Mathematica Pannonica ***New Series*** **29 /NS 3/ (2023) 2, 195–203**
DOI: 10.1556/314.2023.00019

AKADÉMIAI KIADÓ

SIMILARITY FOR 2×2 MATRICES OBTAINED BY CLOCKWISE "ROTATION"

Grigore Călugăreanu[1,*]

[1] Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania

Communicated by Vedran Krčadinac

Original Research Paper
Received: Feb 24, 2023 • Accepted: Aug 30, 2023
First published online: Sep 21, 2023

ABSTRACT

Over integral domains of characteristics different from 2, we determine all the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which are similar to $\begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

KEYWORDS

Rotable matrix, integral domain, quadratic Diophantine equation, 2×2 matrix

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 15B33, 15B99; Secondary 11D09, 16U10

1. INTRODUCTION

There are many studies concerning the similarity of a given matrix and matrices obtained from it by several constructions.

The most familiar such (undergraduate) study is the diagonalization of a matrix, i.e., to find a diagonal matrix similar to a given one.

We also just mention the similarity of a given matrix with its transpose (see [4]) or the similarity of a given matrix with its companion matrix [3].

For any base ring such that any matrix is similar to its Jordan form, it follows that each matrix is similar to its transpose. It is well-known that this happens if the base ring is an algebraic closed field.

If A is an $n \times n$ matrix with entries from some field K, then A is similar to the companion matrix over K of its characteristic polynomial iff there exists a vector $\mathbf{v}$ in $V = K^n$ for A, such that $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{n-1}\mathbf{v}\}$ is a basis of V. Not every square matrix is similar to a companion matrix. But every square matrix A is similar to a matrix made up of blocks of companion matrices.

In this paper, we study another natural construction described as follows. All the rings we consider in this paper (unless otherwise stated) are integral (commutative) domains.

* Corresponding author. E-mail: calu@math.ubbcluj.ro

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$, the latter obtained (say) by a *clockwise "rotation" with* 90°. In the sequel we write $B = \mathrm{rot}(A)$ and, briefly, "*rotation*" will mean "clockwise rotation with 90°". To have a short term at disposal, a matrix A is called *rotable* if it is similar to its rotation.

We mention that, among the multiples of 90°, this is the only one who needs attention. Indeed, A is similar to the (clockwise) "rotation" with 180° over any unital ring: $\mathrm{rot}(\mathrm{rot}(A)) = \begin{bmatrix} d & c \\ b & a \end{bmatrix} = U_2AU_2$ denoting by $U_2 := U_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathrm{rot}(I_2)$. Consequently the similarity with the 270° rotation reduces (via U_2) to that of B. That is, $\mathrm{rot}(\mathrm{rot}(\mathrm{rot}(A))) = U_2BU_2 = U_2\,\mathrm{rot}(A)U_2$.

Obvious examples of rotable matrices are *all matrices with equal entries* (i.e., $a = b = c = d$, that is, multiples of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$), which are precisely the matrices for which $A = \mathrm{rot}(A)$. Again, to have a short term at disposal, these matrices will be called *trivially rotable*.

As for an easy negative example, I_2 is not similar with its rotation (over any nonzero ring), the matrix U_2 above. Actually, it is easy to construct not rotable matrices: it suffices to have a different trace after rotation.

However, similarities of the type considered above, are special: if A is rotable and M is similar to A, then M may not be rotable. An example is given on page 199 (see (2.1)).

Since idempotents, nilpotents and units are respectively invariant to conjugation in any ring, idempotent (or nilpotent or unit) matrices are respectively similar with their rotations only if these are also idempotent (resp. nilpotent or unit).

In this paper, over any integral domain of characteristics different from 2, we determine *all* the rotable 2×2 matrices.

We first show that a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is rotable *only if* $b = c$ or $b = d$ or $c = d$ or $\mathrm{char}(R) = 2$ (see Proposition 2.4). Thus, the rotable matrices have a simple form and are characterized in the following three results (see Proposition 2.3, Theorem 2.5 and Theorem 2.7).

