Exceptional units in matrix rings

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Abstract

A unit u in a ring R is called exceptional if $1-u$ is also a unit. Such units were studied from the Number Theory point of view. In this paper, these are studied from Ring Theory point of view with special emphasis to matrix rings, where several characterizations are given. Replacing units by exceptional units, a special class of clean elements (called ex-clean) is defined. Among other things, ex-clean 2×2 integral matrices are characterized.

1 Introduction

There are three especially important sets of elements in Ring Theory: $\mathrm{Idem}(R)$, the idempotents, $N(R)$, the nilpotents and $U(R)$, the units in a ring R.

For some element a in any of these sets we may wonder if $1-a$ belongs to the same set.

The answer is *affirmative for idempotents* ($1-a$ is the complementary idempotent), is *negative for nilpotents* ($1-a$ is a unit) and for units gives rise of the following

Definition. A unit $u \in U(R)$ is called *exceptional* (*exunit*, for short) if $1-u \in U(R)$. In the sequel we denote by $U_e(R)$ the subset of exunits in R. Equivalently, there is a unit v such that $u+v=1$.

Note that such questions were already addressed in Ring Theory for some other type of elements. The answer is affirmative for nil-clean or clean or exchange elements but may fail for (unit) regular or weakly clean (see [13]) elements.

Considering two units u, v of sum 1 (or the equation $1+u+v=0$) is not new. This equation was considered by Nagell in a series of papers (1959, 1960, 1964, 1968, 1969) where the solvability of this equation in an algebraic extension of the rationals is studied. The solvability is proved (for instance) in the cases of a quadratic extension, a cubic extension with negative discriminant, or a quartic extension satisfying certain conditions. It is also proved (1964, but also proved

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independently in 1961 by S. Chowla) that for an arbitrary algebraic extension, the equation has only finitely many solutions in units u, v.

Thus, Nagell studied exunits from *Number Theory point of view.*

After a 45 years break, recently, exunits were studied in $\mathbb{Z}_n$ and (as a small generalization) in finite commutative rings (see [12], [16], [6]) again from Number Theory point of view. In [12] for instance, Sander determines the number of representations of an arbitrary element of $\mathbb{Z}_n$ as the sum of two exunits, that is, the *sumset* $U_e(\mathbb{Z}_n) + U_e(\mathbb{Z}_n)$ is obtained.

In this paper we study exunits from *Ring Theory point of view*, with a special emphasis to matrix rings.

In Section 2, general facts about exunits in arbitrary (unital) rings are given, while in Section 3 several characterizations of exunits in matrix rings are given. In Section 4, replacing the units in the definition of clean elements in rings, by exunits we define the ex-clean elements, give numerous examples and characterize the 2×2 integral ex-clean matrices.

In order not to further lengthen this paper (and somehow keep the subject essentially about matrices), in the last section some comments and research directions (for rings) are given. These will be addressed in a subsequent study.

All the rings we consider are (associative) with identity. By a triangular matrix we mean an upper triangular one. A unit of type $1 + t$ with $t \in N(R)$ is called *unipotent*. Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text.

2 General facts

As already mentioned in the introduction, $u \in U(R)$ is exceptional iff there is $v \in U(R)$ such that $u + v = 1$, and so clearly v is also exceptional. So actually this definition gives pairs of exunits. In the sequel we use the wording "pair of exunits" only with this meaning.

For a ring R, $U(R)$ is generally not closed under addition. However, exunits appear precisely when a sum of units is a unit. Indeed, if $u + v = w$ with $u, v, w \in U(R)$ then $uw^{-1} + vw^{-1} = 1$.

A ring was termed *2-good* (see [15]) if every element is a sum of two units. As above, in such rings (in particular *fine* rings $\neq \mathbb{F}_2$ - see [5]) every unit provides (at least) one pair of exunits.

In any nonzero ring $1 \in U(R)$ is not exceptional and more general, *unipotents are not exunits.*

Recall that a ring was called UU in [2] if $U(R) = 1 + N(R)$, i.e., all units are unipotent. Hence, *UU rings have no exunits.* In particular *Boolean rings* or the field $\mathbb{F}_2$, *have no exunits.* Actually, this is the only field without exunits.

All units, excepting 1, are obviously exceptional in any division ring D, that is $U_e(D) = D - \{0, 1\}$.

"Exceptional" is invariant to conjugations: for $u \in U_e(R)$ and $v \in U(R)$, $v^{-1}uv \in U_e(R)$ (indeed, $1 - v^{-1}uv = v^{-1}(1 - u)v \in U(R)$).

For $u, v \in U_e(R)$, examples will show that both $-u \in U_e(R)$ or $uv \in U_e(R)$ may fail.

