SOME EXAMPLES RELATED TO SQUARES OF ELEMENTS IN RINGS

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Abstract. We provide a 2×2 integral matrix that is not fine yet its square is fine. Additionally, we characterize the rings in which the square of every element is idempotent.

1. Introduction

In [1], the integral matrix $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ was presented as an example of (uniquely) nil-clean element that is not clean. Recently, Yiqiang Zhou discovered that A^2 is clean (private communication).

While it is easy to find a 2×2 matrix that is not nil-clean but its square is nil-clean, to find a 2×2 matrix that is not fine but its square is fine is harder. Generalizing a result from [2], we provide such an example. Also related to squaring elements in a ring, we characterize the rings where the squares of all elements are idempotent, that is, the rings which have the identity $x^4 = x^2$.

In closing, for the matrix A above, we show that A^2 has precisely 13 clean decompositions.

We recall the following well-known definitions. An element of a ring is: *nil-clean* if it is a sum of an idempotent and a nilpotent, *clean* if it is a sum of an idempotent and a unit and *fine* if it is a sum of a unit and a nilpotent. A nil-clean (or clean, or fine) element is called *strongly* nil-clean (resp. clean, or fine), if the components of the sum commute.

We denote by $U(R)$, the set of all units of a ring R, by $N(R)$, the set of all nilpotents of R and by $J(R)$ the Jacobson radical of R. We solve the quadratic Diophantine equations using [7].

2. A non-fine matrix whose square is fine

As already mentioned in the introduction, it is easy to provide examples of matrices that are not nil-clean but their squares are nil-clean. Clearly, the trivial nil-clean elements (i.e., the idempotent is trivial or the nilpotent is zero) have nil-clean squares. Hence examples must be nontrivial, and for matrices, these should (not) have trace equal to 1.

The matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ has trace $= -1$, so is not nil-clean, but $A^2 = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ is idempotent and hence (trivially) nil-clean.

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Before presenting an example of 2×2 integral matrix that is not fine, but its square is fine, we first prove a characterization which generalizes equation 5.10 in Example 5.9, [2].

Theorem 2.1. *For a 2×2 integral matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote $l := -\det(A) \pm 1$. Then A is fine iff*

(i) at least one of the systems $cx + by = l$, $s^2 + xy = 0$ in unknowns x, y, s has an integer solution, whenever $a = d$, or

(ii) at least one of the (quadratic) Diophantine equations

$$c^2x^2 + [(a-d)^2 + 2bc]xy + b^2y^2 - 2clx - 2bly + l^2 = 0$$

in unknowns x, y has an integer solution such that $-xy$ is a square, whenever $a \neq d$.

Proof. Since nilpotents in $\mathbb{M}_2(\mathbb{Z})$ have the form $\begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ with $s^2 + xy = 0$, A is fine iff $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) = \pm 1$. This condition can be written $s(a-d) = -cx - by + l$. If $a = d$ we get (i) and if $a \neq d$, squaring and eliminating s, we obtain the quadratic Diophantine equation in the statement. Observe that $-s(a-d) = -cx - by + l$ is also suitable since $(-s)^2 + xy = 0$ and so the final step consists of the choice between s and $-s$ (in order to have $s(a-d) = -cx - by + l$). □

Example. Take $A = \begin{bmatrix} 1 & 7 \\ 8 & 0 \end{bmatrix}$, *declared* not fine in [2]. Indeed, for A, the Diophantine equations (ii) are:

$$64x^2 + 113xy + 49y^2 - 16lx - 14ly + l^2 = 0 \quad (*)$$

with $l = 56 \pm 1$, and both have no integer solutions, so A is not fine. More precisely the equations are

$$64x^2 + 113xy + 49y^2 - 880x - 770y + 3025 = 0$$

and

$$64x^2 + 113xy + 49y^2 - 912x - 798y + 3249 = 0.$$

For $A^2 = \begin{bmatrix} 57 & 7 \\ 8 & 56 \end{bmatrix}$, written in l, the Diophantine equations (ii) are the same as (*), but with a different $l = -3136 \pm 1$.