The first one deals with special rotable matrices.

PROPOSITION. Let R be an integral domain of $\mathrm{char}(R) \neq 2$.

(i) No nonzero determinant matrix (in particular, no invertible matrix) is rotable.

(ii) An idempotent 2×2 matrix $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is rotable iff $E \neq I_2$ and $b + c = 1$.

(iii) The only nilpotent rotable matrices are $a\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ or $a\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, for some $a \in R$.

(iv) The only (upper) triangular rotable matrices are of form $b(E_{12} + E_{22})$ or $a(E_{11} + E_{12})$ for some $a, b \in R$. In particular, the only diagonal rotable matrix is 0_2.

The next two theorems characterize the nontrivial rotable matrices.

THEOREM. Let R be an integral domain of $\mathrm{char}(R) \neq 2$, $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \in \mathbb{M}_2(R)$ and $a \neq b$. The matrix A is rotable iff there exists a decomposition $b = b_1b_2$ such that $a + b_1b_2 \mid b_1 - ab_2$ and $b_1 \mid a$.

In particular, A is rotable whenever

(i) $a + b \mid b - a$ and $b \mid a$, or else

(ii) $a + b \mid 1 - ab$.

THEOREM. Let R be an integral domain of $\mathrm{char}(R) \neq 2$, $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix} \in \mathbb{M}_2(R)$ and $a \notin \{\pm c\}$. The matrix A is rotable iff there exists a decomposition $a = a_1a_2$ such that $a_1a_2 + c \mid a_1 + ca_2$ and $a_1 \mid c$.

In particular, A is rotable whenever

(i) $a \mid c$, or else

(ii) $a + c \mid 1 + ac$.

As customarily, by E_{ij} we denote the square matrix with all entries zero, excepting the (i,j)-entry, which is 1, and $|$ denotes the divisibility relation.

2. THE $\mathrm{char}(R) \neq 2$ CASE

First recall that similar matrices have the same determinant and same trace.

Hence $\det(A) = \det(\mathrm{rot}(A))$ so $ad - bc = bc - ad$ and as $\mathrm{char}(R) \neq 2$, both $\det(A) = \det(\mathrm{rot}(A)) = 0$, i.e., $ad = bc$.

Moreover, $\mathrm{Tr}(A) = \mathrm{Tr}(\mathrm{rot}(A))$, i.e., $a + d = b + c$.

Therefore

LEMMA 2.1. Suppose R is commutative ring with identity and $\mathrm{char}(R) \neq 2$. Necessary conditions for a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be rotable are $ad = bc$ and $a + d = b + c$ (i.e., the pairs (a, d) and (b, c) have the same sum and same product).

Some useful properties are gathered in the following

LEMMA 2.2. (a) If A is rotable, so is $\mathrm{rot}(A)$.
(b) A is rotable iff the transpose A^T is rotable.

Proof. (a) Indeed, if $AV = V\,\mathrm{rot}(A)$ for some invertible matrix V, then $\mathrm{rot}(A)(U_2V)^{-1} = (U_2V)^{-1}(U_2AU_2)$, for the involution U_2 considered in the Introduction, and $\mathrm{rot}(\mathrm{rot}(A)) = U_2AU_2$.

(b) Since $(A^T)^T = A$, only one way needs a proof. Suppose A is rotable and $AU = U\,\mathrm{rot}(A)$, for some invertible matrix U. Taking the transposes we first get $A^T(U^T)^{-1} = (U^T)^{-1}\,\mathrm{rot}(A)^T$, for the invertible matrix $(U^T)^{-1}$. Next observe that $\mathrm{rot}(A)^T = U_2A \neq AU_2 = \mathrm{rot}(A^T)$, so the statement finally follows from

$$A^T(U^T)^{-1}U_2 = (U^t)^{-1}U_2\,\mathrm{rot}(A^T). \qquad \square$$

It is easy to determine the rotable matrices, if the matrix is invertible or nilpotent or (upper) triangular but it is harder for idempotent matrices. As it is well-known, the *trace* of a matrix is defined as the sum of the diagonal entries. Just to have a short term at disposal, we call the *secondary trace* of a matrix, the sum of the entries on the secondary diagonal. For a 2×2 matrix $[a_{ij}]$, $1 \leq i, j \leq 2$, this is $a_{12} + a_{21}$.

