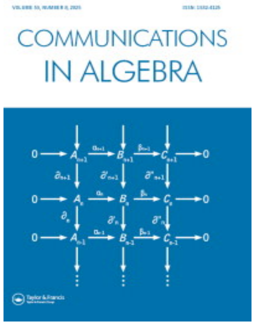

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On perspective Abelian groups

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ABSTRACT
The paper discusses perspective Abelian groups, defined as groups where isomorphic summands have a common complement. This concept is derived from the broader context of perspective R-modules. The paper explores various classes of these groups, providing characterizations and examples.

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1. Introduction

This paper investigates direct summands of Abelian groups. Throughout this study, G denotes an arbitrary Abelian group, and we use $H \leq^{\oplus} G$ to indicate that H is a direct summand of G. All summands considered are direct summands, and the term "complement" refers exclusively to direct complements. To distinguish notations, $\mathbb{Z}(m)$ denotes the Abelian group, while $\mathbb{Z}_m$ denotes the ring, of integers modulo m. For an Abelian group G, End(G) denotes $\text{End}_{\mathbb{Z}}(G)$, the endomorphism ring of G.

Definitions and preliminaries.

We start with a general definition (see [5]).

Let L be a bounded lattice. Two elements $x, y \in L$ are said to be *perspective* (in L) if they have a common (direct) complement, i.e., an element $z \in L$ such that $x \vee z = y \vee z = 1$ and $x \wedge z = y \wedge z = 0$. This definition originates from John von Neumann.

Specializing this to the submodule lattice of a module, two summands A, B of an R-module ${}_RM$ are denoted by $A \sim B$ if they have a common complement, i.e., there exists a submodule C such that $M = A \oplus C = B \oplus C$. Clearly, $A \sim B$ implies $A \cong B$. A module ${}_RM$ is called *perspective* when $A \cong B$ implies $A \sim B$ for any two summands A and B of ${}_RM$.

A module ${}_RM$ over a ring R is said to satisfy *internal cancellation* (IC) if, whenever $M = K \oplus N = K' \oplus N'$ (in the category of R-modules), $N \cong N'$ implies $K \cong K'$ (or $M/N \cong M/N'$).

It is clear that perspective modules satisfy the internal cancellation property, meaning that complements of isomorphic summands are isomorphic (see [6]).

The definition for modules can be extended to rings as follows: a ring R is called *perspective* if isomorphic direct summands of ${}_RR$ have a common (direct) complement. This property for rings is left-right symmetric, that is, ${}_RR$ is perspective if and only if R_R is perspective.

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Main results.

In this paper, we characterize several broad classes of perspective Abelian groups, that is, perspective $\mathbb{Z}$-modules. Hereafter, the term "group" will always refer to an "Abelian group."

1) Many classes of Abelian groups are perspective if and only if they have finite rank. These include:

- Divisible primary groups (Proposition 4.1)
- Divisible torsion-free groups (Corollary 4.12)
- Homogeneous completely decomposable groups with type $\boldsymbol{t}(\mathbb{Q}_p)$ (Corollary 4.17).

2) The rational groups are perspective.
(Proposition 4.14).
Specifically, the only free perspective group is $\mathbb{Z}$ (Proposition 4.13).

3) Any finite p-group is perspective (Proposition 4.3). Any bounded perspective p-group is finite (Corollary 4.4).

4) A torsion group is perspective if and only if all its primary components are perspective (Corollary 3.5).

5) (a) Let $G = T(G) \oplus F$, where $T(G)$ is the torsion part of G and F is torsion-free. Then G is perspective if and only if $T(G)$ and F are perspective.

5) (b) Let $G = D(G) \oplus R$, where $D(G)$ is the divisible part of G and R is reduced. Then G is perspective if and only if $D(G)$ and R are perspective (Corollary 3.6).

6) We describe the perspective torsion-free algebraically compact groups
(Proposition 4.20).

7) For any rational group H (i.e., rank 1 torsion-free group), we characterize when $H \oplus H$ is perspective. Moreover, two general results for PIDs are proved (Theorems 4.7 and 4.11) with important (Abelian group) corollaries (4.8, 4.12, 4.17, and 5.5).

Organization of the Paper.

Section 2 recalls general results on perspective modules from [4], which are utilized in the paper. Sections 3 and 4 contain our results on perspective Abelian groups, divided into:

- Section 3.1: Definitions
- Section 3.2: Reduction theorems
- Section 4.1: Perspective torsion groups
- Section 4.2: Perspective torsion-free groups

The paper concludes with Section 5 which presents applications and examples.

We use the shorthand "iff" for "if and only if" throughout the text for convenience.

2. Generalities on perspective modules

Definition ([8]). Let $\mathcal{P}$ be a module-theoretic property. We say that $\mathcal{P}$ is an *endomorphism ring property* (or **ER**-property for short) if, whenever A_R and A'_S are right modules (over possibly different rings R and S) such that $\mathrm{End}(A_R) \cong \mathrm{End}(A'_S)$ (as rings), A_R satisfies $\mathcal{P}$ implies that A'_S does (and then, of course, also conversely).