For $l = -3135$ we have $64x^2 + 113xy + 49y^2 + 50160x + 43890y + 9828225 = 0$, with no integer solutions, and

for $l = -3137$ we have $64x^2 + 113xy + 49y^2 + 50192x + 43918y + 9840769 = 0$, equation which has an integer solution: $(x, y) = (-2143296, 2798929)$, for which (as desired) the product $-xy$ is a square, that is $s = \pm 2449272$. The final step is to choose s or $-s$ (because of the squaring in the proof of Theorem 2.1).

As $s(a-d) \neq -cx - by + l$ but $-s(a-d) = -cx - by + l$, we have to choose $s = -2449272$.

This gives the fine decomposition

$$\begin{bmatrix} 57 & 7 \\ 8 & 56 \end{bmatrix} = \begin{bmatrix} -2449272 & -2143296 \\ 2798929 & 2449272 \end{bmatrix} + \begin{bmatrix} 2449329 & 2143303 \\ -2798921 & -2449216 \end{bmatrix},$$

since the LHS is nilpotent (zero trace and zero determinant) and the determinant of the RHS is $= -1$ (we used [8], for computation).

3. Rings with idempotent squares

In this section, we describe *the rings where the squares of all elements are idempotent.*

To simplify the discussion, we define a ring as *SI* if all of its squares are idempotent. Formally, a ring R is SI iff $R^2 = Id(R)$, meaning that for every $x \in R$, we have $x^4 = x^2$.

As a trivial example, Boolean rings are SI. An example of SI ring which is not Boolean is $\mathbb{Z}_{12}$, as $\mathbb{Z}_{12}^2 = \{0, 1, 4, 9\} = Id(\mathbb{Z}_{12})$. Clearly, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are not SI.

The study of rings with the polynomial identity $x^4 = x^2$ for every $x \in R$, dates back over 80 years.

Alfred Foster introduced the concept of a Boolean-like ring in his 1946 paper [4]. He defined elements of a ring that satisfy $x^4 = x^2$ as *weakly idempotent*. A *Boolean-like* ring is a *commutative* ring of characteristic 2 with identity in which $(1-a)a(1-b)b = 0$ holds for all elements a, b of the ring. Several well-known properties of Boolean-like rings are as follows: each element is weakly idempotent (i.e., the Boolean-like rings form a special class of SI rings); the nilpotent elements form an ideal; the idempotent elements form a subring; each element can be uniquely written as the sum of an idempotent and a nilpotent (that is, the ring is *uniquely nil-clean*).

The concept of (m, n)-*Boolean* ring ($m > n \geq 1$) was introduced by Maurer and Szigeti (see [9]) as a ring in which every element satisfies the identity $x^m = x^n$. Their paper proves that the structure of (m, n)-Boolean rings depends significantly on the parity of the difference $m - n$. If this difference is odd, a reduction theorem is established. These rings are then $(m - n + 1, 1)$-Boolean and, by Jacobson's theorem, commutative. Moreover, such rings are reduced. For cases where the difference $m - n$ is even, no such reduction theorems exist and rings satisfying the identity $x^{n+2} = x^n$, for some positive integer n, deserve special attention. Specifically, for $n = 2$, these rings are what we refer to as SI rings. For example, the ring of 2×2 upper-triangular matrices over a Boolean ring is a $(4, 2)$-Boolean ring, which is not commutative. Additionally, $\mathbb{Z}_{12}$ is a $(4, 2)$-Boolean ring that is not reduced.

In 1998, Hirano and Tominaga [5] proved that every element of a ring R is a sum of two commuting idempotents iff R satisfies the identity $x^3 = x$. The following characterization was subsequently established (actually, (iii) was added in [11]).

Theorem 3.1. *The following conditions are equivalent for a ring R.*
(i) The ring R has the identity $x^3 = x$.
(ii) Every element of R is a sum of two commuting idempotents.
(iii) Every element of R is a difference of two commuting idempotents.
(iv) R is a direct product $R = A \times B$, where A is zero or a Boolean ring and B is zero or a subdirect product of $\mathbb{Z}_3$'s.