PROPOSITION 2.3. Let R be an integral domain of $\mathrm{char}(R) \neq 2$.

(i) No nonzero determinant matrix (in particular, no invertible matrix) is rotable.

(ii) An idempotent 2×2 matrix is rotable iff $\neq I_2$ and its secondary diagonal equals 1. More detailed, these are precisely the matrices $\begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}$ for some $a \in R$ and their transposes.

(iii) The only nilpotent rotable matrices are $a \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ for some $a \in R$ and their transposes.

(iv) The only upper triangular rotable matrices are of form $b(E_{12} + E_{22})$ or $a(E_{11} + E_{12})$ for some $a, b \in R$. In particular, the only diagonal rotable matrix is 0_2.

Proof. (i) Indeed, $\det(A) = \det(\mathrm{rot}(A)) = 0$ was a necessary condition.

(ii) Since idempotent 2×2 matrices over integral domains are characterized by zero determinant and trace $= 1$, start with $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$. Thus $\mathrm{rot}(E) = \begin{bmatrix} c & a \\ 1-a & b \end{bmatrix}$ has already zero determinant and a necessary (and so also sufficient) condition to be idempotent is to have the trace $b + c = 1$ (any matrix similar to an idempotent is itself idempotent). Clearly $b + c$ is the secondary trace of E.

Replacing $c = 1 - b$ in $a(1-a) = bc$ gives $(a-b)(a+b-1) = 0$ so we have two possibilities: $a = b$ with $E = \begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}$ and $\mathrm{rot}(E) = \begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$ or else $b = 1 - a$ with $F = \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}$ and $\mathrm{rot}(F) = \begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}$. As $F = E^T$, using Lemma 2.2 (b), it suffices to deal with the first possibility, that is, to find the elements $a \in R$, such that $E = \begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}$ is similar to $\mathrm{rot}(E) = \begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$. As it turns out, all these matrices are rotable: for $U = \begin{bmatrix} -a^2+a+1 & a^2-1 \\ a^2-2a & -a^2+a+1 \end{bmatrix}$ we have $\det(U) = 1$ and $EU = U\,\mathrm{rot}(E)$ (see Annex for details).

(iii) For nilpotent matrices, not only $\det(A) = \det(\mathrm{rot}(A)) = 0$ but also $\mathrm{Tr}(A) = \mathrm{Tr}(\mathrm{rot}(A)) = 0$ and so $a+d = b+c = 0$. Thus $A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$ which has zero determinant iff $a^2 - b^2 = 0$. Therefore, $a = b$ and so $A = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ or else $a = -b$ and so $A = a\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, for some $a \in R$.

In both cases, searching for an invertible matrix U such that $AU = U\,\mathrm{rot}(A)$, shows that these matrices are rotable. In detail, $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} U = U \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ for $U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} V = V \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ for $V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(iv) If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, $\det(A) = ad = 0$ and $\mathrm{Tr}(A) = a+d = \mathrm{Tr}(\mathrm{rot}(A)) = b$ is possible in any domain, only if $a = 0$ and $b = d$ or $d = 0$ and $a = b$. That such matrices are indeed rotable will follow as a special case of Theorem 2.5 and Theorem 2.7. For the diagonal case it suffices to take $b = 0$. □

REMARKS. (1) An analogous result holds for lower triangular matrices.

(2) From (iv) above it follows that diagonalization (in particular the Smith or Frobenius normal forms) in the determination of rotable matrices is not suitable.

PROPOSITION 2.4. Over any integral domain R, a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is rotable only in the following cases:

(1) $b = c$ or
(2) $b = d$ or
(3) $c = d$ or else $\mathrm{char}(R) = 2$.