An exhaustive reference on perspective R-modules is [4]. However, not much from [4] can be used when describing perspective Abelian groups.

We list the relevant results from [4].

Theorem 2.1. *For a module M_S with $R = \mathrm{End}_S(M)$, the following conditions are equivalent:*

(1) M_S is perspective.

(4) If $erse = e$ for some $e^2 = e, r, s \in R$, then $erte = e$ for some $t \in R$ such that $ete \in U(eRe)$.

In particular, M_S is perspective iff R_R is perspective, i.e., perspectivity is an ***ER****-property.*

A useful consequence not recorded in [4] is the following

Corollary 2.2. *Arbitrary products of perspective rings are perspective.*

Proof. Suppose R is the direct product of $R_i, i \in I$ and $erse = e$ for some $e^2 = e, r, s \in R$. Writing $e = (e_i)$, $r = (r_i)$ and $s = (s_i)$, we obtain $erse = e$ iff $e_i r_i s_i e_i = e_i$ for each i. As each R_i is perspective, there exists t_i such that $e_i r_i t_i e_i = e_i$ and $e_i t_i e_i \in U(e_i R_i e_i)$. If $t = (t_i)$, then $erte = e$ and $ete \in U(eRe)$. □

Proposition 2.3. *Any direct summand of a perspective module is perspective.*

Corollary 2.4. *If M and N are perspective R-modules with $\mathrm{Hom}_R(M, N) = 0$, then $M \oplus N$ is perspective.*

Proof. [4] As perspectivity is an **ER**-property, it is enough to show that $\mathrm{End}(M \oplus N)$ is a perspective ring. But as

$$\mathrm{End}(M \oplus N) \cong \begin{bmatrix} \mathrm{End}(M) & \mathrm{Hom}_R(N, M) \\ 0 & \mathrm{End}(N) \end{bmatrix},$$

this follows since $\mathrm{End}(M)$ and $\mathrm{End}(N)$ are perspective rings. □

Remark 2.5. The group $\mathbb{Z} \oplus \mathbb{Z}$ is not perspective. For example, $(2, 5)\mathbb{Z}$ and $(1, 0)\mathbb{Z}$ are isomorphic direct summands of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module, which do not have a common complement.

From [9] we recall a *method of constructing common complements* for some special direct sums.

Let $G = H \oplus K$. A subgroup D of G is called a *diagonal* in G (with respect to H and K) if $D + H = G = D + K$ and $D \cap H = 0 = D \cap K$.

Theorem 2.6. *Let $G = H \oplus K$. If $\delta : H \to K$ is an isomorphism then $D(\delta) = D(H, \delta) = \{x + \delta(x) | x \in H\} = (1 + \delta)(H)$ is a diagonal in G (with respect to H and K). Conversely, if D is a diagonal in G (with respect to H and K) there is a unique isomorphism $\delta : H \to K$ such that $D = D(\delta)$.*

Thus, there is a bijection between the diagonals (with respect to H and K) and isomorphisms of H and K.

3. Perspective Abelian groups

3.1. Definitions

First, we define perspective Abelian groups:

As the Abelian groups analogue for $\mathbb{Z}$-modules, an (Abelian group) G is called *perspective* if isomorphic summands of G have a common complement.

In symbols: if $G = A \oplus H = B \oplus K$ with $A \cong B$, there exists (a summand) C such that $G = A \oplus C = B \oplus C$.

Nonexample 3.1. Let N be a group such that $N \ncong N \oplus N$ and let $G = N_1 \oplus N_2 \oplus N_3 \oplus \cdots$ be a direct sum of countably many copies with $N_n = N$. Then G is *not* IC (and so not perspective).

Indeed, $H = N_2 \oplus N_3 \oplus \cdots$ and $S = N_3 \oplus \cdots$ are isomorphic summands, but $G/H \cong N \ncong N \oplus N \cong G/S$.

Remark. 1) Obviously, *if two summands of a group have a common complement, these are isomorphic.*

Indeed, if $G = H \oplus K = L \oplus K$ then $H \cong G/K \cong L$.

Therefore, perspectivity is a converse of this property.

2) *Indecomposable groups are trivially perspective* (e.g., any infinite cyclic group, any cocyclic group or rank 1 torsion-free group). The group G is indecomposable iff $\mathrm{End}(G)$ has only the trivial idempotents (is *connected*).

3) We mention (see Proposition **3.10**, [7]) that *for two idempotent endomorphisms* ε, δ *of* G, $\varepsilon(G) \cong \delta(G)$ *iff there exists endomorphisms* α, β *of* G *such that* $\varepsilon = \alpha\beta$ *and* $\delta = \beta\alpha$. Also equivalently, *the left* $\mathrm{End}(G)$*-modules* $\mathrm{End}(G)\varepsilon$ *and* $\mathrm{End}(G)\delta$ *are isomorphic to each other.* Since we deal only with direct summands, equivalently, we can deal only with the idempotent endomorphisms of the group. Thus, two "*isomorphic*" endomorphisms ε, δ of a group G (i.e., $\mathrm{im}(\varepsilon) \cong \mathrm{im}(\delta)$) are *perspective*, if there is an endomorphism γ such that $\mathrm{im}(\varepsilon) \oplus \mathrm{im}(\gamma) = G = \mathrm{im}(\delta) \oplus \mathrm{im}(\gamma)$.