Thus, these are precisely *the rings all whose elements are tripotents.*

More recently, in [11] (see **Theorem 3.10**), a structure theorem was proved for rings which have the identity $x^6 = x^4$.

Theorem 3.2. *The following conditions are equivalent for a ring R.*
(i) every element of R is a sum of an idempotent and a tripotent that commute,

(ii) R has the identity $x^6 = x^4$,

(iii) $R = A \times B$, where A is zero or $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is zero or a subdirect product of $\mathbb{Z}_3$'s.

We now present a characterization of the SI rings, with some elements of the proof having analogous results in [11]. For the reader's convenience, we provide a complete and detailed exposition below.

First, we recall a result that, for rings, dates back to [5], and for elements, to [10].

Proposition 3.3. *An element a in a ring is strongly nil-clean iff $a - a^2$ is a nilpotent.*

Secondly, we recall from [3] the following characterization.

Theorem 3.4. *A ring R is strongly nil-clean iff $J(R)$ is nil and $R/J(R)$ is Boolean.*

Thirdly, the following result is routine.

Proposition 3.5. *A direct product of rings is SI iff all its components are SI. Any factor ring of a SI ring is SI.*

Next, we outline the prerequisites essential for proving the characterization theorem.

Lemma 3.6. *(i) If $R/J(R)$ is Boolean then*

(a) $2 \in J(R)$,

(b) $U(R) = 1 + J(R)$,

(c) $N(R) \subseteq J(R)$.

(ii) If R has the identity $x^3 = x$, then $R = R_1 \times R_2$, where R_1 is a Boolean ring (a subdirect product of $\mathbb{Z}_2$'s) and R_2 is a subdirect product of $\mathbb{Z}_3$'s.

(iii) Let $A = R/2^2R$ and $B = R/3R$. If $2^2 3 = 0$ in R, then A, B are SI rings with $2^2 = 0$ in A, $3 = 0$ in B, and $R \cong A \times B$.

(iv) If $b^4 = b^2$ and $3 = 0$ in a ring B, then B is a subdirect product of $\mathbb{Z}_3$'s.

(v) If $a^4 = a^2$ and $4 = 0$ in a ring A, then $A/J(A)$ is Boolean.

Proof. (i) If $R/J(R)$ is Boolean, then $r^2 - r \in J(R)$ for every $r \in R$. For (a), we take $r = 2$. For (b), let $u \in U(R)$. Then $u^2 - u \in J(R)$ and since $J(R)$ is an ideal, $u \in 1 + J(R)$. The converse is well-known (e.g., see Corollary **4.5** in [6]). For (c), let $t \in N(R)$. As $1 + t \in U(R)$, by (b) it follows that $t \in J(R)$.

(ii) Let $t \in N(R)$. As $t = t^3 = t^5 = ...$ it follows that $t = 0$, so R is reduced. By Andrunakievich-Ryabukhin theorem (e.g., see Theorem **12.7** in [6]), R is a subdirect product of domains. Since the only suitable domains are $\mathbb{Z}_2$ and $\mathbb{Z}_3$, it follows that R is a subdirect product of $\mathbb{Z}_2$'s and $\mathbb{Z}_3$'s.

(iii) Suppose $2^2 3 = 0$. Then $2^2R \cap 3R = 0$ and $R = 2^2R + 3R$. By the Chinese Remainder theorem, $R \cong R/2^2R \times R/3R$.

(iv) Tripotents have idempotent squares, since $r^4 = r^2$ follows directly from $r^3 = r$. Moreover, if $3 = 0$ in a SI ring, the converse also holds. Indeed, replacing r with $1 + r$ in the equation $r^2 = r^4$ yields $1 + 2r + r^2 = 1 + 4r + 6r^2 + 4r^3 + r^4$ whence $r = r^3$. Hence, $b^4 = b^2$ and $3 = 0$ in a ring B imply $b = b^3$. Thus B is a subdirect product of $\mathbb{Z}_3$'s.

