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3×3 idempotent matrices over some domains and a conjecture on nil-clean matrices

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Abstract A characterization of the 3×3 idempotent matrices over some integral domains is given, in terms of determinant, trace and rank. The conjecture: every nil-clean 3×3 integral matrix is exchange, is revisited. Several new cases are proved.

Keywords exchange · nil-clean · clean · 3×3 integral matrix · similarity · diagonal reduction

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1 Introduction

Expressing the idempotency of a 3×3 matrix amounts to a quadratic system of 9 equations with 9 unknowns, which is clearly hard to handle. As examples in this note show, Cayley-Hamilton's theorem, which for a 3×3 matrix A is

$$A^3 - \mathrm{Tr}(A)A^2 + \frac{1}{2}(\mathrm{Tr}^2(A) - \mathrm{Tr}(A^2))A - \det(A)I_3 = 0_3,$$

does not characterize the idempotents. Therefore a characterization in terms of trace, determinant and rank could be useful.

We did not find any reference for a characterization of the 3×3 idempotent matrices, not over $\mathbb{Z}$, nor over more general conditions on the base ring. In this paper we complete this gap over some special integral (commutative) domains.

We say that a ring R is an *ID* ring (see [4]) if every idempotent matrix over R is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, PID's, elementary divisor rings, unit-regular rings and serial rings.

Recall (see [1]) that, since a matrix over an integral domain may be viewed over the corresponding field of fractions, the definition and properties of the rank are the usual ones, well-known from Linear Algebra.

Since diagonal idempotent matrices over domains have only 0 or 1 on the diagonal, and idempotency is invariant to conjugations (similarity, as for square matrices), it follows that a necessary condition for a matrix E (over an ID domain) to be idempotent is $\mathrm{rank}(E) = \mathrm{Tr}(E)$, that is, the rank equals the trace. Actually, this is the motive for considering in the sequel only matrices over ID domains.

An integral domain is a *GCD* domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$. GCD domains include unique factorization domains, Bezout domains and valuation domains.

In Section 2, our main result is the characterization of the idempotent 3×3 matrices over ID, GCD (commutative) domains (e.g. $\mathbb{Z}$). With this new tool in hand, in Section 3 and 4 we revisit a conjecture made in [2]: Every nil-clean 3×3 integral matrix is exchange.

2 The characterization

First recall the *Sylvester's rank inequality*: if F is a field and $A, B \in \mathbb{M}_n(F)$ then $\mathrm{rank}(A) + \mathrm{rank}(B) - n \leq \mathrm{rank}(AB)$.

As already mentioned, if R is an integral domain with quotient field F and $A \in \mathbb{M}_n(R)$, $\mathrm{rank}_R(A) = \mathrm{rank}_F(A)$ is the largest integer t such that A contains a $t \times t$ submatrix whose determinant is nonzero. Equivalently, this is the maximum number of linearly independent rows (or columns) of A. Therefore Sylvester's rank inequality holds for matrices over integral domains.

So is the *subadditivity* of the rank, that is, $\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$.

Next we mention a predictable.

Lemma 2.1 *Let R be a GCD (commutative) domain and let C_1, C_2 be two 3×1 nonzero columns. If C_1, C_2 are linearly dependent over R there exists a column C and elements $a_1, a_2 \in R$ such that $C_i = a_i C$, $i \in \{1, 2\}$.*

Proof. Denote $C_i = \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix}$, $i \in \{1, 2\}$ and assume $b_1 C_1 = b_2 C_2$ for some $0 \neq b_i \in R$, $i \in \{1, 2\}$. Without loss of generality, suppose $c_{11} \neq 0$ and so $c_{21} \neq 0$. Let $d_1 = \gcd(c_{11}; c_{21})$ and $c_{11} = l_1 d_1$, $c_{21} = l_2 d_1$ with $\gcd(l_1; l_2) = 1$.

Since l_1, l_2 are coprime, from $b_1 l_1 = b_2 l_2$, l_1 divides b_2 and l_2 divides b_1, say $b_1 = l_2 \alpha$, $b_2 = l_1 \beta$. From $b_1 l_1 = b_2 l_2$ it follows that $\alpha = \beta$. Further, since $b_1 c_{12} = b_2 c_{22}$, we obtain $l_2 c_{12} = l_1 c_{22}$. Again, since l_1, l_2 are coprime, l_1 divides c_{12} and l_2 divides c_{22}, which we can write (say), $c_{12} = l_1 d_2$ and $c_{22} = l_2 d_2$. Similarly, since $b_1 c_{13} = b_2 c_{23}$ we show that l_1 divides c_{13} and l_2

divides c_{23}, which we can write $c_{13} = l_1 d_3$ and $c_{23} = l_2 d_3$ for some $d_3 \in R$. Finally, if $C = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ then indeed, $C_i = l_i C$, as desired.

□

An analogous procedure, takes care of the case with three columns.

Recall that for any $n \times n$ matrix A, up to sign, the first three coefficients of the characteristic polynomial are 1, $\mathrm{Tr}(A)$, $\frac{1}{2}(\mathrm{Tr}^2(A) - \mathrm{Tr}(A^2))$ and the last is $\det(A)$. The third coefficient equals the sum of the diagonal 2×2 minors of A, and for $n = 3$ this is $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}$. To simplify the writing, this coefficient will be denoted by t or even t_A, if we need to emphasize the matrix A.

Now we can prove our main result.

