RING PROPERTIES PRESERVED OR REFLECTED BY SQUARES

GRIGORE CĂLUGĂREANU

Abstract. This study explores the relationship between various properties of elements in a ring and their squares. The properties examined include idempotent, nilpotent, unit, regular, quasi-regular, nil-clean, clean, and fine. Finally, we characterize the rings where the squares of all elements are idempotent.

1. Introduction

In [1], the integral matrix $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ was presented as an example of (uniquely) nil-clean element that is not clean. Recently, Yiqiang Zhou discovered that A^2 is clean (private communication). More specifically, using any standard binary quadratic equation solver, such as [7], we show that A^2 is not nil-clean but is clean of index 13 (details are provided in the Appendix). We recall the following well-known definitions.

An element a of a ring R is: *quasi-regular* if $1 - a$ is a unit, *nil-clean* if it is a sum of an idempotent and a nilpotent, *clean* if it is a sum of an idempotent and a unit, *fine* if it is a sum of a unit and a nilpotent. A nil-clean (or clean, or fine) element is called *strongly* nil-clean (resp. clean, or fine) , if the components of the sum commute.

Let R be a ring and define the squaring function $s : R \to R$ by $s(r) = r^2$. We say that a ring property $\mathcal{P}$ is *preserved* by squaring if, for any $r \in R$, $s(r)$ has $\mathcal{P}$ whenever r has it, and, we say that $\mathcal{P}$ is *reflected* by squaring if, for any $r \in R$, r has $\mathcal{P}$ whenever $s(r)$ has it.

In this exposition we examine the main ring theoretic properties that elements of a ring may have, in relationship to squaring (i.e., preservation or reflection of properties). The properties we address include: idempotent, nilpotent, unit, von Neumann regular, quasi-regular, clean, nil-clean, fine.

Summarizing, nilpotents and units are both preserved and reflected by squaring, idempotents are preserved by squaring, but are generally not reflected. Regular elements are generally neither preserved nor reflected by squaring and quasi-regular elements are generally not preserved, but are reflected by squaring. Nil-clean, clean and fine elements are generally neither preserved nor reflected by squaring.

Finally, we characterize the rings where the squares of all elements are idempotent.

We denote by E_{ij}, the $n \times n$ matrix with all entries zero except for the (i, j)-entry, which is 1, by $U(R)$, the set of all units of a ring R and by $N(R)$, the set of all nilpotents of R.

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2. Nilpotent, unit, idempotent

Clearly, both nilpotents and units are preserved and reflected by squaring.

Idempotents are obviously preserved by squaring but are generally not reflected in this way. Simple examples are the nonzero "minus idempotents", i.e., elements e such that $e^2 = -e$. These elements are not idempotent but their squares are idempotent. To illustrate this further, consider matrices over any unital ring with $2 \neq 0$. Take the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$. This matrix is not idempotent, but squaring it gives $A^2 = -A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$, which is idempotent.

3. Regular, quasi-regular

Von Neumann regular elements behave poorly under squaring: in general, regular elements are neither preserved nor reflected by squaring. Since our focus is on providing 2×2 (integral) matrix examples, we first make some observations about regular 2×2 matrices over commutative domains.

Let R be a commutative domain and let $A \in \mathbb{M}_n(R)$. Suppose $A = AXA$, for some matrix X. Taking determinants on both sides, we get $\det(A)(\det(AX) - 1) = 0$. This implies either $\det(A) = 0$ or $\det(AX) = 1$ (and also $\det(XA) = 1$). Hence, if $\det(A) \neq 0$, both AX and XA must be units. Since the matrix ring is Dedekind finite, it follows that both A, X are units. Therefore, regular matrices must either be invertible or have a zero determinant.

Next, we define elements a, b, c, d in a ring to be *coprime* (or, equivalently, the row vector $\begin{bmatrix} a & b & c & d \end{bmatrix}$ is *unimodular*) if there exist elements x, y, z, t such that $ax + cy + bz + dt = 1$.

With this, we can now provide an elementary proof for the following characterization.

Theorem 3.1. *Let R be a commutative domain. A nonzero 2×2 matrix with zero determinant is (von Neumann) regular iff its nonzero entries are coprime*

Proof. Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0_2$ with $ad = bc$ and $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AXA = A$ amounts to a (nonhomogeneous) system, namely

$$\begin{array}{c} a^2x + acy + abz + bct = a \\ abx + ady + b^2z + bdt = b \\ acx + c^2y + adz + cdt = c \\ bcx + cdy + bdz + d^2 = d \end{array}.$$

Since $ad = bc$, the system reduces to

$$\begin{array}{c} a(ax + cy + bz + dt) = a \\ b(ax + cy + bz + dt) = b \\ c(ax + cy + bz + dt) = c \\ d(ax + cy + bz + dt) = d \end{array}.$$

If any of a, b, c, d is zero, the corresponding equality holds for any x, y, z, t.

Since we have assumed $A \neq 0_2$, at least one entry (say a) is nonzero. Cancelling a in the first equation, we get $ax + cy + bz + dt = 1$, which holds iff a, b, c, d are coprime. $\square$

In terms of *preservation*, consider the zero determinant regular integral matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} A$. However, by Cayley-Hamilton's theorem, the square $A^2 = 5A$ is not regular, since its entries are not coprime.

Regarding *reflection*, it is easy to verify that in $\mathbb{Z}_{12}$, $4 = 2^2$ is regular, as it has four inner inverses. On the other hand, 2 is not regular.

It is also important to note that the 2×2 matrices over commutative domains, cannot be given as nonexamples for reflection. As discussed in the remarks preceding the above theorem, a regular matrix is either a unit or a zero determinant matrix with coprime entries. Clearly, in both cases, if A^2 is regular, so is A. Therefore, for a valid matrix example, we require zero divisors.

An easy computation (which reduces to $4 \nmid 2$) shows that the upper triangular matrix $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not regular in $\mathbb{T}_2(\mathbb{Z}_{12})$. However, $B^2 = 4 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is even unit-regular, as it is the product of an idempotent matrix and a unit.

Regarding the *quasiregular* property, it is well-known (and easy to check, as $1 - a^2 = (1-a)(1+a)$) that if a^2 is quasiregular, then a is quasiregular. In other words, quasiregular elements are *reflected* by squaring.

However, quasiregular elements are generally *not preserved* by squaring. For instance, consider the example above over any unital ring with $3 \notin U(R)$. The matrix $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = I_2 + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is quasiregular. However, $B^2 = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} = I_2 + \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$ is not quasiregular.

Remark. Since an element a is quasiregular iff $a = 1 + u$ for some unit u, it follows that $a^2 = 1 + (2+u)u$ is also quasiregular iff $2 + u \in U(R)$.