Simple properties are gathered (for easy reference) in the next proposition and some examples follow.

Proposition 1 *(a) In any ring R, the only possible order 2 exunit is -1. This is the case iff $2 \in U(R)$.*

(b) All possible (order n) exunits are roots of the polynomial $1 + X + X^2 + ... + X^{n-1}$, that is, are roots of unity $\neq 1$.

(c) The number of exunits in $\mathbb{Z}_n$ is $n \prod_{p|n}(1 - \frac{2}{p})$ with prime p.

(d) Any pair of exunits u, v determines another two pairs of exunits. These three pairs are different *if u, v are not roots of the polynomial $X^2 - X + 1 \in R[X]$.*

(e) For any pair of exunits $u^{-1}v = vu^{-1}$ resp. (equivalently) $uv^{-1} = v^{-1}u$ (each exunit commutes with the inverse of its pair). More, exunits in a pair are mutually conjugate.

(f) Inverses of exunits are also exunits.

(g) Products of exunits may not be exunits. Even $u(1-u) \notin U(R)$ is possible.

Proof. (a) Suppose $u^2 = 1$ is an exunit. Then by left multiplying $(1-u)(1+u) = 1 - u^2 = 0$ with $(1 - u)^{-1}$ we get $1 + u = 0$, i.e. $u = -1$. However, -1 is exceptional iff $1 - (-1) = 2$ is a unit. Actually whenever 2 is a unit, it is also an exunit.

(b) If $a^n = 1$ in any (unital) ring R, then $a \in U(R)$ and if $a \in U_e(R)$, by left multiplying $(1 - a)(1 + a + a^2 + ... + a^{n-1}) = 1 - a^n = 0$ with $(1 - a)^{-1}$ we obtain $1 + a + a^2 + ... + a^{n-1} = 0$.

(c) See [12], but very likely far earlier.

(d) Multiplying $u + v = 1$, both sides with u^{-1} resp. v^{-1} gives $1 = u^{-1} + (-u^{-1}v) = u^{-1} + (-vu^{-1})$ and $1 = v^{-1} + (-uv^{-1}) = v^{-1} + (-v^{-1}u)$.

For the second statement, suppose $u+v = 1$, $1-u+u^2 = 0 = 1+v+v^2$. Then $v = -u^2$, $u = -v^2$ and since $u^3 = v^3 = -1$ (by multiplying with $1+u$ resp. $1+v$), $u^{-1} = -u^2 = v$ (and $v^{-1} = -v^2 = u$), so $1 = u^{-1} + (-u^{-1}v) = v^{-1} + (-uv^{-1})$ are the same pair.

(e) Follows from the proof of (d). It is easy to check $v = uvu^{-1}$ and $u = v^{-1}uv$.

(f) If $u, 1 - u \in U(R)$ then $1 - u^{-1} = -(1 - u)u^{-1} \in U(R)$.

(g) Examples are given in the next section. ■

Examples. 1) 4 and 11 are order 2 units in $\mathbb{Z}_{15}$, but none is exceptional. The exunits in $\mathbb{Z}_{15}$ are $2, 8, 14$.

2) For each idempotent $e \in R$, $2e - 1$ is an order 2 unit. By the above proposition, it is an exunit iff $2e - 1 = -1$, i.e. iff $2e = 0$ (e.g. 3 in $\mathbb{Z}_6$).

3) Among other characterizations, a ring R is *local* iff for any $a \in R$, $a \in U(R)$ or $1 - a \in U(R)$. Since the disjuction "or" has *not* an exclusive meaning, *local rings may have exunits*. Indeed, 2 is a unit (so forms a pair of exunits with

-1) in the ring of integers localized at the prime ideal $p\mathbb{Z}$: $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : \gcd(p,n) = 1\}$, for any odd prime p. Clearly $2 \cdot \frac{1}{2} = 1$ with p not dividing 2.

Other properties are given in the next

Theorem 2 *(a) An element is an exunit in a direct product (sum) of rings iff all its components are exunits.*

(b) Let u be an exunit in R, $e^2 = e \in R$ and $\overline{e} + u \in eRe$. Then eue is an exunit in eRe.

(c) Let A be a proper ideal in R. If u is an exunit in R then $u + A$ is an exunit in R/A. However, exunits may not lift in a factor ring R modulo a proper ideal.

Proof. (a) Obvious.

(b) Recall that the units in a corner ring are given by the equality $U(eRe) = (eRe) \cap (\overline{e} + U(R))$. Equivalently, $a = \overline{e} + u$ is a unit in eRe iff $a \in eRe$. Also note that in this case, $a = eae = eue$, and so $eu^{-1}e$ is the inverse of $a = eue$ (both in eRe).