Theorem 2.2 *A 3×3 matrix E over an ID, GCD domain R is* nontrivial *idempotent if and only if* $\det(E) = 0$, $\mathrm{rank}(E) = \mathrm{Tr}(E) = 1 + \frac{1}{2}(\mathrm{Tr}^2(E) - \mathrm{Tr}(E^2))$ *and* $\mathrm{rank}(E) + \mathrm{rank}(I_3 - E) = 3$.

Proof. Suppose $E = [e_{ij}]$, $1 \leq i, j \leq 3$. Then $t := t_E = e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} - e_{12}e_{21} - e_{13}e_{31} - e_{23}e_{32}$.

By Cayley-Hamilton's theorem, we can write

$$E^3 - \mathrm{Tr}(E)E^2 + tE - \det E \cdot I_3 = 0_3.$$

To show *the conditions are necessary*, suppose $E = E^2$. Then $\det(E)^2 = \det(E) \in \{0, 1\}$ and by replacement we get

$$(1 - \mathrm{Tr}(E) + t)E = \det E \cdot I_3.$$

We go into two cases.

If $1 - \mathrm{Tr}(E) + t \neq 0$, then E is a scalar matrix and we can show that $E \in \{0_3, I_3\}$. Indeed, either $\det(E) = 0$ and then $E = 0_3$, or else, $\det(E) = 1$ and if $E = aI_3$, the equality $E = E^2$ gives $a = a^2$ and since $\det E = 1$, $a = 1$ and $E = I_3$ follow.

In the remaining case, $1 - \mathrm{Tr}(E) + t = 0$ and so $\det(E) = 0$, i.e. all nontrivial idempotents satisfy these two (necessary) conditions.

As for the third condition, we use the Sylvester's rank inequality $\mathrm{rank}(E) + \mathrm{rank}(I_3 - E) - 3 \leq \mathrm{rank}(E(I_3 - E)) = 0$, for $\mathrm{rank}(E) + \mathrm{rank}(I_3 - E) \leq 3$ and the subadditivity $\mathrm{rank}(E + I_3 - E) = \mathrm{rank}(I_3) = 3 \leq \mathrm{rank}(E) + \mathrm{rank}(I_3 - E)$, for the opposite inequality.

Next, we show *the conditions are sufficient*. Since $\det(E) = 0$, $\mathrm{rank}(E) \leq 2$. Further, $\mathrm{Tr}(E) = 1 + t$ shows that $E \neq 0_3$, so $\mathrm{rank}(E) \in \{1, 2\}$.

In the first case, notice that if $\mathrm{rank}(E) = 1$ then $t = 0$ and so $\mathrm{Tr}(E) = 1$ follows from $\mathrm{Tr}(E) = 1 + t$.

In this case, by Cayley-Hamilton's theorem, we have $E^3 = E^2$ which generally does not imply $E^2 = E$ (see example 4 below).

However, if $\text{rank}(E) = \text{Tr}(E) = 1$, it does.

A 3×3 matrix A has rank 1 if and only if any two (say) columns are linearly dependent. As shown in the previous lemma, the columns are multiples of a common column. Simplifying the writing, we can suppose E has one of the three following forms: $[C, sC, vC]$, $[0, C, sC]$, $[0, 0, C]$ where s and v are elements of R and C is a column with at least one nonzero entry. If $C = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and we fulfill the condition $\text{Tr}(E) = 1$, it follows that E is in one of the following three forms:

$$E_1 = \begin{bmatrix} 1 - sa_2 - va_3 & s(1 - sa_2 - va_3) & v(1 - sa_2 - va_3) \\ a_2 & sa_2 & va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix},$$

$E_2 = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_3 & s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$. It can be checked that all these (rank 1) matrices are indeed, idempotent.

Notice that in this case, we do not use $\text{rank}(E) + \text{rank}(I_3 - E) = 3$.

In the second case, $\text{rank}(E) = \text{Tr}(E) = 2$ and $\text{Tr}(E) = 1 + t_E$ yields $t_E = 1$.

Observe that in this case $\text{Tr}(I_3 - E) = 3 - 2 = 1$ and $t_{I_3 - E} = t_E + 3 - 2\text{Tr}(E) = 0$.

Since $\text{rank}(E) = 2$ implies $\text{rank}(I_3 - E) = 1$ by the additional hypothesis, this case reduces to the first one. This is because, if $I_3 - E$ is idempotent, so is E (its complementary idempotent).

In this case, by Cayley-Hamilton's theorem, we have $E(E - I_3)^2 = 0_3$ which generally does not imply $E^2 = E$ (see example 5 below).

□

By E_{ij} we denote the 3×3 matrix with all entries zero excepting the (i, j) entry which is 1.

Examples. 1) $E_{11} + E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has trace 1 but rank 2 so it is *not* idempotent: the square is E_{11}.

2) $2E_{11} + E_{23} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2, but $t = 0$ so it is *not* idempotent: the square is $4E_{11}$.

3) $E = E_{11} + E_{22} + E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2 and also $t = 1$. Moreover, $I_3 - E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ so $\text{rank}(E) + \text{rank}(I_3 - E) = 2 + 1 = 3$. It is (indeed) idempotent.

4) Take $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Then $A^3 = A^2 = E_{11} + E_{33} \neq A$ (i.e. A is not idempotent) but $\text{Tr}(A) = 2 = \text{rank}(A)$, $t = 1$ but $\text{rank}(A) = \text{rank}(I_3 - A) = 2$.

5) The matrix $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2 and also $t = 1$. It verifies $C(C - I_3)^2 = 0_3$ but it is *not* idempotent. Again, $\text{rank}(C) = \text{rank}(I_3 - C) = 2$.