4. Squares of nil-clean elements

Regarding *preservation*, since the nil-clean elements are sums of the form $e + t$ where $e^2 = e$ and $t \in N(R)$, it follows that, in particular when $t = 0$ or $e = 0$, *the idempotents and the nilpotents* have idempotent or nilpotent squares, respectively, and so *have nil-clean squares.* Furthermore, if $e = 1$ then $e + t = 1 + t$ is unipotent. Since the square of a unipotent is also unipotent, it follows that *unipotents also have nil-clean squares.*

Thus, the only case left to verify is the preservation for sums of the form $e + t$, where e is a nontrivial idempotent and t is a nonzero nilpotent. Elements of this form are referred to as *nontrivial* nil-clean.

A ring is termed a *GCD* ring if greatest common divisors exist.

For 2×2 matrices over commutative domains we have the following result.

Theorem 4.1. *Let A be a 2×2 nontrivial nil-clean matrix A over a commutative domain D. If A^2 is also nil-clean then A is idempotent or else, the characteristics of D is 2. If every nontrivial idempotent 2×2 matrix is similar to E_{11} (in particular, if D is a GCD domain) then the converse also holds.*

Proof. Over commutative domains, every nontrivial nil-clean 2×2 matrix has trace $= 1$. Hence it has the form $A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ and the square is $A^2 =$

$\begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix}$. A necessary condition for the square to be nil-clean is that the trace $2(a^2-a+bc)+1$ equals 1. This holds iff $a(1-a)=bc$ or $\text{char}(D)=2$. In the first case, $\det(A)=0$, so A is an idempotent, so trivially nil-clean. Thus, for the converse, it remains to show that over a domain of characteristics two, if the nontrivial idempotent 2×2 matrices are similar to E_{11}, the square of a nontrivial nil-clean 2×2 matrix is also (nontrivial) nil-clean. Since the nil-clean property is invariant under similarity, we can start with $A=E_{11}+T$ with $T^2=0_2$ (i.e., $Tr(T)=\det(T)=0$) and so $T=\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2+yz=0$. Then $A^2=E_{11}+E_{11}T+TE_{11}=\begin{bmatrix} 1+2x & y \\ z & 0 \end{bmatrix}=\begin{bmatrix} 1 & y \\ z & 0 \end{bmatrix}=\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}$ is a (nontrivial) nil-clean decomposition ($2x=0$ as $\text{char}(D)=2$). □

Using this, we can easily provide examples which show that the nil-clean property is, in general, *not preserved* by squaring.

Example. Over any domain with $2\neq 0$, consider the unit $U=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then U is nil-clean. However, when squaring, $U^2=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is not nil-clean as its trace is not equal to 1.

To further show that the nil-clean property is also *not reflected* by squaring, we refer to the example given in section 2. Indeed, the matrix $A=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ has trace $=-1$, so is not nil-clean, but $A^2=\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ is idempotent and hence (trivially) nil-clean.

By easy computation we have

Lemma 4.2. *Let A be a 2×2 matrix over any commutative ring. Then $Tr(A^2)=Tr^2(A)-2\det(A)$.*

Corollary 4.3. *Let A be a 2×2 matrix over any commutative ring. Then $Tr(A^2)=1$ iff $Tr^2(A)=1+2\det(A)$.*

Furthermore, one might wonder whether the squares of matrices that are not nil-clean could be nilpotent. The answer to this question is negative.

Proposition 4.4. *Over commutative domains, matrices that are not nil-clean, have no nilpotent squares.*

Proof. By denial, in order to have $Tr(A^2)=\det(A^2)=0$, and so $\det(A)=0$, using Lemma 4.2, we get also $Tr(A)=0$. Hence A is nilpotent and so (trivially) nil-clean. □

In the reminder of this section, we will discuss a straightforward positive result regarding the *preservation* of the nil-clean property.

Proposition 4.5. *Let $e^2=e$ and $t\in N(R)$. If $e+te+et$ is idempotent or $te+et+t^2$ is nilpotent, then $(e+t)^2$ is nil-clean.*

To simplify the writing and wording we introduce the following

Definition. The pair (e,t) with $e^2=e$ and $t\in N(R)$ is called *i-pre-nil-clean* if $e+te+et$ is an idempotent.

Obviously, for any nilpotent t, the pair $(0,t)$ is i-pre-nil-clean so in our study we focus on $e\neq 0$.

We first show that i-pre-nil-cleanness is *invariant under conjugation.*

Lemma 4.6. *If (e,t) is i-pre-nil-clean and $v\in U(R)$ then (vev^{-1},vtv^{-1}) is also i-pre-nil-clean.*

Proof. If $e+te+et$ is idempotent, so is $v(e+te+et)v^{-1}=vev^{-1}+vev^{-1}vtv^{-1}+vtv^{-1}vev^{-1}$. □

Secondly, we exclude the second trivial case.

Proposition 4.7. *The pair $(1,t)$ is i-pre-nil-clean iff $2t=0$.*

Proof. Indeed, $(1+2t)^2=1+2t$ amounts to $1+2t=1$ (by multiplication with $(1+2t)^{-1}$) and so $2t=0$. □

Corollary 4.8. *If R is a domain, $(1,t)$ is i-pre-nil-clean iff $2=0$ and t is an arbitrary nilpotent, or else $2t=0$. In particular, if $R=\mathbb{M}_2(\mathbb{Z})$ there are no i-pre-nil-clean pairs (I_2,T) with $T\neq 0_2$.*

Furthermore, we can prove more general results regarding the preservation of the nil-clean property.

Definition. A ring R will be called *IE*, if the 2×2 nontrivial idempotent matrices over R are similar to the matrix E_{11}. Examples of IE rings include GCD domains and in particular Bézout domains and PIDs.

Theorem 4.9. *Let R be an IE commutative domain with $2\neq 0$. The only i-pre-nil-clean pairs (E,T) with idempotent $E\neq I_2$ in $\mathbb{M}_2(R)$ have $E=0_2$ and an arbitrary nilpotent matrix T.*

Proof. According to Lemma 4.6, over an IE domain, for nontrivial idempotents E, it suffices to show that there are no i-pre-nil-clean pairs (E_{11},T). These would correspond to all pairs with nontrivial idempotent E. Denote $T=[t_{ij}]$, $1\leq i,j\leq 2$. Then $\Sigma=E_{11}+E_{11}T+TE_{11}=\begin{bmatrix}1+2t_{11} & t_{12}\\ t_{21} & 0\end{bmatrix}$ has trace $=1$ iff $t_{11}=0$, and determinant $=0$ iff at least one of t_{12}, t_{21} is zero. In this case T has a zero first row (or a zero first column), so is idempotent iff $t_{22}=1$. But then T has trace 1, so is not nilpotent □

Remarks. 1) If $2=0$ then $\Sigma=\begin{bmatrix}1 & t_{12}\\ t_{21} & 0\end{bmatrix}$, so (E_{11},T) is i-pre-nil-clean if $t_{12}=0$ or $t_{21}=0$. Since also $t_{11}=0$ it follows that $T\in\{E_{12},E_{21}\}$.