More general, the group is *IC* if for any two endomorphisms ε, δ of G, $\mathrm{im}(\varepsilon) \cong \mathrm{im}(\delta)$ implies $G/\mathrm{im}(\varepsilon) \cong G/\mathrm{im}(\delta)$.

Apparently there is another definition one could give for perspective Abelian groups.

Definition. An Abelian group G is called *e-perspective* if its endomorphism ring $\mathrm{End}(G)$ is (left or right) perspective.

However, it follows from Theorem 2.1 that for a module M_S with $R = \mathrm{End}_S(M)$, M_S is perspective iff R_R is perspective, i.e., perspectivity is an **ER**-property. Therefore, for $S = \mathbb{Z}$ it follows that

Proposition 3.2. *An Abelian group is perspective iff it is e-perspective.*

Hence, we can use either definition interchangeably.

3.2. Reduction theorems

These theorems help reduce the problem of perspectivity to simpler cases.

Proposition 3.3. *Summands of perspective groups are perspective.*

Proof. Suppose $G = H \oplus K$ and $H = S \oplus T = L \oplus N$ with $S \cong L$. Since these are direct summands also in G, by hypothesis, there is $M \leq^{\oplus} G$ such that $G = S \oplus M = L \oplus M$. Then, by modularity of the subgroup lattice: $H = G \cap H = (S \oplus M) \cap H \overset{\mathrm{mod}}{=} S \oplus (M \cap H)$ (since $S \leq H$) and similarly $H = L \oplus (M \cap H)$, so $M \cap H$ is a common complement for S and L. □

Proposition 3.4. *Let* $G = \bigoplus_{i \in I} H_i$ *where each summand* H_i *is fully invariant in* G. *Then* G *is perspective iff all* H_i, $i \in I$, *are perspective.*

Proof. By Corollary 2.2, arbitrary products of perspective rings are perspective. It just remains to note that $\mathrm{End}(G) \cong \prod_{i \in I} \mathrm{End}(H_i)$ (see Theorem **106.1**, [3], the $I \times I$ matrices are diagonal). □

Corollary 3.5. *Let* G *be a torsion group. Then* G *is perspective iff so are all its primary components.*

Proof. A straightforward application of the previous proposition. □

This reduces the study of perspective torsion groups to perspective p-groups, for any prime p.

We can use Corollary 2.4 whenever $\mathrm{Hom}(G, H) = 0$, that is, for: (i) G torsion, H torsion-free, (ii) G a p-group and H a q-group with different primes $p \neq q$, (iii) G divisible and H reduced. Thus

Corollary 3.6. *(a) Let $G = T(G) \oplus F$ where $T(G)$ is the torsion part of G and F is torsion-free. Then G is perspective iff $T(G)$ and F are perspective.*

(b) Let $G = D(G) \oplus R$ where $D(G)$ is the divisible part of G and R is reduced. Then G is perspective iff $D(G)$ and R are perspective.

Examples. $\mathbb{Z}_m \oplus \mathbb{Z}$, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ or $\mathbb{Z}_{p^\infty} \oplus \mathbb{Q}$, all are perspective (splitting) mixed groups.

Therefore, the study of splitting mixed perspective (Abelian) groups reduces to perspective primary groups and to perspective torsion-free groups. Moreover, it reduces to perspective divisible groups and to perspective reduced groups.

According to these reductions, the study of perspective (Abelian) groups reduces to reduced perspective p-groups, reduced perspective torsion-free groups and mixed perspective not splitting groups. Some of such results are proved in the reminder of the paper. The reader can convince himself that even for highly predictable (for Abelian groups theorists) results, in the torsion and in the torsion-free cases, the proofs are not at all easy.

Some particularly challenging proofs involving only Abelian groups (as noted in the arXiv version) were greatly simplified through the application of two general theorems, both stated and proven by the referee (Theorems 4.7 and 4.11).

Therefore the nonsplitting mixed case was not addressed. Since all we know about the torsion part of such a group is that it is pure in the whole group, the question of which pure subgroups of perspective groups are also perspective (just partly addressed) becomes central.

4. Specific perspective properties

We now explore further properties of perspective Abelian groups.

4.1. Perspective torsion groups

First, perspective divisible p-groups are described below. For the proof, recall that the socle of $\mathbb{Z}(p^\infty)$ is its smallest nonzero subgroup (having order p) and that each subsocle of divisible p-group D supports some summand of D.

Proposition 4.1. *A divisible p-group is perspective iff it has finite rank. As such, it is isomorphic to a finite direct sum of $\mathbb{Z}(p^\infty)$.*

Proof. As already mentioned, an infinite rank direct sum of $\mathbb{Z}(p^\infty)$ is not perspective (see Nonexample 3.1). Conversely, let D be a divisible p-group. The proof goes by induction on rank of D. If $r(D) = 1$ then D is indecomposable and so trivially perspective.

Let $r(D) = n + 1$, $D = A \oplus B = C \oplus U$ and $A \cong C$. We go into several cases.