(v) Suppose $a^4 = a^2$ and $4 = 2^2 = 0$. For any $a \in A$, as $a^4 = a^2$, we have $(a - a^2)^2 = a^2(1 - a)^2 = a^2(1 - 2a + a^2) = 2(a^2 - a^3)$, which is nilpotent as 2 is nilpotent. Thus, $a - a^2$ is nilpotent and so by Proposition 3.3, a is strongly nil-clean. Therefore, the ring A is strongly nil clean and by Theorem 3.4, $A/J(A)$ is Boolean. □

Now we are ready to characterize the rings all whose squares are idempotent (i.e., the SI rings).

Theorem 3.7. *The following conditions are equivalent for a ring R.*
1) $x^4 = x^2$ for all x in R.
2) R is isomorphic to A, or B, or $A \times B$, where $A/J(A)$ is Boolean and $j^2 = 2j = 0$ for all $j \in J(A)$, and B is a subdirect product of $\mathbb{Z}_3$'s.

Proof. 2) $\Rightarrow$ 1). If $A/J(A)$ is Boolean then $2 \in J(A)$. As $j^2 = 0$ for every $j \in J(A)$, $J(A)$ is nil and so A is strongly nil-clean by Theorem 3.4.

As now $J(A) \subseteq N(A)$, by Lemma 3.6 (i) (c) it follows that $J(A) = N(A)$, so every $a \in A$ is a sum $e+j$ with $e = e^2$, $j^2 = 0$ and $ej = je$. Hence, $a^2 = e+2ej = e$ as $2j = 0$. Thus, $a^4 = a^2$. If B is a subdirect product of $\mathbb{Z}_3$'s then B has the identity $x^3 = x$, and so has also the identity $x^4 = x^2$.

1) $\Rightarrow$ 2) $2^4 = 2^2$ gives $2^2 3 = 0$ in R, so by Lemma 3.6 (iii), $R = A \times B$ where $4 = 0$ in A and $3 = 0$ in B. By Lemma 3.6 (iv), $A/J(A)$ is Boolean and B is a subdirect product of $\mathbb{Z}_3$'s.

If $A/J(A)$ is Boolean, by Lemma 3.6 (i) (a) $2 \in J(A)$ and so by (i) (b), $3 \in U(A)$. For every $a \in A$ we have $a^4 = a^2$ and $(a+1)^4 = (a+1)^2$, whence $2a^2 = 2a$. Finally, for every $j \in J(A)$, we have $(1+j)^4 = (1+j)^2$ and since $(1+j)^2$ is a unit it follows that $(1+j)^2 = 1$. Hence $0 = 2j + j^2 = 3j^2$, so $j^2 = 0$, and so $2j = 0$. □

4. Appendix

Zhou's discovery is particularly intriguing as it follows that for the matrix $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$, which *is (uniquely) nil-clean but not clean* in $\mathbb{M}_2(\mathbb{Z})$, the squared matrix $A^2 = \begin{bmatrix} -54 & 9 \\ -7 & -59 \end{bmatrix}$ *is clean but not nil-clean* in $\mathbb{M}_2(\mathbb{Z})$, since $Tr(A^2) = -113 \notin \{0, 1\}$.

In order to find all the clean decompositions of A, we use the following well-known characterization.

Theorem 4.1. *Let R be a commutative domain and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a matrix over R. Then A is nontrivial clean iff at least one of the systems*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = \pm 1 & (\pm 2) \end{cases}$$

in unknowns x, y, z is solvable over R. If $b \neq 0$ and any of ($\pm$2) holds, then (1) is equivalent to

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0 \qquad (\pm 3).$$

The signs in the equations correspond accordingly.

Recall that the integer solutions of the systems give $\begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$, the idempotent of the clean decomposition. Since $\det(A^2) = 3249$, and $b = 9 \neq 0$, the two pairs of conditions (here $a - d = 5$ and $c = -7$) are the following.

(+3): $9x^2 - 5xy + 7y^2 + 9x + (-59 - 3249 + 1)y = 0$,
(+2): $5x - 7y + 9z + 3249 + 59 = +1$
respectively
(-3): $9x^2 - 5xy + 7y^2 + 9x + (-59 - 3249 - 1)y = 0$,
(-2): $5x - 7y + 9z + 3249 + 59 = -1$.