2) If (E,T) is an i-pre-nil-clean pair so is (E^t,T^t), the pair of transposes.

Returning to the second case in Proposition 4.5, we introduce the following

Definition. The pair (e,t) with $e^2=e$ and $t\in N(R)$ is called *n-pre-nil-clean* if $te+et+t^2$ is nilpotent. Clearly, for any nilpotent t, the pair $(0,t)$ is n-pre-nil-clean. Therefore, in our study, we focus on cases where $e\neq 0$. Unlike i-pre-nil-clean pairs, even in $\mathbb{M}_2(\mathbb{Z})$, *n-pre-nil-clean pairs are abundant.*

A similar computation to the one in Lemma 4.6 shows that the n-pre-nil-clean pairs are also *invariant under conjugations*. Therefore, over any IE domain, for non-trivial idempotent matrices, it suffices to determine the nilpotents T such that (E_{11}, T) is an n-pre-nil-clean pair.

Note that if (E, T) is an n-pre-nil-clean pair, so is (E^t, U^t), the pair of transposes.

Theorem 4.10. *Let R be any commutative domain. For a 2×2 nilpotent $T = [t_{ij}]$ over R, (E_{11}, T) is an n-pre-nil-clean pair iff $t_{11} = 0$, or else $2t_{11} = 0$ and $t_{22} = 0$.*

Proof. Denote $T = [t_{ij}]$, $1 \leq i, j \leq 2$ such that $\det(T) = Tr(T) = 0$ (i.e. $t_{11}t_{22} - t_{12}t_{21} = 0 = t_{11} + t_{22}$). Then

$$\Sigma = E_{11}T + TE_{11} + T^2 = \begin{bmatrix} t_{11}^2 + t_{11}t_{22} + 2t_{11} & t_{12}(t_{11} + t_{22} + 1) \\ t_{21}(t_{11} + t_{22} + 1) & t_{22}^2 + t_{11}t_{22} \end{bmatrix} = \begin{bmatrix} 2t_{11} & t_{12} \\ t_{21} & 0 \end{bmatrix}$$

and by computation $Tr(\Sigma) = 2t_{11}$ and $\det(\Sigma) = -t_{11}t_{22}$. □

Corollary 4.11. *Over $\mathbb{Z}$ there are precisely two n-pre-nil-clean pairs with idempotent component E_{11}: (E_{11}, E_{12}) and (E_{11}, E_{21}).*

Proof. Over the integers $t_{11} = 0$ and so $t_{22} = 0$. Hence also $t_{12} = 0$ or $t_{21} = 0$. □

Remark. By conjugation, we obtain infinitely many such integral n-pre-nil-clean pairs.

5. Squares of clean elements

First, we discuss the trivial clean elements.

Since the clean elements are sums $e + u$ with $e^2 = e$ and $u \in U(R)$, in particular (if $e = 0$), *the units* have unit squares and so *have clean squares*.

If $e = 1$, we have sums $1 + u$ which are known as the *quasiregular* elements of a ring. The preservation and reflection of quasiregular elements was already discussed in Section 3.

Since reflections of quasiregular elements are also quasiregular, these elements are also clean.

Returning to the example of quasiregular 2×2 matrix $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ not preserved by squaring, note that $B^2 = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ is neither quasiregular (if 3 is not a unit) nor clean (if 4 is not a unit). To see this we recall the following characterization.

Theorem 5.1. *Let R be a commutative domain and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a matrix over R. Then A is nontrivial clean iff at least one of the systems*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a - d)x + cy + bz + \det(A) - d = \pm 1 & (\pm 2) \end{cases}$$

in unknowns x, y, z is solvable over R. If $b \neq 0$ and any of ($\pm$2) holds, then (1) is equivalent to

$$bx^2 - (a - d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0 \quad (\pm 3).$$

The signs in the equations correspond accordingly.

For B^2, the equations (± 2) are $4(z+3) = \pm 1$, with no solutions if 4 is not a unit.

This shows that clean elements are (generally) *not preserved* by squaring. The example actually shows more: the square of a quasiregular element may not be clean.

For matrices we have the following result.

Theorem 5.2. *Let R be a commutative ring and $U \in U(\mathbb{M}_2(R))$. The square of $I_2 + U$ is also clean iff $\det(U) + 2(Tr(U) + 2)$ is a unit.*

Proof. We use the final remark in Section 3 and the fact that $\det(U + 2I_2) = \det(U) + 2Tr(U) + 4$. □

For a clean element $e + u$, if $u = 1$, we have sums $1 + e$ with square $1 + 3e$. This square has the same form if 3 is idempotent but not only in this case. Without $3e$ being idempotent, $1 + 3e$ still can be clean.

Actually, we can prove the following result.

Proposition 5.3. *Let E be a nontrivial 2×2 idempotent matrix over a commutative domain. Then $I_2 + 3E$ is clean iff 3 is a unit.*

Proof. Start with a nontrivial idempotent $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ such that $a(1-a) = bc$. Then $I_2 + 3E = \begin{bmatrix} 3a+1 & 3b \\ 3c & 4-3a \end{bmatrix}$ has determinant $= 4$ and the equations (± 2) in the previous theorem become

$$3(2a-1)x + 3cy + 3bz + 4 - 4 + 3a = \pm 1,$$

with no solutions iff $3 \nmid 1$ (i.e., 3 is not a unit). Conversely, if 3 is a unit then $(x, y) = (0, 0)$ is a solution for both systems. □

Therefore, we can give the following

Example. Over $\mathbb{Z}_4$ take $E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $3E$ is not idempotent ($(3E)^2 = E \neq 3E$) but $I_2 + 3E = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$ is idempotent, so clean.

Discarding these exceptions, we are left with squares of clean elements, which are sums of a nontrivial idempotent and a unit $\neq 1$.

Analogous to the previous section, we highlight a straightforward positive result regarding the *preservation* of clean elements.