So every unit a of eRe is determined by a unit u of R. If u is an exunit, we just multiply $u + v = 1$ both sides with e: $eue + eve = e$.

(c) First part is obvious. As seen above $2, 8, 14$ are exunits in $\mathbb{Z}/15\mathbb{Z}$, but do not lift since $\mathbb{Z}$ has no exunits. $\blacksquare$

Remark. The property (b) *does not imply that corners of rings with exunits have exunits*: $\mathbb{Z}$ is a corner for $\mathcal{M}_2(\mathbb{Z})$, which has plenty of exunits (see Proposition 11, next section).

3 Exunits in matrix rings

We (mainly) address this over commutative rings in order to benefit of determinants, Cayley-Hamilton theorem and other ingredients.

We note that a matrix $U \in \mathcal{M}_n(R)$ is an exunit iff so is its transpose U^T.

It is easy to characterize exceptional invertible matrices via the characteristic polynomial.

Proposition 3 *Let R be a commutative ring, $U \in GL_n(R)$ and let $p_U(X) = \det(XI_n - U)$ be the characteristic polynomial of U. Then U is exceptional iff $p_U(1) \in U(R)$.*

Proof. Obvious: $p_U(1) = \det(I_n - U)$. $\blacksquare$

Corollary 4 *(i) A 2×2 matrix U over a commutative ring R is an exunit iff $\det(U) \in U(R)$ and $1 - \mathrm{Tr}(U) + \det(U) \in U(R)$.*

(ii) A 3×3 matrix U over a commutative ring R is an exunit iff $\det(U) \in U(R)$ and $1 - \mathrm{Tr}(U) + \frac{1}{2}(\mathrm{Tr}(U)^2 - \mathrm{Tr}(U^2)) - \det(U) \in U(R)$.

Proof. (i) Indeed $\det(I_2 - U) = 1 - \mathrm{Tr}(U) + \det(U)$.

(ii) Indeed $p_U(X) = X^3 - \mathrm{Tr}(U)X^2 + \frac{1}{2}[\mathrm{Tr}(U)^2 - \mathrm{Tr}(U^2)]X - \det(U)$ and so $\det(I_3 - U) = 1 - \mathrm{Tr}(U) + \frac{1}{2}[\mathrm{Tr}(U)^2 - \mathrm{Tr}(U^2)] - \det(U)$. ∎

Remarks. 1) If the characteristic of R is zero, $p_U(X) = \det(X.I_n - U) = \sum_{k=0}^{n} X^{n-k}(-1)^k \frac{1}{k!}T_k$ where $T_k = \begin{vmatrix} \mathrm{Tr}(U) & k-1 & 0 & ... & \\ \mathrm{Tr}(U^2) & \mathrm{Tr}(U) & k-2 & ... & \\ \vdots & \vdots & & \ddots & \vdots \\ \mathrm{Tr}(U^{k-1}) & \mathrm{Tr}(U^{k-2}) & & ... & 1 \\ \mathrm{Tr}(U^k) & \mathrm{Tr}(U^{k-1}) & & ... & \mathrm{Tr}(U) \end{vmatrix}$.

Therefore $U \in GL_n(R)$ is exceptional iff $\det(I_n - U) = \sum_{k=0}^{n}(-1)^k \frac{1}{k!}T_k \in U(R)$.

2) For 3×3 matrices we get $\frac{1}{2}[\mathrm{Tr}(U)^2 - \mathrm{Tr}(U^2)] = (u_{11}u_{22} - u_{12}u_{21}) + (u_{11}u_{33} - u_{13}u_{31}) + (u_{22}u_{33} - u_{23}u_{32})$, that is, the sum of the diagonal 2×2 minors of U (i.e. the cofactors of the diagonal entries): $A_{33} + A_{22} + A_{11} = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} + \begin{vmatrix} u_{11} & u_{13} \\ u_{31} & u_{33} \end{vmatrix} + \begin{vmatrix} u_{22} & u_{23} \\ u_{32} & u_{33} \end{vmatrix}$.

3) Obviously, all invertible 2×2 matrices with $\mathrm{Tr}(U) = 1$ (over any commutative ring) are exceptional since in this case $\det(I_2 - U) = \det(U)$.

We can easily discard *the triangular case* over any (not necessarily commutative) ring.

Proposition 5 *Let U be a triangular invertible $n \times n$ matrix over any ring R. Then U is exceptional iff the diagonal entries are exceptional. If $2 \in U(R)$ then $2I_n$ is exceptional.*

Proof. A triangular matrix U is invertible iff its diagonal entries are units, that is $u_{11}, ..., u_{nn} \in U(R)$. Same for $I_n - U$, i.e. $1 - u_{11}, ..., 1 - u_{nn} \in U(R)$. So all diagonal entries must be exunits. ∎

Since $U(\mathbb{Z}) = \{\pm 1\}$ and none is exceptional we obtain at once

Corollary 6 *There are no triangular exceptional* integral *invertible matrices.*

However, there exist 2×2 and 3×3 triangular exunits. Moreover, these can be *diagonal*.

