| = \frac{1}{x^2}$ and $\sum_{k=1}^{n} \frac{1}{k^2}$ is convergent, follows that $\sum_{k=1}^{n} \frac{1}{k} - ln(n)$ is also convergent. It's limit is γ, the Euler-Mascheroni constant.

2. ASYMPTOTIC EXPANSION FOR $\zeta(S)$

An asymptotic expansion for $\zeta(s)$ is given below:

THEOREM 5. For $s > 1$ the following relation holds:

$$\zeta(s) = \frac{1}{2} - \frac{1}{1-s} + \sum_{k=2}^{m} \frac{B_k}{k!} \prod_{p=0}^{k-2} (s+p) - R_{m,\infty}(s) \tag{46}$$

if $\exists\ R_{m,\infty}(s) = lim_{n\to\infty} R_{m,n}(s)$

Proof. Using Equation (25) one has:

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{(1+k)^s} &= \frac{(n+1)^{-s+1}}{-s+1} - \frac{1}{-s+1} + \frac{(1+n)^{-s} - 1}{2} \\ &- \sum_{k=2}^{m} \frac{B_k}{k!} \prod_{p=0}^{k-2} (s+p) \left[\frac{1}{(n+1)^{s+k-1}} - 1 \right] \\ &- \frac{1}{m!} \int_0^n \prod_{p=0}^{m-1} (s+p) \frac{1}{(1+x)^{s+m}} P_m(x) dx \end{aligned} \tag{47}$$

From Equation(43) one has:

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{(k+1)^s} - \frac{1}{1-s}(1+n)^{1-s} \right) = \zeta(s) - 1 \tag{48}$$

hence

$$\sum_{k=1}^{n} \frac{1}{(k+1)^s} = Z_n(s) + \frac{1}{-s+1}(1+n)^{1-s} \tag{49}$$

Using both equations one can obtain, denoting $R_{m,n} = \frac{1}{m!}\int_0^n \prod_{p=0}^{m-1}(s+p)\frac{1}{(1+x)^{s+m}}P_m(x)dx$

$$\begin{aligned} Z_n(s) + \frac{1}{-s+1}(1+n)^{1-s} &= \frac{(n+1)^{-s+1}}{-s+1} - \frac{1}{-s+1} + \frac{(1+n)^{-s}-1}{2} \\ &- \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p)\left[\frac{1}{(n+1)^{s+k-1}} - 1\right] \\ &- R_{m,n} \end{aligned} \tag{50}$$

therefore

$$\begin{aligned} Z_n(s) &= -\frac{1}{-s+1} + \frac{(1+n)^{-s}-1}{2} \\ &- \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p)\left[\frac{1}{(n+1)^{s+k-1}} - 1\right] \\ &- R_{m,n} \end{aligned} \tag{51}$$

Letting $n \to \infty$ in Equation (51), because $\exists \lim_{n\to\infty} Z_n(s) = \zeta(s) - 1$ and $\exists \lim_{n\to\infty} R_{m,n}(s) = R_{m,\infty}(s)$ it follows that $\exists$

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p)\left[\frac{1}{(n+1)^{s+k-1}} - 1\right] \\ &= \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p)\left[\lim_{n\to\infty}\frac{1}{(n+1)^{s+k-1}} - 1\right] \\ &= -\sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p) \end{aligned} \tag{52}$$

and

$$\begin{aligned} \zeta(s) - 1 &= \lim_{n\to\infty} Z_n(s) \\ &= -\frac{1}{-s+1} - \frac{1}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p) - R_{m,\infty}(s) \end{aligned} \tag{53}$$

therefore

$$\zeta(s) = \frac{1}{2} - \frac{1}{-s+1} + \sum_{k=2}^{m} \frac{B_k}{k!}\prod_{p=0}^{k-2}(s+p) - R_{m,\infty}(s) \tag{54}$$

□

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