Proposition 5.4. *Let $e^2 = e$ and $u \in U(R)$. If $e + ue + eu$ is idempotent or $ue + eu + u^2$ is a unit, then $(e + u)^2$ is clean.*

To simplify the writing and wording we introduce the following

Definition. The pair (e, u) with $e^2 = e$ and $u \in U(R)$ is called *i-pre-clean* if $e+ue+eu$ is an idempotent. Obviously, for any unit u, the pair $(0, u)$ is i-pre-clean so in our study we focus on $e \neq 0$. A simple computation shows that i-pre-cleanness is *invariant under conjugation.*

As for the second trivial case, the pair $(1, u)$ is i-pre-clean iff $2(1 + 2u) = 0$. Therefore

Proposition 5.5. *If R is a domain, $(1,u)$ is i-pre-clean iff $2=0$ and u is an arbitrary unit, or else $1+2u=0$, which gives only the pair $(1,-2^{-1})$ if 2 is a unit. In particular, if $R=\mathbb{M}_2(\mathbb{Z})$ there are no i-pre-clean pairs (I_2,U).*

Further, over IE rings, we can prove more general results concerning preservation. The following result shows that (nontrivial) i-pre-clean 2×2 matrices are relatively rare.

Theorem 5.6. *Let R be an IE commutative domain with $2\neq 0$. The only i-pre-clean pairs (E,U) with $E\neq I_2$ in $\mathbb{M}_2(R)$ have $E=0_2$ and an arbitrary invertible matrix U.*

Proof. By invariance under conjugation, over an IE domain it suffices to show that there are no i-pre-clean pairs (E_{11},U). These would correspond to all pairs with nontrivial idempotent E. Denote $U=[u_{ij}]$, $1\leq i,j\leq 2$. Then $\Sigma=E_{11}+E_{11}U+UE_{11}=\begin{bmatrix} 1+2u_{11} & u_{12} \\ u_{21} & 0 \end{bmatrix}$ has trace $=1$ iff $u_{11}=0$, and determinant $=0$ iff at least one of u_{12}, u_{21} is zero. But in this case U has a zero row (or a zero column), so is not a unit. □

Remarks. 1) If $2=0$ then $\Sigma=\begin{bmatrix} 1 & u_{12} \\ u_{21} & 0 \end{bmatrix}$, so the pair (E_{11},U) is i-pre-clean for every lower (or upper) triangular unit U. Over $\mathbb{Z}_2$, $U=I_2$ or $U=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are examples of this sort.

2) If (E,U) is an i-pre-clean pair, so is (E^T,U^T), the pair of transposes.

Returning to the second case in Proposition 5.4, we introduce the following

Definition. The pair (e,u) with $e^2=e$ and $u\in U(R)$ is called *u-pre-clean* if $ue+eu+u^2$ is a unit. Clearly, for any unit u, the pair $(0,u)$ is u-pre-clean so in our study we focus on $e\neq 0$.

Different from i-pre-clean pairs, *u-pre-clean pairs are abundant*, even in $\mathbb{M}_2(\mathbb{Z})$.

A simple computation shows that also the u-pre-clean pairs are *invariant under conjugations*. Thus, over any IE domain, for the non-trivial idempotent matrices it suffices to determine the units U such that (E_{11},U) is a u-pre-clean pair.

Note that if (E,U) is an u-pre-clean pair, so is (E^t,U^t), the pair of transposes.

Theorem 5.7. *Let R be any commutative domain. For a 2×2 unit $U=[u_{ij}]$ over R (that is, $\det(U)=1$), (E_{11},U) is a u-pre-clean pair iff $(u_{11}-2)u_{22}=1$.*

Proof. Denote $U=[u_{ij}]$, $1\leq i,j\leq 2$ such that $\det(U)=1$ (i.e. $u_{11}u_{22}-u_{12}u_{21}=1$). Then

$$\Sigma=E_{11}U+UE_{11}+U^2=\begin{bmatrix} u_{11}^2+u_{11}u_{22}+2u_{11}-1 & u_{12}(u_{11}+u_{22}+1) \\ u_{21}(u_{11}+u_{22}+1) & u_{22}^2+u_{11}u_{22}-1 \end{bmatrix}$$

and by computation $\det(\Sigma)=2(u_{22}+1)-u_{11}u_{22}=1$ iff $(u_{11}-2)u_{22}=1$ and $u_{12}u_{21}=u_{11}u_{22}-1$. □

Corollary 5.8. *Over $\mathbb{Z}$ there are precisely eight u-pre-clean pairs (E_{11},U).*

Proof. The equation $(u_{11}-2)u_{22}=1$ has only two solutions over the integers given by $u_{11}=1$ and $u_{11}=3$ with $u_{22}=-1$ and $u_{22}=1$, respectively. The corresponding entries u_{12}, u_{21} satisfy $u_{12}u_{21}=-2$ and $u_{12}u_{21}=2$, respectively. Therefore the possible units U are $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$ and their transposes. □

Remark. By conjugation, we obtain infinitely many such integral u-pre-clean pairs.

An element that is not clean but whose square is clean is $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$, as observed by Y. Zhou (mentioned in the Introduction). This shows that clean elements are *not reflected* by squaring.

As in the previous subsection, we check whether matrices that are not clean may have *some particular clean squares, that is, idempotents or units.*

Since, if x^2 is a unit, so is x, we focus on the idempotent squares of matrices over commutative domains.

Proposition 5.9. *Over commutative domains, matrices that are not clean have no idempotent squares.*

Proof. As $\det(A^2) = 0$ requires $\det(A) = 0$, using Corollary 4.3, $Tr(A^2) = 1$ requires $Tr(A) \in \{\pm 1\}$. Since if $Tr(A) = 1$, A is idempotent, so clean, we focus on finding a matrix that is not clean with zero determinant and trace $= -1$. This is not possible. Indeed, by Cayley-Hamilton's theorem, $A^2 + A = 0$ so A is the negative of a (nonzero) idempotent, so not idempotent. □

6. Squares of fine elements

Since the fine elements are nonzero sums of the form $t + u$ where $t \in N(R)$ and $u \in U(R)$, in particular (if $t = 0$), *the units* have unit squares and thus *have fine squares.* If $u = 1$ then $t + 1$ is unipotent, with unipotent square, so *unipotents have fine squares.*

Now, we turn our attention to the so-called nontrivial fine elements, where $t \neq 0$ and $u \neq 1$. As in the nil-clean and clean cases, the fineness property is generally *neither preserved* nor reflected by squaring. Various examples illustrating this behavior can be found in [2].

It is well-known that 0 is not fine, as this is ruled out by definition - nilpotents cannot be units. Consider the nilpotent matrix $E_{12} \in \mathbb{M}_2(R)$ for any unital ring R. Notably, we can express E_{12} as $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which shows that E_{12} is fine. However, its square $E_{12}^2 = 0_2$, is not fine.