Proof. In order to find the rotable matrices, in the general (zero determinant) case, we consider an invertible matrix $U = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$, such that $AU = U\,\mathrm{rot}(A)$, that is, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}\begin{bmatrix} c & a \\ d & b \end{bmatrix}$. This reduces to

$$\begin{aligned} ax + bz &= xc + yd \\ ay + bu &= xa + yb \\ cx + dz &= zc + ud \\ cy + du &= za + ub, \end{aligned} \tag{S_1}$$

which we write as

$$\begin{aligned} (a-c)x - dy + bz &= 0 \\ ax + (b-a)y - bu &= 0 \\ cx + (d-c)z - du &= 0 \\ cy - az + (d-b)u &= 0 \end{aligned} \tag{S_2}$$

a homogeneous linear system, to which we have to add the equation $xu - yz = 1$.

The matrix associated to this system is

$$M = \begin{bmatrix} a-c & -d & b & 0 \\ a & b-a & 0 & -b \\ c & 0 & d-c & -d \\ 0 & c & -a & d-b \end{bmatrix}.$$

In order to keep the fluency of our exposition, at the end of this paper (see the Annex), we show that $\det(M) = 2(c-b)(b-d)(c-d)$.

According to N. McCoy's theorem (see [2, Th. 5.3]), if $\det(M) \neq 0$, the system has only the trivial solution which is not suitable since $xu - yz = 1$. Therefore, additional necessary conditions for the matrix A to be rotable are $b = c$ or $b = d$ or $c = d$ or $\mathrm{char}(R) = 2$. □

Since we address only the $\mathrm{char}(R) \neq 2$ case and $a + d = b + c$, we distinguish three cases.

CASE 1. $b = c$ so $a + d = 2b$. Here $A = \begin{bmatrix} a & b \\ b & 2b-a \end{bmatrix}$ together with $a(2b - a) = b^2$ or equivalently $(a - b)^2 = 0$.

Since R is a domain, $a = b$ follows and so $a = b = c = d$. Hence only the trivially rotable matrices belong to this case.

CASE 2. $b = d$ and so $a = c$ (for which $ad = bc$ holds for any commutative unital ring). Here $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$.

CASE 3. $c = d$ and so $a = b$ (for which $ad = bc$ holds for any commutative unital ring). Here $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$.

Next, notice that adding all equations of (S_2) gives $2ax + 2(b - a)z - 2bu = 0$, that is, $ax + (b - a)z - bu = 0$, as $\mathrm{char}(R) \neq 2$.

Replacing in the second equation gives

$$(b - a)y = (b - a)z \tag{2.1}$$

There are two possibilities: if $a = b$, this combined with Case 2 above gives $a = b = c = d$, the trivially rotable matrices already found in Case 1.

Hence in the sequel we focus on the remaining possibilities: $a \neq b$ in Case 2 and $a = b$ in Case 3.

First, Case 2.

THEOREM 2.5. Let R be an integral domain of $\mathrm{char}(R) \neq 2$, $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \in \mathbb{M}_2(R)$ and $a \neq b$. The matrix A is rotable iff there exists a decomposition $b = b_1 b_2$ such that $a + b_1 b_2 \mid b_1 - a b_2$ and $b_1 \mid a$.

In particular, A is rotable whenever

(i) $a + b \mid b - a$ and $b \mid a$, or else

(ii) $a + b \mid 1 - ab$.

Proof. As R is a domain, from (1) we infer $y = z$.

The system (S_2) becomes

$$\begin{aligned} b(y - z) &= 0 \\ ax + (b - a)y - bu &= 0 \\ a(y - z) &= 0. \end{aligned}$$

Thus, we just have to solve $ax + (b - a)y - bu = 0$, together with $xu = y^2 + 1$.

Notice that for $bu = ax + (b - a)y = a(x - y) + by$ we need

$$b \mid a(x - y), \tag{$*$}$$

condition we denote by ($*$).