We solve the quadratic Diophantine equations using [7].

(+3) has the solutions: $(0,0), (-1,0)$ and $(300,301)$.
Only $(300,301)$ satisfies (+2), gives $z = -300$ and so yields the following clean decomposition:
$A^2 = \begin{bmatrix} -54 & 9 \\ -7 & -59 \end{bmatrix} = \begin{bmatrix} 301 & 301 \\ -300 & -300 \end{bmatrix} + \begin{bmatrix} -355 & -292 \\ 293 & 241 \end{bmatrix}$, where the LHS matrix is idempotent and the RHS unit matrix has determinant 1.

(-3) has the solutions: $(0,0), (-1,0)$ and another 25:
(115,522), (272,208) [357], (104,520) [21], (-148,259) [84], (-1,472)
(-158,158) [157], (-80,395) [16], (-125,62) [250], (259,182) [370], (-148,108)
(104,27), (272,459), (190,90), (-86,387), (-141,282) [70]
(174,75) [406], (-141,90), (174,522), (252,483) [132], (300,387)
(255,480) [136], (-18,459), (300,300) [301], (-90,27), (207,108)
For each of the above pairs (x,y), instead of checking (-2), equivalently, we can verify whether the fraction $\dfrac{(x+1)x}{y}$ is an integer. If so, this gives $-z$.
Only the underlined pairs satisfy (-2), with the corresponding z added between brackets, so we have another 12 clean decompositions:

$$\begin{bmatrix} 273 & 208 \\ -357 & -272 \end{bmatrix} + \begin{bmatrix} -327 & -199 \\ 350 & 213 \end{bmatrix}, \begin{bmatrix} 105 & 520 \\ -21 & -104 \end{bmatrix} + \begin{bmatrix} -159 & -511 \\ 14 & 45 \end{bmatrix},$$
$$\begin{bmatrix} -147 & 259 \\ -84 & 148 \end{bmatrix} + \begin{bmatrix} 93 & -250 \\ 77 & -207 \end{bmatrix}, \begin{bmatrix} -157 & 158 \\ -157 & 158 \end{bmatrix} + \begin{bmatrix} 103 & -149 \\ 150 & -217 \end{bmatrix},$$
$$\begin{bmatrix} -79 & 395 \\ -16 & 80 \end{bmatrix} + \begin{bmatrix} 25 & -386 \\ 9 & -139 \end{bmatrix}, \begin{bmatrix} -124 & 62 \\ -250 & 125 \end{bmatrix} + \begin{bmatrix} 70 & -53 \\ 243 & -184 \end{bmatrix},$$
$$\begin{bmatrix} 260 & 182 \\ -370 & -259 \end{bmatrix} + \begin{bmatrix} -314 & -173 \\ 363 & 200 \end{bmatrix}, \begin{bmatrix} -140 & 282 \\ -70 & 141 \end{bmatrix} + \begin{bmatrix} 86 & -273 \\ 63 & -200 \end{bmatrix},$$
$$\begin{bmatrix} 175 & 75 \\ -406 & -174 \end{bmatrix} + \begin{bmatrix} -229 & -66 \\ 399 & 115 \end{bmatrix}, \begin{bmatrix} 253 & 483 \\ -132 & -252 \end{bmatrix} + \begin{bmatrix} -307 & -474 \\ 125 & 193 \end{bmatrix},$$
$$\begin{bmatrix} 256 & 480 \\ -136 & -255 \end{bmatrix} + \begin{bmatrix} -310 & -471 \\ 129 & 196 \end{bmatrix}, \begin{bmatrix} 301 & 300 \\ -301 & -300 \end{bmatrix} + \begin{bmatrix} -355 & -291 \\ 294 & 241 \end{bmatrix}.$$

Here the LHS matrices are idempotents and the RHS matrices are units with determinant -1.

Summarizing, A^2 has precisely 13 clean decompositions.

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