Example. Over $\mathbb{Z}_3$, for any a, the matrices $\begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}$ are exunits (in particular $U = 2I_2$ is diagonal). Similarly $U = 2I_3$ is a diagonal 3×3 exunit.

More general, since in any division ring D, $U_e(D) = D - \{0, 1\}$

Proposition 7 *A triangular matrix U over any division ring D is an exunit iff no diagonal entry is* 0 *or* 1.

Having clarified the case of 2×2 matrices with (at least) one zero off diagonal entry (i.e. triangular matrices), we proceed with the matrices which have (at least) *one zero diagonal entry.*

Proposition 8 *Let* $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathcal{M}_2(R)$ *with* $b, c \in U(R)$*. Then*

(i) A is invertible.

(ii) A is an exunit iff $a - 1 + bc \in U(R)$.

(iii) Same conclusions for matrices of form $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ *(here $cb + d - 1 \in U(R)$ gives the exunits).*

Proof. (i) It is readily checked that $A^{-1} = \begin{bmatrix} 0 & c^{-1} \\ b^{-1} & -b^{-1}ac^{-1} \end{bmatrix}$.

(ii) By writing $(A - I_2)W = I_2$ with $W = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ we get

$$\begin{bmatrix} (a-1)x + bz = 1 \\ (a-1)y + bt = 0 \\ cx - z = 0 \\ cy - t = 1 \end{bmatrix},$$

system with solutions iff $a - 1 + bc \in U(R)$ (we replace $z = cx$ in the first equation). If so, we get $(A - I_2)^{-1} = \begin{bmatrix} (a-1+bc)^{-1} & (a-1+bc)^{-1}b \\ c(a-1+bc)^{-1} & c(a-1+bc)^{-1}b - 1 \end{bmatrix}$, i.e., the right inverse turns out to be also left inverse.

(iii) These matrices are obtained from the ones in the statement, conjugating with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. ■

Remarks. 1) If R is a division ring, the conditions are $b \neq 0 \neq c$ and $a - 1 + bc \neq 0$, respectively.

2) Actually $a - 1 + bc = -(1 - \mathrm{Tr}(U) + \det(U))$ if $\det(U) = ad - bc$ for a matrix over a *not necessarily commutative* ring, so same condition as in Corollary 4, in the commutative case.

3) Matrices of this shape *do not* form a subring of $\mathcal{M}_2(R)$ (as triangular do).

A 3×3 version also holds

Proposition 9 *Let* $A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \in \mathcal{M}_3(R)$ *with* $c, e, f \in U(R)$*. Then*

(i) A is invertible.

(ii) A is an exunit iff $b - (a + cf - 1)d^{-1}(e - 1) \in U(R)$.

Proof. (i) As in the previous proposition we search for a 3×3 matrix X such that $AX = I_3$ and check that also $XA = I_3$ holds. The suitable (unique) matrix is $A^{-1} = \begin{bmatrix} 0 & 0 & f^{-1} \\ 0 & e^{-1} & -e^{-1}df^{-1} \\ c^{-1} & -c^{-1}be^{-1} & -c^{-1}af^{-1} + c^{-1}be^{-1}df^{-1} \end{bmatrix}$.

(ii) Analogously, for $(A-I_3)Y=I_3$ we replace a and d by $a-1$, $d-1$ respectively and the SE corner by -1. The system is solvable if $b-(a-1+cf)d^{-1}(e-1)$ has a right inverse. However, since we have to check also $Y(A-I_3)=I_3$ we need this to be a (two-sided) unit. If we denote $\alpha=[b-(a-1+cf)d^{-1}(e-1)]^{-1}$ and $\beta=(a-1+cf)d^{-1}$ we get $(A-I_3)^{-1}=\left[\begin{array}{ccc} -d^{-1}(e-1)\alpha & d^{-1}[1+(e-1)\alpha\beta] & -d^{-1}(e-1)\alpha c \\ \alpha & -\alpha\beta & \alpha c \\ -fd^{-1}(e-1)\alpha & fd^{-1}[1+(e-1)\alpha\beta] & -fd^{-1}(e-1)\alpha c-1 \end{array}\right]$ (just note $b-\beta(e-1)=\alpha^{-1}$). ■

As for the general 2×2 case, *if* $\mathcal{M}_2(R)$ *is Dedekind finite*, we have the following

Proposition 10 *Let* $U=\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ *be a matrix,* $a,d\in U(R)$ *and suppose* $\mathcal{M}_2(R)$ *is Dedekind finite. Then* U *is an unit iff* $a-bd^{-1}c, d-ca^{-1}b\in U(R)$ *and an exunit iff also* $a-1-bd^{-1}c, d-1-ca^{-1}b\in U(R)$.