A less trivial example follows. Before presenting it, we first prove a characterization which generalizes equation 5.10 in Example 5.9, [2].

Theorem 6.1. *For a* 2×2 *integral matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *denote* $l := -\det(A) \pm 1$. *Then* A *is fine iff*

(i) at least one of the systems $cx + by = l$, $s^2 + xy = 0$ *in unknowns* x, y, s *has integer solutions, whenever* $a = d$, *or*

(ii) at least one of the (quadratic) Diophantine equations

$$c^2x^2 + [(a-d)^2 + 2bc]xy + b^2y^2 - 2clx - 2bly + l^2 = 0$$

in unknowns x, y *has integer solutions such that* $-xy$ *is a square, whenever* $a \neq d$.

Proof. Since nilpotents in $\mathbb{M}_2(\mathbb{Z})$ have the form $\begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ with $s^2 + xy = 0$, A is fine iff $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) = \pm 1$. This condition can be written $s(a-d) = -cx - by + l$. If $a = d$ we get (i) and if $a \neq d$, squaring and eliminating s, we obtain the quadratic Diophantine equation in the statement. Observe that $-s(a-d) = -cx - by + l$ is also suitable since $(-s)^2 + xy = 0$ and so the final step consists of the choice between s and $-s$ (in order to have $s(a-d) = -cx - by + l$). □

Remarks. 1) In the case (i), whenever at least one of $-a^2 \pm 1$ is divisible by b or c, the system is solvable with $x = 0$ or $y = 0$, respectively. In both cases, $s = 0$. To see this, recall that we have to solve $cx + by = bc - a^2 \pm 1$.

2) Also in case (i), if $\det(A) = 0$ then $l = \pm 1$ and so $cx + by = \pm 1$ is solvable only if b, c are coprime. As $a^2 = bc$, both b, c must be (coprime) squares (and so $a = \pm\sqrt{bc}$).

Example. $A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ gives $9x + 4y = \pm 1$. For $+1$ the general solution is $x = 1 + 4n$, $y = -2 - 9n$. Here $-xy = (4n+1)(9n+2) = 36n^2 + 17n + 2$ so we are searching for an n to have a square $-xy = s^2$. However, there is no integer n for such a square ($36n^2 + 17n + 2 = s^2$ is a quadratic Diophantine equation, without integer solutions). So A is not fine.

For zero determinant 2×2 matrices (with $a = d$) we can prove the following result.

Proposition 6.2. *Let* $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ *be a zero determinant matrix over a commutative domain, i.e.,* $a^2 = bc$. *If* $2a$ *is not a unit (that is, both* 2 *and* a *are not units) then* A^2 *is not fine.*

Proof. As in case (i), if $\det(A) = 0$ then also $\det(A^2) = 0$ so the $l = \pm 1$ is common for both linear Diophantine equations. For $A^2 = \begin{bmatrix} a^2 + bc & 2ab \\ 2ac & a^2 + bc \end{bmatrix}$, which is also in case (i), the corresponding equation is $2acx + 2aby = 1$, with no solutions if $2a$ is not a unit. □

To provide another example of a fine element whose fineness property is not preserved under squaring, we first recall **Corollary 5.4** from [2].

Corollary 6.3. *A matrix* $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$ *is fine iff* $b \equiv \pm 1$ *(mod* a*).*

Using this, it follows that $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ is fine (over $\mathbb{Z}$), but its square $\begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$ is not.

Actually this example can be generalized using an easy result.

Lemma 6.4. *Let* $a, b \in \mathbb{Z}$. *If* $ab \equiv \pm 1$ *(mod* a^2*) then* $a \in \{\pm 1\}$.

Combining with the previous corollary we obtain

Proposition 6.5. *The integral matrix* $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}^2$ *is not fine whenever* $a \notin \{\pm 1\}$.

Corollary 6.6. *For all the fine matrices* $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ *with* $a \notin \{\pm 1\}$*, the square is not fine.*

Demonstrating that fineness is not reflected by squaring is more challenging. We present a 2×2 integral matrix that is not fine, yet its square is fine.

Example. Take $A = \begin{bmatrix} 1 & 7 \\ 8 & 0 \end{bmatrix}$, declared not fine in [2]. Indeed, for A, the Diophantine equations (ii) are:

$$64x^2 + 113xy + 49y^2 - 16lx - 14ly + l^2 = 0 \quad (*)$$

with $l = 56 \pm 1$, and both have no integer solutions, so A is not fine. More precisely the equations are

$$64x^2 + 113xy + 49y^2 - 880x - 770y + 3025 = 0$$

and

$$64x^2 + 113xy + 49y^2 - 912x - 798y + 3249 = 0.$$

For $A^2 = \begin{bmatrix} 57 & 7 \\ 8 & 56 \end{bmatrix}$, written in l, the Diophantine equations (ii) are the same as (*), but with a different $l = -3136 \pm 1$.

For $l = -3135$ we have $64x^2 + 113xy + 49y^2 + 50160x + 43890y + 9828225 = 0$, with no integer solutions, and

for $l = -3137$ we have $64x^2+113xy+49y^2+50192x+43918y+9840769 = 0$. The second equation has an integer solution: $(x, y) = (-2143296, 2798929)$, for which (as desired) the product $-xy$ is a square, that is $s = \pm 2449272$. The final step is to choose s or $-s$ (because of the squaring in the proof of Theorem 6.1).

As $s(a - d) \neq -cx - by + l$ but $-s(a - d) = -cx - by + l$, we have to choose $s = -2449272$.

This gives a huge fine decomposition

$$\begin{bmatrix} 57 & 7 \\ 8 & 56 \end{bmatrix} = \begin{bmatrix} -2449272 & -2143296 \\ 2798929 & 2449272 \end{bmatrix} + \begin{bmatrix} 2449329 & 2143303 \\ -2798921 & -2449216 \end{bmatrix},$$

since the LHS is idempotent and the determinant of the RHS is $= -1$ (we used [8], for computation).

Analogous with the previous two sections, we can consider the following definitions for any unital ring R.

The pair (t, u) with $t \in N(R)$ and $u \in U(R)$ is called *n-pre-fine* if $t + ut + tu$ is nilpotent, and *u-pre-fine* if $ut + tu + u^2$ is a unit. Clearly

Proposition 6.7. *Let* $t \in N(R)$ *and* $u \in U(R)$*. If* $t + ut + tu$ *is nilpotent or* $tu + ut + u^2$ *is a unit, then* $(t + u)^2$ *is fine.*

However, without the advantage of similarity to E_{11} for nontrivial idempotent 2×2 matrices, determining such pairs becomes more challenging. Nevertheless, we can establish the following notable result.