1) Let $A + C \neq D$. Then $D = (A + C) \oplus D'$ for some $D' \neq 0$ and so A and C are summands in a divisible group $A + C$ of rank $\leq n$. By induction $A + C = A \oplus K = C \oplus K$ for some K, whence $D = A \oplus (K \oplus D') = C \oplus (K \oplus D')$.

2) Let $A + C = D$.

a) If $A \cap C = 0$ then $D = A \oplus C$, so $A \cong C$. Denote an isomorphism by $f : A \to C$. Using Theorem 2.6, the subgroup $A' = \{a + f(a) \mid a \in A\}$ is a diagonal, so a summand of D such that $A \cap A' = C \cap A' = 0$ and $D = A \oplus A' = C \oplus A'$.

b) Let $A \cap C \neq 0$, so $(A \cap C)[p] \leq A[p], C[p]$ where $A[p] \cong C[p]$.

If $(A \cap C)[p] = A[p]$ then also $(A \cap C)[p] = C[p]$. Hence $A[p] = C[p]$, and by [2], $D = A \oplus B = C \oplus B$.

Next assume that $0 \neq (A \cap C)[p] < A[p]$, so $(A \cap C)[p] < C[p]$. There exist a summand A_1 of A with $A_1[p] = (A \cap C)[p]$, $A = A_1 \oplus A_2$. Similarly $C = C_1 \oplus C_2$, where $C_1[p] = (A \cap C)[p]$. Since $A \cong C$ and $A_1[p] = C_1[p]$ it follows that $A_1 \cong C_1$ and so $A_2 \cong C_2$.

3) Let $A_2 + C_2 \neq D$. Then as in case **1)**, $D = A_2 \oplus V = C_2 \oplus V$ for some V and so $A = A_2 \oplus (A \cap V)$ and $C = C_2 \oplus (C \cap V)$. Here V has rank $\leq n$ and $A \cap V$, $C \cap V$ are isomorphic summands, so by induction $V = (A \cap V) \oplus L = (C \cap V) \oplus L$ for some L. Finally $D = [A_2 \oplus (A \cap V)] \oplus L = [C_2 \oplus (C \cap V)] \oplus L$.

4) Let $A_2 + C_2 = D$. Since $A_2 \cap C_2 = 0$, as in case **2 a)**, $D = A_2 \oplus M = C_2 \oplus M$ for some M. Since $r(M) \leq n$, by induction, $A \cap M$ and $C \cap M$ are perspective in M, and as in case **3)**, A and C are perspective in D. □

If m is a cardinal and G is a group then $G^{(m)}$ denotes the direct sum of m copies of G.

The previous proposition has the following

Corollary 4.2. *For any cardinal m and any natural number n, the group $G = \mathbb{Z}(p^n)^{(m)}$ is perspective iff m is finite.*

Proof. As already mentioned, an infinite rank direct sum of $\mathbb{Z}(p^n)$ is not perspective (see Nonexample 3.1). Conversely, if D is a divisible hull of G then $G = D[p^n]$. If $G = A \oplus B = C \oplus K$, $A \cong C$, then by Proposition 4.1, $D = D_A \oplus U = D_C \oplus U$ for some $U \leq D$, where D_A, D_C are the divisible hulls of A, C, respectively. So $G = D[p^n] = D_A[p^n] \oplus U[p^n] = D_C[p^n] \oplus U[p^n]$, where $D_A[p^n] = A$, $D_C[p^n] = C$. □

Next, about finite or bounded p-groups, we have

Proposition 4.3. *The finite p-groups are perspective.*

Proof. The proof goes by induction on the order $|G|$ of the group G. Let $G = G_1 \oplus G_2$, where G_1 is a direct sum of finitely many groups $\mathbb{Z}(p^n)$, where p^n is the maximal order of elements in G, $G = A \oplus B = C \oplus K$ and $A \cong C$.

By Corollary 4.2, G_1 is perspective. Note that if $A \cap G_1 \neq 0$ then also $C \cap G_1 \neq 0$. Otherwise, if $C \cap G_1 = 0$, since G_1 is an absolute direct summand (see [2], Exercise 8 of Section 9) of G, we can suppose that $C \leq G_2$. However, in this case C would not have any element of order p^n but in A such elements exist in view of $A \cap G_1 \neq 0$. This would contradict the isomorphism $A \cong C$.

Since in a direct sum of cyclic groups each subsocle supports some summand of this group (see [3], Exercise 3 of Section 66) it follows that $A = A_1 \oplus A_2$, $C = C_1 \oplus C_2$, where $A_1[p] = (A \cap G_1)[p]$, $C_1[p] = (C \cap G_1)[p]$. These direct decompositions of cyclic groups are isomorphic, so from $A \cong C$ it follows that $A_1 \cong C_1$ and so $A_2 \cong C_2$. Hence by Corollary 4.2, $G_1 = A_1 \oplus U = C_1 \oplus U$ for some U. Then $A = A_1 \oplus A_3$, $C = C_1 \oplus C_3$, where $A_3 = [A \cap (U \oplus G_2)]$, $C_3 = [C \cap (U \oplus G_2)]$. So A_3, C_3 are isomorphic direct summands in $U \oplus G_2$. Since $|U \oplus G_2| < |G|$, by induction $U \oplus G_2 = A_3 \oplus V = C_3 \oplus V$. Hence $G = G_1 \oplus G_2 = (A_1 \oplus U) \oplus G_2 = (A_1 \oplus A_3) \oplus V = (C_1 \oplus C_3) \oplus V$, where $A_1 \oplus A_3 = A$ and $C_1 \oplus C_3 = C$, as desired. □