Multiplying by b and replacing bu we get $x[ax + (b - a)y] = by^2 + b$ or $ax^2 + (b - a)xy - by^2 = b$, a (Diophantine like) quadratic equation with $\Delta = (b - a)^2 + 4ab = (a + b)^2$. Hence we can decompose this equation in two linear factors, namely $(ax + by)(x - y) = b$.

Such equations are actually solved by decomposing the RHS, that is, if $b = b_1 b_2$ with $ax + by = b_1$, $x - y = b_2$ then $(a + b_1 b_2)y = b_1 - a b_2$.

Thus, the linear equations are solvable iff $a+b_1b_2 \mid b_1-ab_2$. Finally, u is given by $bu = a(x-y)+by = ab_2 + by$ iff $b_1 \mid a$. Hence the invertible matrix U which satisfies $AU = U\,\mathrm{rot}(A)$ is

$$U = \begin{bmatrix} \dfrac{b_1 - ab_2}{a+b} + b_2 & \dfrac{b_1 - ab_2}{a+b} \\ \dfrac{b_1 - ab_2}{a+b} & \dfrac{b_1 - ab_2}{a+b} + \dfrac{a}{b_1} \end{bmatrix} = \begin{bmatrix} \dfrac{b_1(1+b_2^2)}{a+b} & \dfrac{b_1 - ab_2}{a+b} \\ \dfrac{b_1 - ab_2}{a+b} & \dfrac{b_1^2 + a^2}{b_1(a+b)} \end{bmatrix}.$$

(i) If $b_2 = 1$ then for $a+b \mid b-a$ and $b \mid a$ we get $U = \begin{bmatrix} \dfrac{2b}{a+b} & \dfrac{b-a}{a+b} \\ \dfrac{b-a}{a+b} & \dfrac{a^2+b^2}{a+b} \end{bmatrix}$.

(ii) If $b_1 = 1$, then for $a+b \mid 1-ab$ (which also implies $a+b \mid 1+a^2$ and $a+b \mid 1+b^2$) we get $U = \begin{bmatrix} \dfrac{1+b^2}{a+b} & \dfrac{1-ab}{a+b} \\ \dfrac{1-ab}{a+b} & \dfrac{1+a^2}{a+b} \end{bmatrix}$. □

REMARK. In the above case (i), more details are available.

If $a+b \mid b-a$ and $b \mid a$ with $a = bt$ and $(a+b)y = b-a$ then we get $(1+t)y = 1-t$, a (Diophantine like) quadratic equation which can be written $(t+1)(y+1) = 2$.

COROLLARY 2.6. An integral matrix $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ with $b \mid a$ and $a+b \mid b-a$ is rotable iff $a \in \{b, 0, -2b, -3b\}$.

Proof. As noticed in the previous remark, if $a = bt$ and $(a+b)y = b-a$ then $(t+1)(y+1) = 2$ which, for $R = \mathbb{Z}$, is a Diophantine quadratic equation with *only* 4 (symmetric) solutions: $(y,t) \in \{(1,0),(0,1),(-3,-2),(-2,-3)\}$.

If $t = 0$, $y = z = 1$, $xu = 2$, then $a = 0$ and indeed,

$$AU = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & \frac{2}{x} \end{bmatrix} = \begin{bmatrix} b & \frac{2b}{x} \\ b & \frac{2b}{x} \end{bmatrix} = \begin{bmatrix} x & 1 \\ 1 & \frac{2}{x} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} = U\,\mathrm{rot}(A) \qquad \text{for } x = 2 \text{ (and so } u = 1).$$

If $t = 1$, $y = z = 0$, $xu = 1$, then $a = b$ and so $A = \mathrm{rot}(A)$.