Theorem 6.8. *Let* $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ *be a nilpotent matrix and let* $U = [u_{ij}]$*,* $i, j \in \{1, 2\}$ *be an invertible matrix over a commutative ring. Then the pair* (T, U) *is n-pre-fine iff* $x(u_{11} - u_{22}) + zu_{12} + yu_{21} = 0$.

Proof. By computation, using $x^2+yz=0$ and $\det(U)=1$, for $\Sigma=T+TU+UT$, we get $\det(\Sigma)=\alpha^2$ and $Tr(\Sigma)=2\alpha$ where $\alpha:=x(u_{11}-u_{22})+zu_{12}+yu_{21}$. □

Example. While obviously (T,I_2) is n-pre-fine for every nilpotent matrix T, for given $T=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$ and given $u_{21}=1$, there are only two n-pre-fine pairs (T,U) over the integers: $\left(T,\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\right)$ and $\left(T,\begin{bmatrix}-2 & -1\\ 1 & 0\end{bmatrix}\right)$.

As for u-pre-fine pairs we have the following result.

Theorem 6.9. *Let* $T=\begin{bmatrix}x & y\\ z & -x\end{bmatrix}$ *be a nilpotent matrix and let* $U=[u_{ij}]$, $i,j\in\{1,2\}$ *be an invertible matrix over a commutative ring. Then the pair* (T,U) *is u-pre-fine iff* $[x(u_{11}-u_{22})+zu_{12}+yu_{21}-1]^2=1$.

Proof. By computation, using $x^2+yz=0$ and $\det(U)=1$, for $\Sigma_1=TU+UT+U^2$, $\det(\Sigma_1)=1$. By computation, $\det(\Sigma)=(\alpha-1)^2$ with the same $\alpha:=x(u_{11}-u_{22})+zu_{12}+yu_{21}$, as in the previous theorem. □

Example. While obviously $(0_2,U)$ are u-pre-fine for every unit U, for given $T=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$ and given $u_{21}=1$, there are only two n-pre-fine pairs (T,U) over the integers: $\left(T,\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)$ and $\left(T,\begin{bmatrix}-2 & -5\\ 1 & 2\end{bmatrix}\right)$.

7. Rings with idempotent squares

Another approach to addressing the topic of this exposition is to examine the rings in which all squares exhibit a particular property.

Since units and nilpotents are preserved and reflected by squaring, the rings in which only unit squares (excluding 0) exist are precisely the division rings, while the rings in which only nilpotent squares exist are the nil rings. In the remaining of this section, we focus on *the rings in which all squares are idempotent.*

To simplify the discussion, we define a ring as *SI* if all of its squares are idempotent. Formally, a ring R is SI iff $R^2=Id(R)$, meaning that for every $r\in R$, we have $r^4=r^2$.

The following straightforward result will be helpful.

Proposition 7.1. *A direct product of rings is SI iff all its components are SI. Any factor ring of a SI ring is SI.*

As a trivial example, Boolean rings are SI. An example which is not Boolean is $\mathbb{Z}_{12}$, as $\mathbb{Z}_{12}^2=\{0,1,4,9\}=Id(\mathbb{Z}_{12})$. Clearly, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are not SI.

Some additional examples are provided in the following result, where the direct proofs are straightforward. A ring is said to be *connected* if it contains only the trivial idempotents.

Proposition 7.2. *(i) A connected ring is SI iff it is local, has only zerosquare nilpotents and only order two units.*

(ii) A domain is SI iff it is a division ring of exponent 2. In particular, the only domains $\mathbb{Z}_n$ *that are SI are the fields* $\mathbb{Z}_2$ *and* $\mathbb{Z}_3$.

(iii) A local ring is SI iff it has only zerosquare nilpotents and only order two units.

(iv) Let p be an odd prime number. Then $\mathbb{Z}_{2^s}$ is SI iff $s \in \{1, 2\}$ and $\mathbb{Z}_{p^s}$ is SI iff $s = 1$.

(v) Let $n \geq 2$ be a positive integer. The ring $\mathbb{Z}_n$ is SI iff $n \in \{6, 12\}$.

Proof. (v) If a ring is SI, it has units of order at most two. Suppose $n = 2^s p_1^{r_1} ... p_k^{r_k}$. It is well known that, including the identity, $\mathbb{Z}_n$ has 2^k order 2 units if $0 \leq s \leq 1$, 2^{k+1} order 2 units if $s = 2$ and 2^{k+2} order 2 units if $s \geq 3$. Since the total number of units is given by the Euler's totient function $\phi(n) = 2^{s-1} p_1^{r_1 - i}(p_1 - 1) ... p_k^{r_k - 1}(p_k - 1)$, it follows that $\mathbb{Z}_n$ has units of order greater than 2 iff $k \geq 2$ ($\phi(n) \geq 2^{s+1}$ for $k \geq 2$). Therefore, we have units of order at most 2 only for $n = 2^s p^t$ for some odd prime p. More precisely, this holds iff $p = 3$ and $t = 1$. □

Remarks. 1) For $s = 1$ we have $\mathbb{Z}_6^2 = \{0, 1, 3, 4\} = Id(\mathbb{Z}_6)$ and the case $s = 2$ (i.e., $n = 12$) was mentioned above.

2) The ring $\mathbb{Z}_n$ with $n = 2^s \cdot 3$ has only 4 idempotents but more than 4 elements in $\mathbb{Z}_n^2$ if $s > 2$.

3) If s is odd, $Id(\mathbb{Z}_{2^s \cdot 3)} = \{0, 1, 2^s + 1, 2^{s+1}\}$ and if s is even, $Id(\mathbb{Z}_{2^s \cdot 3)} = \{0, 1, 2^s, 2^{s+1} + 1\}$.

4) *Tripotents* (i.e., elements r satisfying $r^3 = r$) have idempotent squares, since $r^4 = r^2$ follows directly from $r^3 = r$. Moreover, if $3 = 0$ in a SI ring, the converse also holds. Replacing r with $1 + r$ in the equation $r^2 = r^4$ yields $1 + 2r + r^2 = 1 + 4r + 6r^2 + 4r^3 + r^4$ whence $r = r^3$.

The study of rings with the polynomial identity $x^4 = x^2$ for every $x \in R$, dates back over 80 years.

Alfred Foster introduced the concept of a Boolean-like ring in his 1946 paper [4]. He defined elements of a ring that satisfy $x^4 = x^2$ as *weakly idempotent*. A *Boolean-like* ring is a *commutative* ring of characteristic 2 with identity in which $(1 - a)a(1 - b)b = 0$ holds for all elements a, b of the ring. Several well-known properties of Boolean-like rings are as follows: each element is weakly idempotent (i.e., Boolean-like rings are a special class of SI rings); the nilpotent elements form an ideal; the idempotent elements form a subring; each element can be uniquely written as the sum of an idempotent element and a nilpotent element (that is, the ring is *uniquely nil-clean*).