Since summands of perspective groups are perspective, it follows from the Nonexample 3.1 that the Ulm-Kaplanski invariants $f_n(G)$ of perspective reduced p-groups G are finite for all integer $n \geq 0$. Thus a basic subgroup of G is countable (if G is non-bounded) and so $|G| \leq 2^{\aleph_0}$ [2]. Therefore

Corollary 4.4. *Any perspective bounded p-group is finite.*

For (Abelian) groups we can introduce the following

Definition. A group is called *finitely perspective* if it is perspective with respect to finite (direct) summands. Then we can prove a surprising (specific for Abelian groups) result.

Proposition 4.5. *Each p-group G is finitely perspective.*

Proof. Let $A \cong C$ be finite summands of G. Then $p^m A = 0$ for some integer $m \geq 1$, and so also $p^m C = 0$ and $p^m(A + C) = 0$. Hence $A + C \leq H$ for a $p^m G$-high subgroup H. By a theorem of Khabbaz (see [2], Theorem 27.7), H is a summand of $G = H \oplus F$. We have $H = H_1 \oplus \cdots \oplus H_m$, where H_i is a direct sum of groups $\mathbb{Z}(p^i)$ whenever $H_i \neq 0$. Let $\pi_i : G \to H_i$ be the projections for $i \in \{1, \ldots, m\}$. Since $A + C$ is finite, it follows that each $\pi_i(A + C)$ is finite, and $A + C \leq \pi_1(A + C) \oplus \cdots \oplus \pi_m(A + C)$. Each $\pi_i(A + C)$ is contained in some finite summand H'_i of H_i and so $H' = H'_1 \oplus \cdots \oplus H'_m$ is a finite summand in $H = H' \oplus H''$. By Proposition 4.3, $H' = A \oplus U = C \oplus U$ for some U. Then $G = A \oplus (U \oplus H'' \oplus F) = C \oplus (U \oplus H'' \oplus F)$. □

We just mention that if $G = A \oplus C$, where $A \cong C$ then $G = A \oplus U = C \oplus U$, for any diagonal U with respect to A and C.

Let $\mathfrak{A}$ be a class of (Abelian) groups and $G \in \mathfrak{A}$. A relativization of our main property can be defined.

Definition. We call A *perspective in class* $\mathfrak{A}$ if for $G = A \oplus B = C \oplus K$, where $A \cong C$ and $G \in \mathfrak{A}$, it follows that $G = A \oplus U = C \oplus U$ for some U.

Then we can prove a result on *torsion-complete* (for several equivalent definitions for reduced groups, see [3], Theorem 68.4) p-groups.

Proposition 4.6. *The torsion-complete p-groups A with finite Ulm-Kaplanski invariants are perspective in the class of separable p-groups.*

Proof. Let $G = A \oplus B = C \oplus K$, where $A \cong C$ and let G be a separable p-group. Then $G = (A_1 \oplus \cdots \oplus A_n) \oplus (A^*_n \oplus B) = (C_1 \oplus \cdots \oplus C_n) \oplus (C^*_n \oplus K)$, where $A = (A_1 \oplus \cdots \oplus A_n) \oplus A^*_n$, $C = (C_1 \oplus \cdots \oplus C_n) \oplus C^*_n$, and $A_1 \oplus \cdots \oplus A_n$, $C_1 \oplus \cdots \oplus C_n$, respectively, are summands of the basic subgroups of A, C (A_k, C_k are direct sums of cyclic groups of order p^k). By Proposition 4.5, $G = (A_1 \oplus \cdots \oplus A_n) \oplus U^{(n)}$, $G = (C_1 \oplus \cdots \oplus C_n) \oplus U^{(n)}$, where we can choose the $U^{(n)}$'s, such that $U^{(n+1)}$ is a summand in $U^{(n)}$ and $U^{(n)}/U^{(n+1)}$ is a direct sum of cyclic groups of order p^{n+1}. So G has a basic subgroup of type $(\bigoplus_{n\geq 1} A_n) \oplus (\bigoplus_{n\geq 1} V^{(n)}_n) = (\bigoplus_{n\geq 1} C_n) \oplus (\bigoplus_{n\geq 1} V^{(n)}_n)$, where $V^{(n)}_n$ is a summand in $U^{(n)}_n$, each $U^{(n)}_n$ is a summand in $U^{(n)}$ and is a direct sum of cyclic groups of order p^n. So by [3], Theorem 72.2, $G = A \oplus U = C \oplus U$, and if $\overline{X}$ denotes the torsion completion of X then $A = \overline{(\bigoplus_{n\geq 1} A_n)}$, $C = \overline{(\bigoplus_{n\geq 1} C_n)}$, $U = \overline{(\bigoplus_{n\geq 1} V^{(n)}_n)}$. Hence $G = A \oplus (G \cap U) = C \oplus (G \cap U)$, and the proof is complete. □

4.2. *Perspective torsion-free groups*

As in the module case, conditions for a perspective group G which assure that $G \oplus G$ is also perspective, are of interest (and difficult to find).