Actually, all matrices of type $C = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$ satisfy $\text{rank}(C) = \text{Tr}(C) = 2$ and $t = 1$ but $C^2 = \begin{bmatrix} 1 & 2a & b + ac \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} \neq C$ for many choices of a, b, c.

6) Observe that if $\text{char}(R) = 2$, there are idempotents $E \neq 0_3$ with $\det(E) = \text{Tr}(E) = 0$. An example is $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ with $E^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $\det E = \text{Tr}(E) = 0$ and $\text{Tr}(E^2) = 2 = 0$.

3 A conjecture revisited

In [2], we can find the following

Conjecture 3.1 Every nil-clean 3×3 integral matrix is exchange.

When writing the paper, this characterization of 3×3 idempotents was not known to the authors.

The characterization allows a different approach in order *to prove* this conjecture. Indeed, idempotents appear twice in this conjecture: in the definition of nil-clean matrices, i.e. these are sums of idempotents and nilpotents, and in the characterization of exchange elements, i.e. in a ring R, $a \in R$ is exchange if and only if there exists $m \in R$ (called *exchanger* in [2]) such that $a + m(a - a^2)$ is idempotent.

Since

Proposition 3.2 *Let R be any ring, $a \in R$, and suppose that $a = e + t$ where $e^2 = e$ and $t^2 = 0$. Then a is exchange in R.*

in the remaining nonzero case, we will assume the nilpotent, in the nil-clean decomposition of the matrix A, has index 3, i.e. $A = E + T$ with $E^2 = E$ and $T^2 \neq 0_3 = T^3$. As for E we can suppose it is *nontrivial* idempotent: indeed, nilpotents and unipotents are clean and so exchange.

Recall that every nilpotent matrix *over a field* is similar to a block diagonal matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes). Actually, this form is a special case of the Jordan canonical form for matrices. A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix.

The following result is proved in [3]:

Theorem 3.3 *The following are equivalent for a ring R:*
(i) Every nilpotent matrix over R is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).
(ii) R is a division ring.

In the sequel, we prove the conjecture for all nil-clean matrices whose nilpotent (of index 3) is similar to the 3×3 shift.

This is a special case (over $\mathbb{Z}$), because over any commutative domain D, there are plenty of nilpotent nonzero matrices which are not similar to the corresponding shift. For example, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is a nonzero nilpotent of $\mathbb{M}_2(\mathbb{Z})$ which is not similar to E_{12}, the nonzero 2×2 shift.

However, it can be proved that

Proposition 3.4 *Every nonzero nilpotent 2×2 matrix over a commutative GCD domain R is similar to rE_{12}, for some $r \in R$.*

Proof. We are looking for an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and $x^2 + yz = 0$.

Let $d = \gcd(x; y)$ and denote $x = dx_1$, $y = dy_1$ with $\gcd(x_1; y_1) = 1$. Then $d^2x_1^2 = -dy_1z$ and since $\gcd(x_1; y_1) = 1$ implies $\gcd(x_1^2; y_1) = 1$, it follows y_1 divides d. Set $d = y_1y_2$ and so $T = \begin{bmatrix} x_1y_1y_2 & y_1^2y_2 \\ -x_1^2y_2 & -x_1y_1y_2 \end{bmatrix} = y_2 \begin{bmatrix} x_1y_1 & y_1^2 \\ -x_1^2 & -x_1y_1 \end{bmatrix} = y_2T'$.

Since $\gcd(x_1; y_1) = 1$ there exist $s, t \in R$ such that $sx_1 + ty_1 = 1$. Take $U = \begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}$ which is invertible (indeed, $U^{-1} = \begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}$). One can check $T'U = \begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix} = UE_{12}$, so $r = y_2$.

□

The 3×3 analogue is

Proposition 3.5 *Every index 3 nilpotent* 3×3 *matrix over a GCD domain* R *is similar to* $rE_{12} + uE_{23}$*, for some* $r, u \in R$.

Notice that the possible nonzero 3×3 block diagonal matrices with each block a shift matrix are $S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $S' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where S' has index two and only S has index three ($S^2 = E_{13} \neq 0_3$).

Here is what we prove

Theorem 3.6 *The nil-clean* 3×3 *integral matrices whose nilpotent (of index 3) is similar to the shift* S*, are exchange.*

Proof. For $A = E + S$ we have to find an exchanger M such that $A + M(A - A^2)$ is an idempotent. As observed in the previous section, it suffices to consider E any (nontrivial) trace = rank = 1, 3×3 idempotent matrix. Also noticed in the previous section, it suffices to find exchangers for E, any of the following matrices: $[0, 0, C]$, $[0, C, sC]$, $[C, sC, vC]$ where s and v are some integers and C is a column with at least one nonzero entry.

There are three cases to discuss.

Case 1. The idempotent is of form $[0, 0, C]$, that is, $E = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 1+b \\ 0 & 0 & 1 \end{bmatrix}$. Here $ES = 0_3$, $SE = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = E + SE + E_{13} = \begin{bmatrix} 0 & 0 & a+b+1 \\ 0 & 0 & b+1 \\ 0 & 0 & 1 \end{bmatrix}$, $A - A^2 = \begin{bmatrix} 0 & 1 & -1-b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Denoting $M = [m_{ij}]$, $1 \leq i, j \leq 3$ we get $A + M(A - A^2) = \begin{bmatrix} 0 & 1+m_{11} & a-(1+b)m_{11} \\ 0 & m_{21} & (1+b)(1-m_{21}) \\ 0 & m_{31} & 1-(1+b)m_{31} \end{bmatrix}$. We choose $m_{21} = m_{31} = 0$ in order to have trace $= 1$, and $m_{11} = -1$ in order to vanish the second column. Since the second and third columns of M play no rôle, we choose these zero. Hence for $M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A + M(A - A^2) = \begin{bmatrix} 0 & 0 & a+b+1 \\ 0 & 0 & b+1 \\ 0 & 0 & 1 \end{bmatrix}$ which is indeed idempotent *of the same type as* E.