Proof. Let $W=\left[\begin{array}{cc} x & y \\ z & t \end{array}\right]$ be such that $UW=I_2$. This amounts to the system

$$\begin{array}{c} ax+bz=1 \\ ay+bt=0 \\ cx+dz=0 \\ cy+dt=1 \end{array}$$

and the middle equations can be solved as $y=-a^{-1}bt$ and $z=-d^{-1}cx$. By replacement we get $(a-bd^{-1}c)x=1$ and $(d-ca^{-1}b)t=1$ so W exists iff $a-bd^{-1}c, d-ca^{-1}b\in U(R)$. In order to have an exunit we also need a matrix P such that $(U-I_2)P=I_2$, that is the same as above, replacing a by $a-1$ and d by $d-1$ (this explains why we prefer $U-I_2$ instead of I_2-U). So $a-1-bd^{-1}c, d-1-ca^{-1}b\in U(R)$. ■

Remarks. 1) Over a division ring the conditions become $a-bd^{-1}c\neq 0, 1\neq d-ca^{-1}b$.

2) There are *another three propositions* which analogously give 2 exunits:

$a, a-1, c\in U(R)$ (so a is exunit) together with $b-ac^{-1}d$, $d-ca^{-1}b\in U(R)$ and $b-(a-1)c^{-1}(d-1)$, $(d-1)-c(a-1)^{-1}b\in U(R)$,

$b, c\in U(R)$ together with $b-ac^{-1}d$, $c-db^{-1}a\in U(R)$ and $b-(a-1)c^{-1}(d-1)$, $c-(d-1)b^{-1}(a-1)\in U(R)$,

$b, d, d-1\in U(R)$ (so d is exunit) together with $a-bd^{-1}c$, $c-db^{-1}a\in U(R)$ and $a-1-b(d-1)^{-1}c$, $c-(d-1)b^{-1}(a-1)\in U(R)$.

The remaining possibilities $a, b\in U(R)$ or $c, d\in U(R)$ are also covered by transpose.

3) For $b, c\in U(R)$ and $d=0$ we need $b+(a-1)c^{-1}$, $c+b^{-1}(a-1)\in U(R)$, both equivalent to $bc+a-1\in U(R)$, as in the previous proposition.

For *integral* matrices more details are given in the following characterization

Proposition 11 *A 2×2 integral matrix U is an exunit iff* $\det(U) = 1$ *and* $\mathrm{Tr}(U) \in \{1, 3\}$ *or else* $\det(U) = -1$ *and* $\mathrm{Tr}(U) \in \{-1, 1\}$.

Proof. Immediate from Corollary 4. ■

Corollary 12 *A matrix ring over a ring without exunits may have exunits.*

Corollary 13 *For 2×2 integral exunit U, $-U$ is also an exunit iff* $\det(U) = -1$.

Proof. Just note that $\mathrm{Tr}(-U) = -\mathrm{Tr}(U)$ and $\det(-U) = \det(U)$. ■

Example. $U = \begin{bmatrix} -1 & 5 \\ -1 & 4 \end{bmatrix}$ is an integral exunit but $-U = \begin{bmatrix} 1 & -5 \\ 1 & -4 \end{bmatrix}$ is not (the trace is -3).

This example shows that *the opposite of an exunit may not be an exunit.*

Moreover, U and $I_2 - U$ are exunits but $U(I_2 - U) = \begin{bmatrix} 3 & -10 \\ 2 & -7 \end{bmatrix}$ is not: the trace is -4 (or directly: $I_2 - \begin{bmatrix} 3 & -10 \\ 2 & -7 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ -2 & 8 \end{bmatrix}$ is not invertible).

Hence, *products of exunits may not be exunits.*

It is considerably harder to determine the 3×3 invertible integral exceptional matrices. We analyze some special cases.

In an invertible integral matrix, since the determinant must be ± 1, the (three) entries in any row or in any column must be coprime. More precisely

Proposition 14 *Let a, b, c be entries in any row (or any column) of an invertible integral 3×3 matrix U and let a be the diagonal entry. Two necessary conditions for U to be an exunit are: a, b, c are coprime and so are $1 - a, b, c$.*

So if a, b, c are entries in a row (or column) and a is the diagonal entry, we should have $\gcd(a; \gcd(b; c)) = 1 = \gcd(1 - a; \gcd(b; c))$.