The concept of (m, n)-*Boolean* ring ($m > n \geq 1$) was introduced by Maurer and Szigeti (see [9]) as a ring in which every element satisfies the identity $x^m = x^n$. Their paper proves that the structure of (m, n)-Boolean rings depends significantly on the parity of the difference $m - n$. If this difference is odd, a reduction theorem is established. These rings are then $(m - n + 1, 1)$-Boolean and, by Jacobson's theorem, commutative. Moreover, such rings are reduced. For cases where the difference $m - n$ is even, no such reduction theorems exist and rings satisfying the identity $x^{n+2} = x^n$, for some positive integer n, deserve special attention. Specifically, for $n = 2$, these rings are what we refer to as SI rings. For example, the ring of 2×2 upper-triangular matrices over a Boolean ring is a $(4, 2)$-Boolean ring, which is not commutative. Additionally, $\mathbb{Z}_{12}$ is a $(4, 2)$-Boolean ring that is not reduced.

In 1998, Hirano and Tominaga [5] proved that every element of a ring R is a sum of two commuting idempotents iff R satisfies the identity $x^3 = x$. The following characterization was subsequently established (actually, (iii) was added in [11]).

Theorem 7.3. *The following are equivalent for a ring R.*
(i) The ring R has the identity $x^3 = x$,
(ii) Every element of R is a sum of two commuting idempotents,
(iii) Every element of R is a difference of two commuting idempotents
(iv) R is a direct product $R = A \times B$, where A is zero or a Boolean ring and B is zero or a subdirect product of $\mathbb{Z}_3$s.

Thus, these are precisely *the rings all whose elements are tripotents.*

More recently, in [11] (see **Theorem 3.10**), a structure theorem was proved for rings which have the identity $x^6 = x^4$.

Theorem 7.4. *The following are equivalent for a ring R.*
(i) every element of R is a sum of an idempotent and a tripotent that commute,
(ii) R has the identity $x^6 = x^4$,
(iii) $R = A \times B$, where A is zero or $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is zero or a subdirect product of $\mathbb{Z}_3$s.

In the following, we provide a characterization of SI rings. Some elements of the proof have analogous results in [11], but for reader's convenience, below we present all the details here.

First, we recall a result that, for rings, dates back to [5], and for elements, to [10] (see **Proposition 3.2** and the accompanying remark).

Proposition 7.5. *An element a in a ring is strongly nil-clean iff $a - a^2$ is a nilpotent.*

Secondly, we recall from [3] the following characterization.

Theorem 7.6. *A ring R is strongly nil-clean iff $J(R)$ is nil and $R/J(R)$ is Boolean.*

We mention that in a nil-clean ring, the element 2 is a (central) nilpotent and, as such, is always contained in $J(R)$, and, that by definition, for any positive integer n, $2 \in J(R/2^nR)$.

Next, in the following lemma, we present some prerequisites necessary in the proof of the characterization theorem.

Lemma 7.7. *(i) If $R/J(R)$ is Boolean then*
(a) $2 \in J(R)$,
(b) $U(R) = 1 + J(R)$,
(c) $N(R) \subseteq J(R)$.
(ii) If R has the identity $x^3 = x$, then $R = R_1 \times R_2$, where R_1 is a Boolean ring (a subdirect product of $\mathbb{Z}_2$s) and R_2 is a subdirect product of $\mathbb{Z}_3$s.
(iii) Let $A = R/2^2R$ and $B = R/3R$. If $2^2 3 = 0$ in R, then A, B are SI rings with $2^2 = 0$ in A, $3 = 0$ in B, and $R \cong A \times B$.
(iv) If $b^4 = b^2$ and $3 = 0$ in a ring B, then B is a subdirect product of $\mathbb{Z}_3$s.
(v) If $a^4 = a^2$ and $4 = 0$ in a ring A then $A/J(A)$ is Boolean.

Proof. (i) If $R/J(R)$ is Boolean, then $r^2 - r \in J(R)$ for every $r \in R$. For (a), we take $r = 2$. For (b), let $u \in U(R)$. Then $u^2 - u \in J(R)$ and since $J(R)$ is an ideal, $u \in 1 + J(R)$. The converse is well-known (e.g., see Corollary **4.5** in [6]). For (c), let $t \in N(R)$. As $1 + t \in U(R)$, by (b) it follows that $t \in J(R)$.

(ii) Let $t \in N(R)$. As $t = t^3 = t^5 = ...$ it follows that $t = 0$, so R is reduced. By Andrunakievich-Ryabukhin theorem (e.g., see Theorem **12.7** in [6]), R is a subdirect product of domains. The only suitable domains are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (see Proposition 7.2). Hence R is a subdirect product of $\mathbb{Z}_2$s and $\mathbb{Z}_3$s.

(iii) Suppose $2^2 3 = 0$. Then $2^2 R \cap 3R = 0$ and $R = 2^2 R + 3R$. By the Chinese Remainder theorem, $R \cong R/2^2 R \times R/3R$.

(iv) This was already mentioned in Remark 4, after Proposition 7.2. That is, $b^4 = b^2$ and $3 = 0$ in a ring B imply $b = b^3$. Thus B is a subdirect product of $\mathbb{Z}_3$s.

(v) Suppose $a^4 = a^2$ and $4 = 2^2 = 0$. For any $a \in R$, as $a^4 = a^2$, we have $(a - a^2)^2 = a^2(1 - a)^2 = a^2(1 - 2a + a^2) = 2(a^2 - a^3)$, which is nilpotent as 2 is nilpotent. Thus, $a - a^2$ is nilpotent and so by Proposition 7.5, a is strongly nil-clean. Therefore, the ring R is strongly nil clean and by [3], $R/J(R)$ is Boolean. □

Now we are ready to characterize the rings all whose squares are idempotent (i.e., the SI rings).

Theorem 7.8. *The following conditions are equivalent for a ring R.*

1) $x^4 = x^2$ for all x in R.

2) R is isomorphic to A, or B, or $A \times B$, where $A/J(A)$ is Boolean and $j^2 = 2j = 0$ for all $j \in J(A)$, and B is a subdirect product of $\mathbb{Z}_3 s$.

Proof. 2) ⇒ 1). If $A/J(A)$ is Boolean then $2 \in J(A)$. As $j^2 = 0$ for every $j \in J(A)$, $J(A)$ is nil and so A is strongly nil-clean by [3].