Case 2. Take $E = [0, C, sC] = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1-sa_3 & s(1-sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}$. Now $A - A^2 = (E+S)-(E+S)^2 = S - E_{13} - ES - SE = \begin{bmatrix} 0 & sa_3 & -1-a_1-s(1-sa_3) \\ 0 & -a_3 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}$ and, denoting $b := -1 - a_1 - s(1-sa_3)$ we obtain

$M(A-A^2) = \begin{bmatrix} 0 & (m_{11}s - m_{12})a_3 & m_{11}b - m_{13}a_3 \\ 0 & (m_{21}s - m_{22})a_3 & m_{21}b - m_{23}a_3 \\ 0 & (m_{31}s - m_{32})a_3 & m_{31}b - m_{33}a_3 \end{bmatrix}$.

Here $\mathrm{Tr}(M(A-A^2)) = (m_{21}s - m_{22} - m_{33})a_3 + m_{31}b$. An exchanger must be found for arbitrary a_1, a_3 and s. For any choice such that a_3 and b are *not* coprime, there are no m_{ij}'s such that $\mathrm{Tr}(M(A-A^2)) = 1$ (e.g., $a_1 = -3$, $a_3 = 2$, $s = 0$ and so $b = 2$).

Hence the m_{ij}'s must be chosen to give $\mathrm{Tr}(M(A-A^2)) = 0$ for arbitrary a_1, a_3 and s. Hence

$$m_{21}s = m_{22} + m_{33} \text{ and } m_{31} = 0\,.$$

Moreover, since then $\mathrm{Tr}(A + M(A-A^2)) = 1$ we also need $\mathrm{rank}(A + M(A-A^2)) = 1$.

Here $A + M(A-A^2) =$
$\begin{bmatrix} 0 & 1 + a_1 + (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 0 & 1 + (m_{33} - s)a_3 & 1 + s(1-sa_3) + m_{21}b - m_{23}a_3 \\ 0 & (1-m_{32})a_3 & (s - m_{33})a_3 \end{bmatrix}$
has trace 1. For rank 1, we need dependent columns (or rows).

We will choose the other entries in the third row of M, in order to have zero 3-rd row in $A + M(A-A^2)$, that is $m_{32} = 1$ and $m_{33} = s$.

Then $m_{22} = (m_{21} - 1)s$ and $A + M(A-A^2) =$
$\begin{bmatrix} 0 & 1 + a_1 + (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 0 & 1 & 1 + s(1-sa_3) + m_{21}b - m_{23}a_3 \\ 0 & 0 & 0 \end{bmatrix}$
and we have to choose $m_{11}, m_{12}, m_{13}, m_{21}$ and m_{23} in order to get the rank 1, that is,

$\det \begin{bmatrix} 1 + a_1 + (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 1 & 1 + s(1-sa_3) + m_{21}b - m_{23}a_3 \end{bmatrix} = 0.$

Equivalently, $sa_1 + m_{11}b - m_{13}a_3 = [1 + a_1 + (m_{11}s - m_{12})a_3][1 + s(1-sa_3) + m_{21}b - m_{23}a_3]$.

Further we choose

$$m_{21} = 1 \text{ and } m_{11} = a_1$$

(and so $m_{22} = 0$). The equality reduces to $sa_1 - a_1[1 + a_1 + s(1-sa_3)] - m_{13}a_3 = [1 + a_1 + (a_1 s - m_{12})a_3](-a_1 - m_{23}a_3)$ and, by taking

$$m_{23} = 0$$

to (dividing by a_3) $m_{13} = s^2 a_1 + s a_1^2 - m_{12} a_1$ with infinitely many possible choices for m_{12}. For

$$m_{12} = 0$$

we get $m_{13} = s a_1 (s + a_1)$.

Hence finally $M = \begin{bmatrix} a_1 & 0 & s a_1(s+a_1) \\ 1 & 0 & 0 \\ 0 & 1 & s \end{bmatrix}$ and

$A + M(A - A^2) = \begin{bmatrix} 0 & 1 + a_1 + s a_1 a_3 & -a_1(1 + a_1 + s a_1 a_3) \\ 0 & 1 & -a_1 \\ 0 & 0 & 0 \end{bmatrix}$. The conditions in Theorem 2.2 can be easily checked: rank = trace =1, $t = 0$ and

rank ($\begin{bmatrix} \mathbf{-1} & 1 + a_1 + s a_1 a_3 & -a_1(1 + a_1 + s a_1 a_3) \\ 0 & 0 & -a_1 \\ \mathbf{0} & 0 & \mathbf{-1} \end{bmatrix}$) = 2.

One can verify directly that $\begin{bmatrix} 0 & c & -a_1 c \\ 0 & 1 & -a_1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & c & -a_1 c \\ 0 & 1 & -a_1 \\ 0 & 0 & 0 \end{bmatrix}$, so matrices of this form are indeed idempotent, *of the same type as E*.

