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SOCLE AND RADICAL AS PURE SUBGROUPS OF ABELIAN GROUPS

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1. We shall use the following notations for an abelian group A: $S(A)$ the socle, $T(A)$ the torsion part, $D(A)$ the maximal divisible subgroup, $F(A)$ the radical (the Frattini subgroup), A_p the p-component (for any prime number p) and $L(A)$ the lattice of all subgroups of A. All the notions we use are standard.

In this paper, two classes of abelian groups are characterized : those whose socle is a pure subgroup and those whose radical (the Frattini subgroup) is a pure subgroup. For the sake of completeness we first recall the following known results

a. The only pure, essential subgroup of A is A itself.

b. $L(A)$ is complemented iff A is an elementary group.

c. A torsion group T has the property that every mixed group with torsion part T splits iff it is a direct sum of a divisible group and a bounded group.

d. $F(A) = \cap \{pA \mid p \text{ prime number}\}$.

e. Every subfunctor F of the identity commutes with direct sums.

f. If in a p-group every element of order p is of infinite height then this group is divisible.

2. The first problem is solved by the next result

THEOREM 1. *For an abelian group A the following conditions are equivalent :*

(i) *the socle $S(A)$ is a pure subgroup of A ;*

(ii) *the torsion part $T(A)$ of A is elementary or zero.*

Proof. If the group A is torsion-free then $S(A) = T(A) = 0$ is trivially pure in A. If A is a torsion group, the socle is essential in A, so that if $S(A)$ is also pure, using (a) $S(A) = A$. Hence A is elementary. Conversely, if A is elementary, by (*b*) the lattice $L(A)$ is complemented so that every subgroup is a direct summand. Finally, if A is mixed and $S(A)$ is pure in A, $T(A)$ being also pure in A, $S(A)$ is pure in $T(A)$. But $S(A)$ is also essential in $T(A)$ so that $S(A) = T(A)$ and $T(A)$ is elementary. Conversely, if $S(A) = T(A)$ or $S(A) = 0$, $S(A)$ is trivially pure in A.

Remark. We can easily characterize also the abelian groups whose socle is a direct summand : only the mixed case needs a change : if A is mixed and $S(A)$ is a direct summand, i.e. $A = S(A) \oplus B$ we deduce $S(B) = S(A) \cap B = 0$ so that $T(B) = 0$. Hence A splits and by (c) the class we are looking for is formed by the abelian groups whose torsion part is a bounded elementary group. Conversely, if $T(A)$ is bounded elementary, by (c) A splits and $S(A) = T(A)$ so that $S(A)$ is a direct summand.

3. The characterization of the abelian groups whose Frattini subgroup is a pure subgroup needs informations about abelian groups whose Frattini subgroup is zero. So we start with some lemmas about such groups. We remark at once that such groups are reduced. Indeed, using (e) $F(A) = F(D(A) \oplus R) = F(D(A)) \oplus F(R) = D(A) \oplus F(R)$ where R is reduced, because the Frattini subgroup of a group is the group itself iff the group is divisible. Hence $D(A) \subseteq F(A)$ so that $D(A) \neq 0$ implies $F(A) \neq 0$.

LEMMA 1. *Let R be a reduced torsion-free group. The Frattini subgroup $F(R) = 0$ iff for each $a \in R$ the characteristic $\chi(a)$ contains an infinite number of zeros.*

Proof. It suffices to observe that $F(R) = \{a \in R \mid h_p(a) \geqslant 1$ for each prime number $p\}$ (see (d)). If an element a has only a finite number of zero heights and $S = \{p \text{ prime} \mid h_p(a) = 0\}$ then $0 \neq (\prod_{p \in S}) a \in F(R)$.

LEMMA 2. *Let R be a reduced torsion group $F(R) = 0$ iff R is elementary.*

Proof. If R is a torsion group, R has only prime order elements. We distinguish the following three cases:

(i) thereis an element $a \in R$ such that $o(a) = p$ and $h_p(a) = k$ a non-zero positive integer. In this case there is an element $b \in R$ such that $p^k b = a$ and $\langle b \rangle$ is cyclic of order p^{k+1}. Using (c) $F(\langle b \rangle) = p\langle b \rangle \neq 0$ so that $F(R) \neq 0$.

(ii) there is an element $a \in R$ such that $o(a) = p$ and $h_p(a)$ is infinite, but all the elements in $R[p]$ have zero or infinite heights; in $R_p \cap pR \neq 0$ all the elements have infinite heights. Using (*f*) $R_p \cap pR$ would be divisible, which is impossible (R is reduced).

(iii) all the elements in $R[p]$ have zero heights, i.e. $R[p] \cap pR = 0$. In this case R_p is elementary so that $S(R) = R$ and R is elementary. Conversely, if R is elementary, $F(R) = 0$ follows by (d).

LEMMA 3. *Let R be a mixed reduced group. If $F(R) = 0$ then $T(R)$ is elementary. Conversely, if $T(R)$ is elementary bounded and $R/T(R)$ satisfies the condition from Lemma 1, then $F(R) = 0$.*

Proof. The first part follows in a way similar to Lemma 2. As for the second, if $T(R)$ is elementary bounded, by (c) R splits, so that according to Lemma 2, $F(R) = F(T(R)) \oplus F(R/T(R)) = 0$ iff $F(R/T(R)) = 0$. One now uses Lemma 1.

We are ready to prove our result about pure Frattini subgroups:

THEOREM 2. *For an abelian group A the following conditions are equivalent.*

(i) $F(A)$ *is a pure subgroup of* A;

(ii) $F(A)$ *is a direct summand of* A;

(iii) $F(A) = D(A)$;

(iv) $F(A/D(A)) = 0$;

(v) $A = D \oplus R$ *with* D *divisible and* R *reduced,* R *satisfying the condition* $F(R) = 0$.

Proof. If $F(A)$ is pure in A, using (d) we can deduce $pF(A) = pA \cap F(A) = F(A)$ for each prime number p. Hence $F(A)$ is p-divisible for each prime number p, so even divisible and $F(A) \subseteq D(A)$. The converse inclusion was proved above so that $D(A) = F(A)$ and $F(R) = 0$. For the implication (v) $\Rightarrow$ (iv) one uses again (e); the rest is obvious.

We finally record an open question : let R be a mixed reduced with $T(R)$ elementary unbounded. Find sufficient conditions for $F(R) = 0$.

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