In this subsection, we address this problem (in particular) for rational groups, which are perspective as indecomposable groups.

Suppose G is a torsion-free group of rank one and $\mathcal{Q}$ be the set of all primes such that $pG \neq G$. It follows that the localization $E := \mathbb{Z}_{(\mathcal{Q})}$ at $\mathcal{Q}$ is isomorphic to the endomorphism ring of G. Clearly E is actually a PID (as a localization of a PID) and not only is E isomorphic to the endomorphism ring of E, itself, but the endomorphism ring of $G \oplus G$ is naturally isomorphic to the endomorphism ring of $E \oplus E = E^2$ (both are the 2×2 matrices with coefficients in E). Since G and E have torsion-free rank 1, both are perspective. In addition, it follows from Theorem 2.1 that $G \oplus G$ is perspective iff E^2 is perspective. So there is no loss of generality in assuming that $G = E$ is actually a unitary subring of $\mathbb{Q}$.

Next, a general result

Theorem 4.7. *Suppose E is a PID. Then $E^2 = E \oplus E$ is a perspective E-module iff for every relatively prime pair $a, b \in E$, there is an $x \in E$ such that $a + xb$ is a unit of E. That is, whenever $(a) + (b) = E$, there is $x \in E$ such that $(a + xb) = E$.*

Proof. Suppose $E^2 = A \oplus B = A' \oplus B'$ and $A \cong A'$. If $A \cong A' \cong 0$ we may let $C = E^2$ and if $A \cong A' \cong E^2$, we may let $C = 0$; so we may assume $A \cong A' \cong E$, so that $B \cong B' \cong E$. Choose a basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ for E^2 such that $A = E\boldsymbol{e}_1$ and $B = E\boldsymbol{e}_2$. It follows that $A' = E(a\boldsymbol{e}_1 + b\boldsymbol{e}_2)$ where a, b are relatively prime in E. Note that any complementary summand for A in E^2 will be of the form $C_x = E(-x\boldsymbol{e}_1 + 1\boldsymbol{e}_2)$ for some $x \in E$. Clearly, C_x will also be a complementary summand of A' iff $a + xb$ is a unit of E, completing the proof. □

Corollary 4.8. *Let G be a torsion-free group of rank 1 such that $G \oplus G$ is perspective. Then G is p-divisible at least for one prime number p.*

Proof. If G is not p-divisible for any prime p, then $E \oplus E = \mathbb{Z} \oplus \mathbb{Z}$ is not perspective. □

The converse fails as shows the following

Example 4.9. If G is a torsion-free group of rank 1, with $2G, 5G \neq G$ and G is divisible only by 11, then $G \oplus G$ is not perspective.

By contradiction, suppose $G \oplus G$ is perspective. Since 11 is unit in $E = \mathrm{End}(G)$ then according to Theorem 4.7, $5l - 2s = \pm 11^a$, $kl - st = \pm 11^b$, for some integers $a, b \geq 0$. Taking $l = 0$ we can suppose $t = 1$ and so $s = \pm 11^b$ and $5l \pm 2 \cdot 11^b = \pm 11^a$. Since $(l, 11^b) = 1$ we get $b = 0$ or $a = 0$. If $b = 0$ then the equation $5l \pm 2 = \pm 11^a$ has no solutions since the last digit of the RHS is 1 but is 2 or 7 in the LHS. If $a = 0$ then the equation $5l \pm 1 = \pm 2 \cdot 11^b$ has no solutions since the last digit of the RHS is 2 but is 1 or 6 in the LHS.

Theorem 4.7 leads to an improvement of this example. Note that the proof is essentially identical with the exception of the use of Dirichlet's theorem.

Example 4.10. If α is either a non-negative integer or ω, then there is a subring $E \subset \mathbb{Q}$ that is p-divisible for exactly α primes p, such that E^2 is not perspective.

Using Dirichlet's theorem on primes in an arithmetic progression, we can find a set $\mathcal{Q}$ of primes with exactly α elements that for every $p \in \mathcal{Q}$, $p \equiv 1 \pmod 5$.

Let E be divisible by precisely the primes in $\mathcal{Q}$. Let $V \subset \mathbb{Z}$ be the collection of all possible products of elements of $\mathcal{Q}$; we include $1 \in V$. So the units of E are precisely the elements of the form $\pm\frac{y}{z}$, where $y, z \in V$. Clearly, 2 and 5 are relatively prime in $\mathbb{Z}$, and so in E. Assuming E^2 is perspective, find $x \in E$ as in Theorem 4.7. It follows that $x = \frac{c}{v}$ for some $c \in \mathbb{Z}$ and $v \in V$, and there are $x, y \in V$ such that

$$2 + 5\frac{c}{v} = \pm\frac{y}{z}.$$

It follows that $2vz + 5cz = \pm yv$. However, modulo 5, the left side of this equation is congruent to 2 and the right side is congruent to ± 1, giving our desired contradiction.