Case 3. Take $E = [C, sC, vC] =$

$= \begin{bmatrix} 1 - s a_2 - v a_3 & s(1 - s a_2 - v a_3) & v(1 - s a_2 - v a_3) \\ a_2 & s a_2 & v a_2 \\ a_3 & s a_3 & v a_3 \end{bmatrix}$ and so

$A = E + S = \begin{bmatrix} 1 - s a_2 - v a_3 & 1 + s(1 - s a_2 - v a_3) & v(1 - s a_2 - v a_3) \\ a_2 & s a_2 & 1 + v a_2 \\ a_3 & s a_3 & v a_3 \end{bmatrix}$. As above $A - A^2 = S - E_{13} - ES - SE =$

$\begin{bmatrix} -a_2 & v a_3 & -1 - s(1 - s a_2 - v a_3) - v a_2 \\ -a_3 & -a_2 - s a_3 & 1 - s a_2 - v a_3 \\ 0 & -a_3 & -s a_3 \end{bmatrix}$ and denoting $b = 1 - s a_2 - v a_3$,

$= \begin{bmatrix} -a_2 & v a_3 & -1 - sb - v a_2 \\ -a_3 & -a_2 - s a_3 & b \\ 0 & -a_3 & -s a_3 \end{bmatrix}$.

Finally the columns of $A + M(A - A^2)$ are

$\begin{bmatrix} 1 - s a_2 - v a_3 - m_{11} a_2 - m_{12} a_3 \\ a_2 - m_{21} a_2 - m_{22} a_3 \\ a_3 - m_{31} a_2 - m_{32} a_3 \end{bmatrix}$,

$\begin{bmatrix} \mathbf{1} + s(1 - s a_2 - v a_3) + m_{11} v a_3 - m_{12}(a_2 + s a_3) - m_{13} a_3 \\ s a_2 + m_{21} v a_3 - m_{22}(a_2 + s a_3) - m_{23} a_3 \\ s a_3 + m_{31} v a_3 - m_{32}(a_2 + s a_3) - m_{33} a_3 \end{bmatrix}$ and

$\begin{bmatrix} v(1 - s a_2 - v a_3) - m_{11}(1 + sb + v a_2) + m_{12} b - m_{13} s a_3 \\ \mathbf{1} + v a_2 - m_{21}(1 + sb + v a_2) + m_{22} b - m_{23} s a_3 \\ v a_3 - m_{31}(1 + sb + v a_2) + m_{32} b - m_{33} s a_3 \end{bmatrix}$.

Using computer aid, we chose $M = \begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$.

Replacing we get $A + M(A - A^2) = \begin{bmatrix} & & \\ 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$ with (so far) the same first row.

Moreover with $m_{11} = -s$, $m_{12} = -v$ we obtain $A + M(A - A^2) =$
$\begin{bmatrix} 1 & 1 + s + (v - s^2)a_2 - (sv+m_{13})a_3 & s[1 + s + (v - s^2)a_2 - (sv+m_{13})a_3] \\ 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$.

Finally m_{13} is arbitrary since matrices of type $\begin{bmatrix} 1 & \alpha & s\alpha \\ 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$ are idempotent for any α. Indeed $\mathrm{Tr}(A+M(A-A^2)) = \mathrm{rank}(A+M(A-A^2)) = 2$, $t = 1$ and $\mathrm{Tr}(I_3 - A - M(A - A^2)) = \mathrm{rank}(I_3 - A - M(A - A^2)) = 1$. Therefore (choosing $m_{13} = 0$) the exchanger in this case is $M = \begin{bmatrix} -s & -v & 0 \\ 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$.

□

Example. For $A = \begin{bmatrix} -13 & -25 & -39 \\ 1 & 2 & 4 \\ 4 & 8 & 12 \end{bmatrix}$ and $M = \begin{bmatrix} -2 & -3 & -3 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ we have $M(A - A^2) = \begin{bmatrix} 14 & 15 & 19 \\ -1 & 0 & -2 \\ -4 & -9 & -13 \end{bmatrix}$, $A + M(A - A^2) = \begin{bmatrix} 1 & -10 & 20 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$ (here $a_2 = 1$, $a_3 = 4$, $s = 2$, $v = 3$; $b = -13$).

As already noticed in the previous section, any 3×3 index 3 nilpotent is similar to a generalized shift $S_g = rE_{12} + uE_{23}$.

In trying to prove the (whole) conjecture, one has to replace the shift S by S_g.

We were able to do this in the first case of the previous proof, and made some progress with the second and third case.

Proposition 3.7 *The nil-clean* 3×3 *integral matrices with idempotent of form* $[0, 0, C]$ *are exchange.*

Proof. The proof goes along the lines of the (previous) special case $r = v = 1$. Take $A = E + S_g$ with $E = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$ and $S_g = \begin{bmatrix} 0 & r & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}$ ($S_g^2 = ruE_{13}$).

Then $ES_g = 0_3$, $S_gE = \begin{bmatrix} 0 & 0 & rb \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}$, $A - A^2 = \begin{bmatrix} 0 & r & -r(u+b) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Denoting $M = [m_{ij}]$, $1 \leq i,j \leq 3$ we get

$A + M(A - A^2) = \begin{bmatrix} 0 & r + rm_{11} & a - r(u+b)m_{11} \\ 0 & rm_{21} & b + s - r(u+b)m_{21} \\ 0 & rm_{31} & 1 - r(u+b)m_{31} \end{bmatrix}$. We choose $m_{21} = m_{31} = 0$ in order to have trace = 1, and $m_{11} = -1$ in order to vanish the second column. Since the second and third columns of M play no rôle, we choose these zero. Hence for $M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (the same exchanger), $A + M(A - A^2) = \begin{bmatrix} 0 & 0 & a + r(b+u) \\ 0 & 0 & b+u \\ 0 & 0 & 1 \end{bmatrix}$ which is an idempotent *of the same type as E*.