If $t = -3$, $y = z = -2$, $xu = 5$, then $a = -3b$ and indeed,

$$AU = \begin{bmatrix} -3b & b \\ -3b & b \end{bmatrix} \begin{bmatrix} x & -2 \\ -2 & \frac{5}{x} \end{bmatrix} = \begin{bmatrix} -3bx - 2b & 6b + \frac{5b}{x} \\ -3bx - 2b & 6b + \frac{5b}{x} \end{bmatrix}$$

$$= \begin{bmatrix} x & -2 \\ -2 & \frac{5}{x} \end{bmatrix} \begin{bmatrix} -3b & -3b \\ b & b \end{bmatrix} = U\,\mathrm{rot}(A) \qquad \text{for } x = -1.$$

If $t = -2$, $y = z = -3$, $xu = 10$, then $a = -2b$ and indeed,

$$AU = \begin{bmatrix} -2b & b \\ -2b & b \end{bmatrix} \begin{bmatrix} x & -3 \\ -3 & \frac{10}{x} \end{bmatrix} = \begin{bmatrix} -2bx - 3b & 6b + \frac{10b}{x} \\ -2bx - 3b & 6b + \frac{10b}{x} \end{bmatrix}$$

$$= \begin{bmatrix} x & -3 \\ -3 & \frac{10}{x} \end{bmatrix} \begin{bmatrix} -2b & -2b \\ b & b \end{bmatrix} = U\,\mathrm{rot}(A) \qquad \text{for } x = -2.$$

Therefore, including the trivially rotable ones, we have precisely the matrices $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ with $a \in \{b, 0, -2b, -3b\}$. □

EXAMPLES (Over any commutative ring). (1) $A = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}$, that is $a = 4, b = -2$. Then $y = -3 = z$, $x = -2$ (for $x - y = 1$) and so $u = -5$. Then $AU = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ -3 & -5 \end{bmatrix} = -2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} = U \operatorname{rot}(A)$.

This was chosen with $b \mid a$ and $a + b \mid b - a$ (that is $b - a = (a + b)y$ and $x - y = 1$).

In the Introduction we claimed that a matrix similar to a rotable matrix may not be rotable. Indeed, consider $M = U^{-1}AU$ for $U = I_2 + E_{12}$. Then $M = \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}$ is similar to A (which is rotable) but, as $\operatorname{Tr}(M) = 2 \neq 4 = \operatorname{Tr}(\operatorname{rot}(M))$, M is not rotable over any ring with $2 \neq 0$.

(2) $A = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$, that is $a = -3, b = 1$. Then $y = -2 = z$, $x = -1$ and so $u = -5$. The result for $U = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$ is $AU = U \operatorname{rot}(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Also chosen as above (incl. $x - y = 1$).

(3) $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$, that is $a = 2, b = 3$. Then $y = z = -1$, $x = y + b = 2$ and so $u = 1$. The result for $U = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is $AU = U \operatorname{rot}(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

This was chosen with $a + b \mid 1 - ab$ (here $x - y = b$ and $xu = y^2 + 1$).

REMARK. The fact that in the examples above $AU = U \operatorname{rot}(A)$ is a multiple of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (i.e., has equal entries) follows easily by replacing $c = a$, $d = b$ in the system (S_1).

Since the matrices in Case 3 (we exclude the trivially rotable matrices and the nilpotents in Proposition 2.3 (iii)) are just the transposes of those in Case 2, using Lemma 2.2, we obtain at once

THEOREM 2.7. Let R be an integral domain of $\operatorname{char}(R) \neq 2$, $A = \begin{bmatrix} a & a \\ c & c \end{bmatrix} \in \mathbb{M}_2(R)$ and $a \notin \{\pm c\}$. The matrix A is rotable iff there exists a decomposition $a = a_1 a_2$ such that $a_1 a_2 + c \mid a_1 + c a_2$ and $a_1 \mid c$.

In particular, A is rotable whenever

(i) $a \mid c$, or else
(ii) $a + c \mid 1 + ac$.

Proof. We get $AU = U \operatorname{rot}(A)$ for $U = \begin{bmatrix} \frac{a_1 + ca_2}{a + c} & a_2 - \frac{a_1 + ca_2}{a + c} \\ \frac{c}{a_1} - \frac{a_1 + ca_2}{a + c} & \frac{a_1 + ca_2}{a + c} \end{bmatrix} = \begin{bmatrix} \frac{a_1 + ca_2}{a + c} & \frac{a_1(a^2 - 1)}{a + c} \\ \frac{c^2 - a_1^2}{(a + c)c_1} & \frac{a_1 + ca_2}{a + c} \end{bmatrix}$.