As a special case

Proposition 15 *Let U be a 3×3 invertible* integral *matrix.*

(i) If U has two not diagonal even *entries in the same row or in the same column, then U is not exceptional.*

(ii) There are infinitely many exceptional invertible matrices with two even entries in the same row or in the same column, one being diagonal.

Proof. (i) If two not diagonal entries in the same row (or column) are even the corresponding diagonal entry must be odd. Then in $I_3 - U$, the entries in the same row (or column) are even and so $\det(I_3 - U) \in 2\mathbb{Z}$. Hence U is not an exunit.

(ii) We discuss the case $u_{11} = u_{31} = 0$ and (since $\det(U) = \pm 1$) so $u_{21} = \pm 1$ case.

If $\det(U) = 1$ and $u_{21} = 1$, then for $u_{33} = u_{13} = u_{32} = 1$, $u_{12} = 0$ and $u_{23} = -2$, we have infinitely many exunits: $U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & a & -2 \\ 0 & 1 & 1 \end{bmatrix}$. The case $u_{21} = -1$ is analogous.

If $\det(U) = -1$ and $u_{21} = 1$, then for $u_{33} = u_{13} = 1$, $u_{32} = -1$, $u_{12} = 0$ and $u_{23} = -2$, we have infinitely many exunits: $U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & a & -2 \\ 0 & -1 & 1 \end{bmatrix}$. The case $u_{21} = -1$ is analogous. ■

Example. $U = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $U^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\det(U) = 1$, $\det(I_3 - U) = -1$ covers several cases addressed in the previous proof.

4 Ex-clean

In a ring R with exunits an element a is (strongly) *ex-clean* if $a = e + u$ with $e^2 = e \in R$ and $u \in U_e(R)$ (and e, u commute).

As previously noticed, if $u \in U_e(R)$, $-u$ may not be exceptional. Hence $1 - a = (1 - e) - u$ is clean but may not be ex-clean.

If $u + v = 1$ then $a - 1 = e - v$ is also clean (and may not be ex-clean).

We start this section with some easy observations on such elements in arbitrary (unital) rings.

In any (unital) ring, *units*, *nilpotents* and *idempotents* are known to be *clean*. More precisely, for $u \in U(R)$, $u = 0 + u$ is a (trivial) clean decomposition, for $t \in N(R)$, $t = 1 - (1 - t)$ is a (trivial) clean decomposition and for $e = e^2$, $e = (1 - e) + (2e - 1)$ is a clean decomposition.

As for ex-clean elements, since $u = 0 + u$, *exunits* $u \in U_e(R)$, still are (trivial) ex-clean, but these have also the other trivial clean decomposition $u = 1 - (1 - u) = 1 - v$ if $u + v = 1$ and $v = 1 - u \in U_e(R)$. However, as already noticed, if $v \in U_e(R)$, $-v$ may not be an exunit, so the last decomposition is clean but may not be ex-clean.

Rephrasing, *a unit has both trivial clean decompositions* ($u = 0 + u = 1 - (1 - u)$) *iff it is an exunit.*

The above given clean decomposition of the *idempotents*, is an ex-clean decomposition iff $2e = 0$ (see Proposition 1, (a)).

As for the trivial idempotents, we mention the following.

In any ring R, $0 = e + u$ implies $e = -u \in U(R)$ so $e = 1$ and $0 = 1 + (-1)$ is ex-clean iff $2 = 0$. Hence, 0 *is ex-clean only in rings of characteristic 2.*

In any ring R, $1 = e + u$ implies $1 - e = u \in U(R)$ so $e = 0$ and $1 = 0 + 1$ is not ex-clean. Hence, 1 *is not ex-clean in any ring.*

In any ring R, *a nontrivial idempotent cannot be trivial ex-clean.* Indeed, $e = 0 + u$ implies $e = 1$ and $e = 1 + u$ implies $e = 0$.

In the sequel ex-clean matrices in $\mathcal{M}_2(\mathbb{Z})$ are discussed and finally characterized.

4.1 Exunits

Note that for any pair of exunits $U, V \in \mathcal{M}_2(\mathbb{Z})$ (i.e. $U + V = I_2$) we have $\mathrm{Tr}(V) = 2 - \mathrm{Tr}(U)$ and $\det(V) = \det(U) - \mathrm{Tr}(U) + 1$.

The 2×2 *exunits* over $\mathbb{Z}$ were characterized in Proposition 11 (these are trivial ex-clean $0_2 + U$).

Proposition 16 *There are no uniquely clean exunits in* $\mathcal{M}_2(\mathbb{Z})$.