□

4 The other cases

The second (general) case. $E = [0, C, sC] = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_3 & s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}$ and $A = E + S_g$, again going along the lines of the previous proof, the following can be done.

For $S_g = \begin{bmatrix} 0 & r & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}$, $S_g^2 = ruE_{13}$, $A = \begin{bmatrix} 0 & r + a_1 & sa_1 \\ 0 & 1 - sa_3 & u + s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}$, $ES_g = \begin{bmatrix} 0 & 0 & ua_1 \\ 0 & 0 & u(1 - sa_3) \\ 0 & 0 & ua_3 \end{bmatrix}$ and $S_gE = \begin{bmatrix} 0 & r(1 - sa_3) & rs(1 - sa_3) \\ 0 & ua_3 & usa_3 \\ 0 & 0 & 0 \end{bmatrix}$.

So $A - A^2 = S_g - ruE_{13} - ES_g - S_gE =$
$\begin{bmatrix} 0 & rsa_3 & -ua_1 - rs(1 - sa_3) - ru \\ 0 & -ua_3 & 0 \\ 0 & 0 & -ua_3 \end{bmatrix}$. Denoting $M = [m_{ij}]$, $1 \leq i,j \leq 3$ and $b = -ua_1 - rs(1 - sa_3) - ru$ we get

$M(A - A^2) = \begin{bmatrix} 0 & (m_{11}rs - m_{12}u)a_3 & m_{11}b - m_{13}ua_3 \\ 0 & (m_{21}rs - m_{22}u)a_3 & m_{21}b - m_{23}ua_3 \\ 0 & (m_{31}rs - m_{32}u)a_3 & m_{31}b - m_{33}ua_3 \end{bmatrix}$ and $A + M(A - A^2) =$

$$\begin{bmatrix} 0 & r + a_1 + (m_{11}rs - m_{12}u)a_3 & sa_1 + m_{11}b - m_{13}ua_3 \\ 0 & 1 - sa_3 + (m_{21}rs - m_{22}u)a_3 & u + s(1 - sa_3) + m_{21}b - m_{23}ua_3 \\ 0 & a_3 + (m_{31}rs - m_{32}u)a_3 & sa_3 + m_{31}b - m_{33}ua_3 \end{bmatrix}.$$

Here $\mathrm{Tr}(M(A - A^2)) = (m_{21}rs - m_{22}u - m_{33}u)a_3 + m_{31}b$. An exchanger must be found for arbitrary a_1, a_3 and s. For any choice such that a_3 and

b are *not* coprime, there are no m_{ij}'s such that $\mathrm{Tr}(M(A-A^2)) = 1$ (e.g., $a_1 = -3$, $a_3 = 2$ and $s = 0$: $b = 2$).

Hence the m_{ij}'s must be chosen to give $\mathrm{Tr}(M(A-A^2)) = 0$ for arbitrary a_1, a_3 and s. Hence

$$m_{21}rs = (m_{22} + m_{33})v \text{ and } m_{31} = 0\,.$$

Moreover, since then $\mathrm{Tr}(A + M(A - A^2)) = 1$ we also need $\mathrm{rank}(A + M(A - A^2)) = 1$.

Here $A + M(A - A^2) =$
$$\begin{bmatrix} 0 & r + a_1 + (m_{11}rs - m_{12}u)a_3 & sa_1 + m_{11}b - m_{13}ua_3 \\ 0 & 1 + (m_{33}u - s)a_3 & u + s(1 - sa_3) + m_{21}b - m_{23}ua_3 \\ 0 & (1 - m_{32}u)a_3 & (s - m_{33}u)a_3 \end{bmatrix}$$
has trace 1. For rank 1, we need dependent columns (or rows).

This reduces to
$$\det \begin{bmatrix} r + a_1 + (m_{11}rs - m_{12}u)a_3 & sa_1 + m_{11}b - m_{13}ua_3 \\ (1 - m_{32}u)a_3 & (s - m_{33}u)a_3 \end{bmatrix} =$$
$$\det \begin{bmatrix} 1 + (m_{33}u - s)a_3 & u + s(1 - sa_3) + m_{21}b - m_{23}ua_3 \\ (1 - m_{32}u)a_3 & (s - m_{33}u)a_3 \end{bmatrix} = 0, \text{ that is}$$
$[r + a_1 + (m_{11}rs - m_{12}u)a_3](s - m_{33}u) = [sa_1 + m_{11}b - m_{13}ua_3](1 - m_{32}u)$ and
$[1 + (m_{33}u - s)a_3](s - m_{33}u) = [u + s(1 - sa_3) + m_{21}b - m_{23}ua_3](1 - m_{32}u)$ [both equalities divided by a_3].

Notice that, as in the special $r = u = 1$ case, the vanishing of the third row of $A + M(A - A^2)$ cannot be done, unless $u = 1$.

We were not able to determine the entries of a suitable exchanger.

By computer aid, the third row of M, $[0, m_{32}, m_{33}]$ could be $[0, 1, 1]$ or $[0, 1, s]$ or $[0, 1, 0]$ or some others. In each case, computation yields a complementary condition on a_1, a_3, r, u and s.

Trying to find a *counterexample for the conjecture*, with $E = [0, C, sC]$ and $S_g = rE_{12} + uE_{23}$, we have successively gathered the following *non-conditions*:

$u \neq 1$, u not dividing s, a_3 not dividing u, a_3 not dividing rs, a_3 not dividing $u + s - 2$, $s + u \neq a_3$ and a_3 not dividing $ua_1 - 1$.