As now $J(A) \subseteq N(A)$, by Lemma 7.7 (i) (c) it follows that $J(A) = N(A)$, so every $a \in A$ is a sum $e + j$ with $e = e^2$, $j^2 = 0$ and $ej = je$. Hence, $a^2 = e + 2ej = e$ as $2j = 0$. Thus, $a^4 = a^2$. If B is a subdirect product of $\mathbb{Z}_3$s then B has the identity $x^3 = x$, and so has also the identity $x^4 = x^2$.

1) ⇒ 2) $2^4 = 2^2$ gives $2^2 3 = 0$ in R, so by Lemma 7.7 (iv), $R = A \times B$ where $4 = 0$ in A and $3 = 0$ in B. By Lemma 7.7 (v), $A/J(A)$ is Boolean and B is a subdirect product of $\mathbb{Z}_3$s.

If $A/J(A)$ is Boolean, by Lemma 7.7 (i) (a) $2 \in J(A)$ and so by (i) (b), $3 \in U(A)$. For every $a \in A$ we have $a^4 = a^2$ and $(a+1)^4 = (a+1)^2$, whence $2a^2 = 2a$. Finally, for every $j \in J(A)$, we have $(1+j)^4 = (1+j)^2$ and since $(1+j)^2$ is a unit it follows that $(1 + j)^2 = 1$. Hence $0 = 2j + j^2 = 3j^2$, so $j^2 = 0$, and so $2j = 0$. □

8. Appendix

Zhou's discovery is particularly intriguing as it follows that for the ma trix $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$, which *is (uniquely) nil-clean but not clean* in $\mathbb{M}_2(\mathbb{Z})$, the squared matrix $A^2 = \begin{bmatrix} -54 & 9 \\ -7 & -59 \end{bmatrix}$ *is clean but not nil-clean* in $\mathbb{M}_2(\mathbb{Z})$, since $Tr(A^2) = -113 \notin \{0, 1\}$.

In order to find all the clean decompositions of A, we use Theorem 5.1. Recall that the integer solutions of the systems give the idempotent $\begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ of the clean decomposition. Since $\det(A^2) = 3249$, and $b = 9 \neq 0$, the two pairs of conditions (here $a - d = 5$ and $c = -7$) are the following.

(+3): $9x^2 - 5xy + 7y^2 + 9x + (-59 - 3249 + 1)y = 0$,
(+2): $5x - 7y + 9z + 3249 + 59 = +1$
respectively
(-3): $9x^2 - 5xy + 7y^2 + 9x + (-59 - 3249 - 1)y = 0$,
(-2): $5x - 7y + 9z + 3249 + 59 = -1$.

We solve the quadratic Diophantine equations using [7].

(+3) has the solutions: $(0, 0), (-1, 0)$ and one more $(300, 301)$.
Only $(300, 301)$ satisfies (+2), gives $z = -300$ and so yields the following clean decomposition:
$A^2 = \begin{bmatrix} -54 & 9 \\ -7 & -59 \end{bmatrix} = \begin{bmatrix} 301 & 301 \\ -300 & -300 \end{bmatrix} + \begin{bmatrix} -355 & -292 \\ 293 & 241 \end{bmatrix}$, where the LHS matrix is idempotent and the RHS unit matrix has determinant 1.

(-3) has the solutions: $(0, 0), (-1, 0)$ and another 25:
(115,522), $\underline{(272,208)}$ [357], $\underline{(104,520)}$ [21], $\underline{(-148,259)}$ [84], (-1,472)
$\underline{(-158,158)}$ [157], $\underline{(-80,395)}$ [16], $\underline{(-125,62)}$ [250], $\underline{(259,182)}$ [370], (-148,108)
(104,27), (272,459), (190,90), (-86,387), $\underline{(-141,282)}$ [70]
$\underline{(174,75)}$ [406], (-141,90), (174,522), $\underline{(252,483)}$ [132], (300,387)
$\underline{(255,480)}$ [136], (-18,459), $\underline{(300,300)}$ [301], (-90,27), (207,108)
For each of the above pairs (x, y), instead of checking (-2), equivalently, we can verify whether the fraction $\frac{(x+1)x}{y}$ is an integer. If so, this gives $-z$.
Only the underlined pairs satisfy (-2), with the corresponding z added between brackets, so we have another 12 clean decompositions:

$$\begin{bmatrix} 273 & 208 \\ -357 & -272 \end{bmatrix} + \begin{bmatrix} -327 & -199 \\ 350 & 213 \end{bmatrix}, \begin{bmatrix} 105 & 520 \\ -21 & -104 \end{bmatrix} + \begin{bmatrix} -159 & -511 \\ 14 & 45 \end{bmatrix},$$
$$\begin{bmatrix} -147 & 259 \\ -84 & 148 \end{bmatrix} + \begin{bmatrix} 93 & -250 \\ 77 & -207 \end{bmatrix}, \begin{bmatrix} -157 & 158 \\ -157 & 158 \end{bmatrix} + \begin{bmatrix} 103 & -149 \\ 150 & -217 \end{bmatrix},$$
$$\begin{bmatrix} -79 & 395 \\ -16 & 80 \end{bmatrix} + \begin{bmatrix} 25 & -386 \\ 9 & -139 \end{bmatrix}, \begin{bmatrix} -124 & 62 \\ -250 & 125 \end{bmatrix} + \begin{bmatrix} 70 & -53 \\ 243 & -184 \end{bmatrix},$$
$$\begin{bmatrix} 260 & 182 \\ -370 & -259 \end{bmatrix} + \begin{bmatrix} -314 & -173 \\ 363 & 200 \end{bmatrix}, \begin{bmatrix} -140 & 282 \\ -70 & 141 \end{bmatrix} + \begin{bmatrix} 86 & -273 \\ 63 & -200 \end{bmatrix},$$
$$\begin{bmatrix} 175 & 75 \\ -406 & -174 \end{bmatrix} + \begin{bmatrix} -229 & -66 \\ 399 & 115 \end{bmatrix}, \begin{bmatrix} 253 & 483 \\ -132 & -252 \end{bmatrix} + \begin{bmatrix} -307 & -474 \\ 125 & 193 \end{bmatrix},$$
$$\begin{bmatrix} 256 & 480 \\ -136 & -255 \end{bmatrix} + \begin{bmatrix} -310 & -471 \\ 129 & 196 \end{bmatrix}, \begin{bmatrix} 301 & 300 \\ -301 & -300 \end{bmatrix} + \begin{bmatrix} -355 & -291 \\ 294 & 241 \end{bmatrix}.$$

Here the LHS matrices are idempotents and the RHS matrices are units with determinant -1.

Summarizing, A^2 is an index 13 clean integral matrix.

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Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, 400084, Romania