Proof. We use the main result in [1]: *an invertible* 2×2 *integral matrix is uniquely clean iff it is similar to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since exunit is a similarity invariant, it suffices to check that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is not exunit. Indeed, $I_2 - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is not invertible. ■

Examples. 1) $U = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$ is an exunit over $\mathbb{Z}$, so is (its pair) $V = I_2 - U = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ and, by Corollary 13, so is $-V$. Therefore, U has both trivial *ex-clean* decompositions $U = 0_2 + U = I_2 + \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$.

2) $U = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}$ is an integral exunit, so is its pair $V = I_2 - U = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ and $U = I_2 + \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$ is a clean but not ex-clean decomposition. Indeed, here $-V$ is not an exunit.

3) *Exunits may have clean decompositions which are not ex-clean.*

$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = 0_2 + \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = I_2 + \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$ are the trivial ex-clean decompositions but

$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (and the computer gives other infinitely many clean decompositions) are clean but not ex-clean since $I_2 - \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ and $I_2 - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, none is a unit.

3) Since $\det(I_2 + U) = \det(U) + \mathrm{Tr}(U) + 1$ and $\mathrm{Tr}(I_2 + U) = 2 + \mathrm{Tr}(U)$, *the other trivial ex-clean matrices*, i.e., $A = I_2 + U$ with exunit U are also characterized by Proposition 11: $\det(A) = 3 = \mathrm{Tr}(A)$ or $\det(A) = 5 = \mathrm{Tr}(A)$ or $\det(A) = -1$ with $\mathrm{Tr}(A) = 1$ or $\det(A) = 1$ and $\mathrm{Tr}(A) = 3$.

4) *The exunits have at most two ex-clean decompositions*: the trivial ones. Indeed, any nontrivial idempotent E has trace 1. Hence, $\mathrm{Tr}(E+U) = 1 + \mathrm{Tr}(U) \in \{0, 2, 4\}$ and the claim follows from Proposition 11,

4.2 Nilpotents

The above given clean decomposition of the *nilpotents*, is not ex-clean since $1-t$ is an unipotent.

However

Proposition 17 *A nilpotent 2×2 integral matrix T is ex-clean iff there exist integers x, y, x, a, b, c such that $x^2 + x + yz = 0$, $a^2 + a + bc - 1 = 0$ and $(2a+1)x + cy + bz = -a - 2$. In this case $T = \begin{bmatrix} a+x+1 & y+b \\ z+c & -a-x-1 \end{bmatrix}$.*

Proof. Suppose $T = E + U$ with exunit U. Since $\mathrm{Tr}(E) \in \{0, 1, 2\}$, $\mathrm{Tr}(U) \in \{-1, 1, 3\}$ and $\mathrm{Tr}(T) = 0$ the only possible combination is $\mathrm{Tr}(E) = 1$ (i.e. nontrivial idempotent) with $\det(E) = 0$, and $\mathrm{Tr}(U) = -1$ (i.e. exunit) with $\det(U) = -1$. Thus $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ with $x(x+1) + yz = 0$ and $U = \begin{bmatrix} a & b \\ c & -a-1 \end{bmatrix}$ with $a^2+a+bc = 1$. Then $\det(E+U) = 0$ iff $(2a+1)x+cy+bz = -a-2$. ■

Examples. With the exunit $U = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$ (i.e. $a = -2$, $b = 1$, $c = -1$) we get $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$, or $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$ since the Diophantine equation $x^2+3xy+y^2+x = 0$ (together with $z = 3x+y$; see Theorem 19 below) has (among other infinitely many solutions) the solutions $(0,0)$, $(-1,0)$ and $(1,-1)$.

4.3 Idempotents

Proposition 18 *No idempotent matrix in $\mathcal{M}_2(\mathbb{Z})$ is ex-clean.*

Proof. As seen above we have to discuss only nontrivial ex-clean decompositions of nontrivial idempotents, that is $E = E' + U$ with $\mathrm{Tr}(E) = \mathrm{Tr}(E') = 1$. But this implies $\mathrm{Tr}(U) = 0$ and Proposition 11 shows that such ex-clean integral matrices do not exist. ■

4.4 The nontrivial ex-clean matrices.

We first recall (from [1]) the characterization of the nontrivial clean 2×2 integral matrices

Theorem 19 *A* 2×2 *integral matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is nontrivial clean iff the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = \pm 1 & (2) \end{cases}$$

with unknowns x, y, z*, has at least one solution over* $\mathbb{Z}$*. If* $b \neq 0$ *and (2) holds, then (1) is equivalent to*

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0 \quad (3).$$

Proof. Any nontrivial idempotent is characterized by zero determinant and trace $= 1$. The general matrix A is clean iff there is a nontrivial idempotent $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ i.e., $\mathrm{Tr}(E) = 1$ and $-\det(E) = x^2 + x + yz = 0$, that is (1), such that $\det(A - E) = \pm 1$. If (1) holds, the last condition amounts to $(a-d)x + cy + bz + \det(A) - d = \pm 1$, that is (2).