The selection $a_1 = 2$, $a_3 = 7$, $s = 3$, $r = 2$ and $u = 5$ satisfies all these. The resulting 3×3 matrix is $A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & -20 & -55 \\ 0 & 7 & 21 \end{bmatrix}$ which still is exchange: among the exchangers we find $\begin{bmatrix} -1 & x & y \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$ with $[x, y] \in \{[-2, -2], [2, -5], [-6, 1]\}$.

Next attempt: $m_{32} = 1$, $m_{33} = 0 = m_{21}$.

The second equation: $(1 - sa_3)s = [v + s(1 - sa_3) + m_{21}b - m_{23}ua_3](1 - v)$ or $v + m_{21}b - m_{23}ua_3 = u[u + s(1 - sa_3) + m_{21}b - m_{23}ua_3]$

If $m_{21}=0$ (as in example), $1-m_{23}a_3=u+s(1-sa_3)-m_{23}ua_3$ (divided by u). Or $(1-u)(1-m_{23}a_3)=s(1-sa_3)$ so now $1-u$ divides $s(1-sa_3)$.

Here $a_1=2$, $a_3=7$, $s=3$, $r=2$ and $u=5$: indeed 4 divides 20.

So we add *another non-condition*: $1-u$ not dividing $s(1-sa_3)$: $u=10$. So $A=\begin{bmatrix}0 & 4 & 6\\ 0 & -20 & -50\\ 0 & 7 & 21\end{bmatrix}$.

Nothing until $z=6$ (inclusive), but for $z=7$ we found $M=\begin{bmatrix}6&3&6\\0&3&4\\7&2&5\end{bmatrix}$, but also $M=\begin{bmatrix}7 & x & y\\ -5 & -5 & -7\\ -7 & 1 & -6\end{bmatrix}$ $[x,y]\in\{[-5,5],[-2,-6],[1,7]\}$.

We did not continue our attempts in this case.

The third case. The computation goes along the lines of the $r=u=1$ case. $A=E+S_g=\begin{bmatrix}1-sa_2-va_3 & r+s(1-sa_2-va_3) & v(1-sa_2-va_3)\\ a_2 & sa_2 & u+va_2\\ a_3 & sa_3 & va_3\end{bmatrix}$,

$ES_g=\begin{bmatrix}0 & r(1-sa_2-va_3) & us(1-sa_2-va_3)\\ 0 & ra_2 & usa_2\\ 0 & ra_3 & usa_3\end{bmatrix}$,

$S_gE=\begin{bmatrix}ra_2 & rsa_2 & rva_2\\ ua_3 & usa_3 & uva_3\\ 0 & 0 & 0\end{bmatrix}$ and $A-A^2=S_g-ruE_{13}-ES_g-S_gE=$

$\begin{bmatrix}-ra_2 & rva_3 & -us(1-sa_2-va_3)-rva_2-ru\\ -ua_3 & -ra_2-usa_3 & u(1-sa_2-va_3)\\ 0 & -ra_3 & -usa_3\end{bmatrix}$.

Denoting $b=u(1-sa_2-va_3)$ we have

$A-A^2=\begin{bmatrix}-ra_2 & rva_3 & -sb-rva_2-ru\\ -ua_3 & -ra_2-usa_3 & b\\ 0 & -ra_3 & -usa_3\end{bmatrix}$. Denoting $M=[m_{ij}]$, $1\le i,j\le 3$ the columns

of $A+M(A-A^2)$ are

$\begin{bmatrix}1-sa_2-va_3-m_{11}ra_2-m_{12}ua_3\\ a_2-m_{21}ra_2-m_{22}ua_3\\ a_3-m_{31}ra_2-m_{32}ua_3\end{bmatrix}$,

$\begin{bmatrix}r+s(1-sa_2-va_3)+m_{11}rva_3-m_{12}(ra_2+usa_3)-m_{13}ra_3\\ sa_2+m_{21}rva_3-m_{22}(ra_2+usa_3)-m_{23}ra_3\\ sa_3+m_{31}rva_3-m_{32}(ra_2+usa_3)-m_{33}ra_3\end{bmatrix}$,

and $\begin{bmatrix}v(1-sa_2-va_3)-m_{11}(sb+rva_2+ru)+m_{12}b-m_{13}usa_3\\ u+va_2-m_{21}(sb+rva_2+ru)+m_{22}b-m_{23}usa_3\\ va_3-m_{31}(sb+rva_2+ru)+m_{32}b-m_{33}usa_3\end{bmatrix}$.

Continuation with $M = \begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$ is not very bad but *seems not likely* [unlikely to get rank=trace =2]: $A + M(A - A^2) =$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ (1-r)a_2 & sa_2 & (-s+r-1)u + [s^2u + (1-r)v]a_2 \\ (1-u)a_3 & -ra_2 + (1-u)sa_3 & u(1-sa_2) + (1-u)va_3 \end{bmatrix}.$$

For $r = u = 1$ this was already $A + M(A - A^2) = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & sa_2 & -s(1-sa_2) \\ 0 & -a_2 & 1-sa_2 \end{bmatrix}$.

We did not continue our attempts in this case.