Another general result with important consequences is the following

Theorem 4.11. *Suppose E is a PID with only a finite number of primes (up to associates). Then E^2 is perspective, so that E^n is perspective for all $n < \omega$.*

Proof. Suppose $p_1, \dots, p_k$ is a list of all of the primes in E. Considering Theorem 4.7, let $a, b \in E$ be relatively prime. For each $j = 1, \dots, k$, we consider two cases. First, if p_j divides a, then it does not divide b, and we let $x_j = 1$. Otherwise, p_j does not divide a, and we let $x_j = 0$. In either case, we will be able to conclude that p_j does not divide $a + x_j b$. By the "Chinese Remainder Theorem" we can find $x \in E$ such that for each such j, x is congruent to x_j modulo p_j. It follows that $a + xb \in E$ will not be divisible by any p_j. Therefore, $a + xb$ will be a unit of E, so that by Theorem 4.7, E^2 is perspective. The last statement follows using (the forthcoming) Lemma 4.18, where we show that for a rational group G, if G^2 is perspective, then G^n is perspective whenever n is finite. □

Thus we infer

Corollary 4.12. *A torsion-free divisible group is perspective iff it has finite rank. As such, it is isomorphic to a finite direct sum of* $\mathbb{Q}$.

It was already mentioned that $\mathbb{Z}$ is perspective since it is indecomposable and that $\mathbb{Z} \oplus \mathbb{Z}$ is not perspective (see Remark 2.5). Hence

Proposition 4.13. *The only perspective free group is* $\mathbb{Z}$.

Since the rational groups (the subgroups of $\mathbb{Q}$) are also indecomposable it follows

Proposition 4.14. *The rational groups are perspective.*

Clearly

Proposition 4.15. *If a torsion-free group has a summand isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$*, it is not perspective.*

Next we characterize some perspective homogeneous completely decomposable groups.

Note that, similarly to Proposition 3.4, we can prove the following

Proposition 4.16. *Let* $G = \prod_{i \in I} H_i$*, where each subgroup* H_i *is fully invariant in* G*. Then* G *is perspective iff all* H_i*,* $i \in I$*, are perspective.*

Let p be a prime number and $\mathbb{Q}_p$ the group (ring) of all rational numbers with denominator coprime with p.

Another notable consequence of Theorem 4.11 is the following

Corollary 4.17. *Let* G *be a homogeneous completely decomposable group with type* $\boldsymbol{t}(G) = \boldsymbol{t}(\mathbb{Q}_p)$*. The group* G *is perspective iff* G *has finite rank.*

Remark. Since the pure subgroups of the group G in the previous proposition are summands of G, this is a trivial example of group with all pure subgroups being perspective.

The next result draws attention on rank 2 summands of homogeneous completely decomposable torsion-free groups of finite rank.

Lemma 4.18. *If* G *is a torsion-free homogeneous completely decomposable group of finite rank then* G *is perspective iff* G *has a perspective rank* 2 *summand.*

Proof. The condition is necessary since summands of perspective groups are perspective.

Conversely, since equal rank summands of G are isomorphic, it suffices to focus on any summand. Let $G = A \oplus B = C \oplus K$, where $A \cong C$. As in Corollary 4.17, we can assume that $A \cap C = A \cap K = 0$. Let $0 \neq a \in A$ and $a = c + x$, where $c \in C$, $x \in K$ $(c, x \neq 0)$. If $A_1 = \langle a \rangle_*$, $C_1 = \langle c \rangle_*$, $K_1 = \langle x \rangle_*$ then A_1, C_1 and K_1 are (rank 1) summands, so $A = A_1 \oplus A_2$, $C = C_1 \oplus C_2$, $K = K_1 \oplus K_2$ and $G = (C_1 \oplus K_1) \oplus (C_2 \oplus K_2)$. By hypothesis, $C_1 \oplus K_1 = C_1 \oplus W_1 = A_1 \oplus W_1$ for some W_1, so $G = C_1 \oplus (W_1 \oplus C_2 \oplus K_2) = A_1 \oplus (W_1 \oplus C_2 \oplus K_2)$. Since $A = A_1 \oplus A \cap (W_1 \oplus C_2 \oplus K_2)$, consider $A_2 = A \cap (W_1 \oplus C_2 \oplus K_2)$. Hence $W_1 \oplus C_2 \oplus K_2 = A_2 \oplus L$ for some L and we complete the proof by induction: $W_1 \oplus C_2 \oplus K_2 = C_2 \oplus W = A_2 \oplus W$ so $G = C_1 \oplus (W_1 \oplus C_2 \oplus K_2) = (C_1 \oplus C_2) \oplus W = A_1 \oplus (W_1 \oplus C_2 \oplus K_2) = (A_1 \oplus A_2) \oplus W$. □

Using Propositions 3.4 and 4.16 it follows that

Corollary 4.19. *Let $G = \bigoplus_{p \in P} G_p$ $(G = \prod_{p \in P} G_p)$, where P is some subset of prime numbers and G_p are homogeneous completely decomposable groups of finite rank with type $t(G_p) = t(\mathbb{Q}_p)$. Then the group G is perspective.*

Next, we describe the perspective torsion-free algebraically compact groups. Denote by $\hat{\mathbb{Z}}_p$ be the ring (group) of p-adic integers and by $\mathbb{P}$ the set of all prime numbers.