As for the particular cases,

(i) if $a_2 = 1$ and $a \mid c$ then we have $AU = U \operatorname{rot}(A)$ for $U = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} - 1 & 1 \end{bmatrix}$.

(ii) If $a_1 = 1$ and $a + c \mid 1 + ac$ then we have $AU = U \operatorname{rot}(A)$ for $U = \begin{bmatrix} \frac{1 + ac}{a + c} & \frac{a^2 - 1}{a + c} \\ \frac{c^2 - 1}{(a + c)c_1} & \frac{1 + ac}{a + c} \end{bmatrix}$. □

EXAMPLES. (1) $A = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$. As above $c = a(z + 1)$ and indeed, for $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ we get $AU = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} = U \operatorname{rot}(A)$.

(2) $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. Now $x = u$, $x + y = a$, $az = (c - a)x + cy$ and so, for $U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ we obtain $AU = \begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix} = U\,\mathrm{rot}(A)$.

(3) $A = \begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$. Here $U = \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix}$ and $AU = \begin{bmatrix} -6 & 4 \\ 9 & -6 \end{bmatrix} = U\,\mathrm{rot}(A)$.

REMARK. Now $AU = U\,\mathrm{rot}(A) = \begin{bmatrix} c(x+y) & a(x+y) \\ c(x+z) & c(x+y) \end{bmatrix}$ with $a(x + z) = c(x + y)$.

3. THE $\mathrm{char}(R) = 2$ CASE

As previously mentioned, $\det(A) = \det(\mathrm{rot}(A))$ is equivalent to $2(ad - bc) = 0$. Note that this always holds if $\mathrm{char}(R) = 2$.

Therefore, in this case, in order to find rotable matrices, we can *only* use $\mathrm{Tr}(A) = \mathrm{Tr}(B)$, that is $a + d = b + c$.

Moreover, $\det(M) = 2(c - b)(b - d)(c - d) = 0$, always holds if $\mathrm{char}(R) = 2$, that is, the system (S_2) has always nonzero solutions. However, in order to give rotable matrices, these solutions must verify $xu - yz = 1$.

The determination of the rotable matrices in the $\mathrm{char}(R) = 2$ is harder and is not addressed here.

For just a glimpse, compared to Proposition 2.3 (ii), we discuss the 8 idempotent matrices of $\mathbb{M}_2(\mathbb{F}_2)$.

We have noticed the invariance to conjugations for idempotent (or nilpotent or unit) matrices. Since $\mathbb{M}_2(\mathbb{F}_2)$ has 8 idempotents, 4 nilpotents and 6 units, rotation similarities may occur only in these subclasses of elements.

The idempotents are: 0_2, I_2, E_{11}, E_{22}, $E_{11} + E_{12}$, $E_{11} + E_{21}$, $E_{12} + E_{22}$, $E_{21} + E_{22}$.

Since I_2 is similar only with itself, but its rotation is $U_2 \neq I_2$, we infer that I_2 is not rotable. Clearly, 0_2 is (trivially) rotable.

All the nontrivial idempotents have trace 1.

For the next two idempotent matrices (i.e., E_{11}, E_{22}), the trace of their rotations is zero, so these are not rotable.

For the other four we have a "cyclic" rotation, that is, $\mathrm{rot}(E_{11} + E_{12}) = E_{12} + E_{22}$, $\mathrm{rot}(E_{12} + E_{22}) = E_{21} + E_{22}$, $\mathrm{rot}(E_{21} + E_{22}) = E_{11} + E_{21}$ and $\mathrm{rot}(E_{11} + E_{21}) = E_{11} + E_{12}$. According to the Lemma 2.2 (a), it suffices to check whether $E_{11}+E_{12}$ is rotable. It is indeed: $(E_{11}+E_{12}) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = E_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (E_{12}+E_{22})$. Hence all these four idempotent matrices are rotable.