In trying to find a counterexample for the conjecture, we made the following selection:

Example. $A = E + S_g =$

$\begin{bmatrix} -13 & -26 & -39 \\ 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -13 & -21 & -39 \\ 1 & 2 & 9 \\ 4 & 8 & 12 \end{bmatrix}$ no exchanger until (incl.) $z = 11$. Here $a_2 = 1, a_3 = 4, s = 2, v = 3, r = 5, u = 6$ and $b = u(1 - sa_2 - va_3) = -78$. Now $A - A^2 =$

$$\begin{bmatrix} -ra_2 & rva_3 & -sb - rva_2 - ru \\ -ua_3 & -ra_2 - usa_3 & b \\ 0 & -ra_3 & -usa_3 \end{bmatrix} = \begin{bmatrix} -5 & 60 & 111 \\ -24 & -53 & -78 \\ 0 & -20 & -48 \end{bmatrix}.$$

Denoting $M = [m_{ij}]$, $1 \leq i, j \leq 3$ we get $M(A - A^2) =$

$$\begin{bmatrix} -5m_{11} - 24m_{12} & 60m_{11} - 53m_{12} - 20m_{13} & 111m_{11} - 78m_{12}b - 48m_{13} \\ -5m_{21} - 24m_{22} & 60m_{21} - 53m_{22} - 20m_{23} & 111m_{21} - 78m_{22}b - 48m_{23} \\ -5m_{31} - 24m_{32} & 60m_{31} - 53m_{32} - 20m_{33} & 111m_{31} - 78m_{32}b - 48m_{33} \end{bmatrix}$$

and the columns of $D := A + M(A - A^2)$ are

$\begin{bmatrix} -13 - 5m_{11} - 24m_{12} \\ 1 - 5m_{21} - 24m_{22} \\ 4 - 5m_{31} - 24m_{32} \end{bmatrix}$, $\begin{bmatrix} -21 + 60m_{11} - 53m_{12} - 20m_{13} \\ 2 + 60m_{21} - 53m_{22} - 20m_{23} \\ 8 + 60m_{31} - 53m_{32} - 20m_{33} \end{bmatrix}$ and

$\begin{bmatrix} -39 + 111m_{11} - 78m_{12}b - 48m_{13} \\ 9 + 111m_{21} - 78m_{22}b - 48m_{23} \\ 12 + 111m_{31} - 78m_{32}b - 48m_{33} \end{bmatrix}$.

The trace is

$\mathrm{Tr}(D) = 1 + \mathrm{Tr}(M(A - A^2) = 1 - 5m_{11} - 24m_{12} + 60m_{21} - 53m_{22} - 20m_{23} + 111m_{31} - 78m_{32}b - 48m_{33}$.

$\mathrm{Tr}(I_3 - D) = 2 - \mathrm{Tr}(M(A - A^2) = 2 - (-5m_{11} - 24m_{12} + 60m_{21} - 53m_{22} - 20m_{23} + 111m_{31} - 78m_{32}b - 48m_{33})$.

How to prove this cannot be idempotent ?

In [2], the nil-clean matrices discussed had (by similarity) the idempotent E_{11} or $E_{11} + E_{22}$.

Since $\mathrm{Tr}(E) = \mathrm{rank}(E) = 1$, E is similar to E_{11}. We look for a conjugation.

$EU = UE_{11}$ amounts to

$$\begin{bmatrix} -13(u_{11}+2u_{21}+3u_{31}) & -13(u_{12}+2u_{22}+3u_{32}) & -13(u_{13}+2u_{23}+3u_{33}) \\ u_{11}+2u_{21}+3u_{31} & u_{12}+2u_{22}+3u_{32} & u_{13}+2u_{23}+3u_{33} \\ 4(u_{11}+2u_{21}+3u_{31}) & 4(u_{12}+2u_{22}+3u_{32}) & 4(u_{13}+2u_{23}+3u_{33}) \end{bmatrix}$$

$= \begin{bmatrix} u_{11} & 0 & 0 \\ u_{21} & 0 & 0 \\ u_{31} & 0 & 0 \end{bmatrix}$ with $\det(U) = \pm 1$. Hence

$-13(u_{11}+2u_{21}+3u_{31}) = u_{11}$ or $14u_{11}+26u_{21}+39u_{31} = 0$

$u_{11}+2u_{21}+3u_{31} = u_{21}$ or $u_{11}+u_{21}+3u_{31} = 0$

$4(u_{11}+2u_{21}+3u_{31}) = u_{31}$ or $4u_{11}+8u_{21}+11u_{31} = 0$

and

$u_{12}+2u_{22}+3u_{32} = u_{13}+2u_{23}+3u_{33} = 0$.

The first 3 equation form a homogeneous linear system with zero determinant, so we can choose only

$u_{11}+u_{21}+3u_{31} = 0$ (multiplied by -4 and added to the next)

$4u_{11}+8u_{21}+11u_{31} = 0$ or

$4u_{21} = u_{31}$ and $u_{11} = -13u_{21}$.

An example is $U = \begin{bmatrix} -13 & 2 & -1 \\ 1 & -1 & -1 \\ 4 & 0 & 1 \end{bmatrix}$ for which $U^{-1}E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $U^{-1}EU = E_{11}$.

Then the similar nil-clean matrix with E_{11} idempotent is $A' = E_{11} + U^{-1}S_gU = E_{11} + \begin{bmatrix} 53 & -5 & 7 \\ 241 & -25 & 29 \\ -212 & 20 & -28 \end{bmatrix}$.

Here $S_g^2 = \begin{bmatrix} 120 & 0 & 30 \\ 600 & 0 & 150 \\ -480 & 0 & -120 \end{bmatrix}$ and (indeed) $S_g^3 = 0_3$.

However, for $A' = \begin{bmatrix} 53 & -5 & 7 \\ 241 & -25 & 29 \\ -212 & 20 & -28 \end{bmatrix}$, an exchanger was fast found for $z = 6$: $M = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -1 & 6 \\ -4 & 0 & -1 \end{bmatrix}$.

The idempotent is $A' + M(A' - A'^2) = \begin{bmatrix} -119 & -5 & -23 \\ 2856 & 120 & 552 \\ 0 & 0 & 0 \end{bmatrix}$.

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