If $b \neq 0$ (the case $c \neq 0$ is symmetric), multiplying (1) by b and eliminating z, we get the Diophantine equation $bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0$, that is (3). ■

Corollary 20 *A* 2×2 *integral matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is nontrivial ex-clean iff at least one of the systems*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = 1 & (2) \\ a + d \in \{2, 4\} & (4) \end{cases}$$

or

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = -1 & (2') \\ a + d \in \{0, 2\} & (4') \end{cases}$$

with unknowns x, y, z*, has at least one solution over* $\mathbb{Z}$*. Accordingly (if* $b \neq 0$*) we associate* (3) *or* (3′)*:*

$$\begin{cases} bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) + 1)y = 0 & (3') \\ (a-d)x + cy + bz + \det(A) - d = 1 & (2') \\ a + d \in \{2, 4\} & (4') \end{cases} .$$

$$\begin{cases} bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) - 1)y = 0 & (3') \\ (a-d)x + cy + bz + \det(A) - d = -1 & (2') \\ a + d \in \{0, 2\} & (4') \end{cases} .$$

Proof. We just have to add the conditions given in Proposition 11, for the exunit $A - E$. ■

Examples. 1) $\begin{bmatrix} 0 & 6 \\ -1 & 4 \end{bmatrix}$ has precisely two clean decompositions:

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$. Both are ex-clean decompositions.

2) $\begin{bmatrix} 0 & 4 \\ -1 & 4 \end{bmatrix}$ has infinitely many clean decompositions: for any nonnegative integer k,

$$\begin{bmatrix} -8k^2+2k+1 & 16k^2-1 \\ -4k^2+2k & 8k^2-2k \end{bmatrix} + \begin{bmatrix} 8k^2-2k-1 & -16k^2+5 \\ 4k^2-2k-1 & -8k^2+2k+4 \end{bmatrix} =$$
$$= \begin{bmatrix} -8k^2-6k & 16k^2+16k+3 \\ -4k^2-2k & 8k^2+6k+1 \end{bmatrix} + \begin{bmatrix} 8k^2+6k & -16k^2-16k+1 \\ 4k^2+2k-1 & -8k^2-6k+3 \end{bmatrix}.$$

All are ex-clean decompositions.

Remark. According to Corollary 6, there are no triangular exunits, that is, the (ex)unit in any ex-clean decomposition cannot be triangular.

5 Comments and open questions

1) *Examples of clean rings* include semiperfect rings, unit-regular rings and endomorphism rings of continuous modules.

Is there any relation between ex-clean and special classes of clean rings ? (incl. strongly clean or uniquely clean).

Give *examples of ex-clean rings*.

2) The strongly property is preserved by the pair of any exunit: $eu = ue$ and $u + v = 1$ imply $ev = ve$.

Determine the (strongly) ex-clean rings.

3) The proof given in [4] for: *the class of clean rings is closed under extensions (and in particular for matrix rings)*, cannot be adapted in the ex-clean situation. Even the particular case of matrix rings seems unlikely.

Give an example of ex-clean ring R such that $\mathcal{M}_n(R)$ is not ex-clean (for some $n \geq 2$).

4) The proof given in [3] for: *corners of strongly clean rings are strongly clean*, cannot be adapted in the ex-clean situation.

Give an example of ex-clean ring R and a (full) idempotent $e \in R$ with not ex-clean corner eRe.

5) In [14] it is proved that *if R is a clean ring and I an ideal such that R/I has only trivial idempotents, then units in R/I lift to units in R.*

An analogous proof for exunits does not work: in the final case of this proof, for $u, v \in U_e(R)$ we need $-uv^{-1} \in U_e(R)$. But products of exunits may not be exunits, and opposites of exunits may not be exunits. Give an example.

6) Even if units lift in a factor ring R modulo a proper ideal, exunits may not lift. Give an example.

7) A nonzero element a in a ring R is (strongly) *ex-fine* if $a = u + t$ with $u \in U_e(R)$, $t \in N(R)$ (and u, t commute).

If a is fine, so is $-a$, but not necessary $1 - a$. If a is ex-fine then $1 - a = (1 - u) - t$ is (ex)fine.

In [5] it is proved that the *strongly fine elements (in a nonzero ring) are precisely the units*. Analogously, *strongly ex-fine elements are precisely the exunits*. (One way is clear. Conversely, if $a \in R\backslash(0)$ has a strongly ex-fine decomposition $u+t$, then $a = u(1+u^{-1}t) \in U(R)$ (since $ut = tu$ implies that $u^{-1}t \in N(R)$)).

Characterize the ex-fine rings.

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