Proposition 4.20. *A non-zero torsion-free algebraically compact group G is perspective iff $G = \prod_{p \in \pi} G_p$, where $\varnothing \neq \pi \subseteq \mathbb{P}$ and G_p is a finite direct product of copies of the group $\hat{\mathbb{Z}}_p$.*

Proof. To show that the condition is necessary, first recall that any torsion-free algebraically compact group G has the form $G = \prod_{p \in \pi} G_p$, where G_p is a p-adic algebraically compact group (see [2], Proposition **40.1**), and in particular, is a $\hat{\mathbb{Z}}_p$-module. So the rank of each G_p is finite, i.e. G_p is a free $\hat{\mathbb{Z}}_p$-module of finite rank.

To show that the condition is sufficient, also recall that intersections of summands in torsion-free p-adic algebraically compact groups are also summands of this group. Next note that in the ring $\hat{\mathbb{Z}}_p$, all elements of $\hat{\mathbb{Z}}_p \setminus p\hat{\mathbb{Z}}_p$ are invertible, and that in any $\hat{\mathbb{Z}}_p$-module of finite rank, pure submodules are summands. As in Corollary 4.17, one can prove that p-adic algebraically compact modules of finite rank are perspective. Then using Proposition 4.16, the proof is complete. □

Example 4.21. Let $G = G_1 \oplus \cdots \oplus G_n$, where G_i, $1 \leq i \leq n$ are perspective groups and $\mathrm{Hom}(G_i, G_j) = 0$ for $i = 2, \ldots, n$ and $1 \leq j \leq i - 1$. Then G is perspective.

Proof. The proof goes by induction on n. If $G_2 \oplus \cdots \oplus G_n$ is perspective then since it is fully invariant in G, it is perspective by Corollary 2.4. □

5. Applications and examples

We present several examples and nonexamples illustrating the perspective property in various contexts.

Recall that a torsion-free group G is called *cohesive* if G/H is divisible for all pure subgroups $H \neq 0$ of G. For some facts about such groups see [3], **88**, exercise 17.

Example 5.1. A perspective direct sum of pure subgroups of a perspective group.

By Proposition 4.20, the group $G = \prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}_p$ is perspective. Let A_i, $i \in \{1, \ldots, n\}$, be pure subgroups of G such that $p^{\omega} A_i = 0$ for all $p \in \mathbb{P}$ and $r(A_i) = m_i$, where $1 \leq m_1 < \cdots < m_n \leq 2^{\aleph_0}$. Then A_i are cohesive groups (see [7], **32**). If $A = A_1 \oplus \cdots \oplus A_n$, then A is perspective.

Indeed, each A_i is perspective as indecomposable group and $\bigoplus_{j=i}^{n} A_j$ is fully invariant in A for all $i \in \{2, \ldots, n-1\}$. This follows from the fact that $\mathrm{Hom}(A_j, A_i) = 0$ for every $i < j$, owing that since $r(A_i) < r(A_j)$, each homomorphism $f : A_j \to A_i$ has a nonzero kernel and so $f(A_j)$ is divisible, whence $f = 0$.

In general, we can ask the following

Question. Which *pure subgroups* of a perspective group are perspective?

Example 5.2. A subgroup of a perspective group which is not perspective.

Let G be the torsion-free group of rank $n \geq 3$ as in [3], **88**, exercise 8. Then G is perspective as an indecomposable group but all the subgroups in G of rank $n-1$ are free. So the proper subgroups of rank ≥ 2 of G are not perspective.

Example 5.3. A factor group of perspective groups which is not perspective.

Let G be a torsion-free cohesive group with $r(G) \geq \aleph_0$ and let H be a pure subgroup of G such that $r(G/H) \geq \aleph_0$ (e.g., a pure subgroup H of rank 1). Then G/H is not perspective being divisible torsion-free group of infinite rank, but G and H are perspective being indecomposable groups.

Example 5.4. A group which is not perspective may have a perspective factor group.

Let $X = \{a_n : n \in \mathbb{N}^*\}$ and let $G = \langle X \rangle$ be free of countable rank. Consider the function $f : X \to \mathbb{Q}$, $f(a_n) = \dfrac{1}{n!}$ for every $n \in \mathbb{N}^*$. The group G being free, f extends to a group homomorphism $\overline{f} : G \to \mathbb{Q}$, obviously surjective (as $\mathbb{Q} = \left\langle \dfrac{1}{n!} : n \in \mathbb{N}^* \right\rangle$). So G is not perspective but $\mathbb{Q} \cong G/\ker(\overline{f})$ is perspective.

In closing, we mention another consequence of Theorem 4.11.

Corollary 5.5. *If G is a torsion-free homogeneous group of rank 1 such that G is divisible for all prime numbers except two coprime numbers p and q then $G \oplus G$ is perspective.*

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