This is to be compared with Proposition 2.3 (ii).

4. ANNEX

In this section we first give the details of the proof of Proposition 2.4, that is, the computation for $\det(M) = 2(c - b)(b - d)(c - d)$.

To compute its determinant, some elementary transformations performed on M, yield

$$\overset{\mathrm{row}_3+\mathrm{row}_1}{\sim} \begin{bmatrix} a & -d & b-c+d & -d \\ a & b-a & 0 & -b \\ c & 0 & d-c & -d \\ 0 & c & -a & d-b \end{bmatrix} \overset{-\mathrm{row}_1+\mathrm{row}_2}{\sim} \begin{bmatrix} a & -d & b-c+d & -d \\ 0 & b-a+d & -b+c-d & -b+d \\ c & 0 & d-c & -d \\ 0 & c & -a & d-b \end{bmatrix}.$$

Since $a+d = b+c$, the latter becomes $\begin{bmatrix} a & -d & 2b-a & -d \\ 0 & 2d-c & a-2b & -b+d \\ c & 0 & d-c & -d \\ 0 & c & -a & d-b \end{bmatrix}$, using $b-a = d-c$, $a-c = b-d$.

Therefore,

$$\det(M) = a\det\begin{bmatrix} 2d-c & a-2b & -b+d \\ 0 & d-c & -d \\ c & -a & d-b \end{bmatrix} + c\det\begin{bmatrix} -d & 2b-a & -d \\ 2d-c & a-2b & -b+d \\ c & -a & d-b \end{bmatrix}.$$

The first determinant simplifies if we add minus third row to the first:

$$\det\begin{bmatrix} 2(d-c) & 2(a-b) & 0 \\ 0 & d-c & -d \\ c & -a & d-b \end{bmatrix} = 2d(d-c)(d-b).$$

The second determinant simplifies if we add the second row to the first and minus the third row to the second:

$$\det\begin{bmatrix} d-c & 0 & -b \\ 2(d-c) & 2(a-b) & 0 \\ c & -a & d-b \end{bmatrix} = 2(d-c)\det\begin{bmatrix} d-c & 0 & -b \\ 1 & -1 & 0 \\ c & -a & d-b \end{bmatrix} = 2(d-c)(d-b)(a-2b)$$

using also $ad = bc$ and $b - d = a - c$ and $a - b = c - d$.

Finally, $\det(M) = 2(c-b)(b-d)(c-d)$.

Secondly, we provide the details for finding the matrix U in Lemma 2.3 (ii).

It is easy to find a similarity if $a = 0$:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and for $a = 1$:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

For $a \notin \{0, 1\}$ we have to find an invertible matrix $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ (depending on a) such that $\begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$. This equality reduces to $x = w$, $a(2x + y + z) = x + y$ together with $x^2 - yz = 1$. Multiplying the latter by a, we can eliminate z and obtain $(x + y)[a(x + y) - y] = a$. Since $a \mid x + y$ we can write $t := x + y = at'$ and $t'(a^2t' - y) = 1$. Choosing $t' = 1$ we finally get $U = \begin{bmatrix} -a^2+a+1 & a^2-1 \\ a^2-2a & -a^2+a+1 \end{bmatrix}$ with $\det(U) = 1$ and $\begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}U = U\begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$, as desired. Notice that this covers also $a \in \{0, 1\}$.

REFERENCES

[1] Atiyah, M. F. and Macdonald, I. G. *Introduction to Commutative Algebra.* Addison-Wesley, Reading, Massachusetts, 1969.

[2] Brown, W. C. *Matrices over commutative rings.* Marcel Dekker Inc., New-York, Basel, Hong Kong 1993.

[3] Jacobson, N. *Basic Algebra I.* Second edition, W. H. Freeman, New York, 1985.

[4] Taussky, O. and Zassenhaus, H. On the similarity transformation between a matrix and its transpose. *Pacific J. Math.* 9 (1959), 893–896.