International Journal of Algebra, Vol. 20, 2026, no. 3, 115 - 124
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2026.92033

Ring Multiples of (not) Clean Elements

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Abstract

An element of a unital ring is called strongly not clean if all of its unit multiples fail to be clean and very strongly not clean if all its nonzero products with arbitrary (nonzero) elements fail to be clean. In this paper, we investigate such elements, with a particular focus on matrices over commutative rings. In our main result we show that a 2×2 matrix over any Bézout domain with almost stable range 1 is vsn-clean iff its entries are not coprime, excepting $2I_2$.

Mathematics Subject Classification: 16U10, 16U30, 16U40, 16S50

Keywords: clean, very strongly not clean, almost stable range one, Bézout ring, pre-Schreier ring, matrix, quadratic Diophantine equation

1 Introduction

It is well known that cleanness behaves in a delicate way with respect to multiplication. On the one hand, one easily finds clean elements whose products with certain elements fail to be clean. On the other hand, there are non-clean elements whose products with suitable elements turn out to be clean. Thus, cleanness is far from being multiplicatively stable, and this instability naturally raises the question of how far such phenomena can go.

From a multiplicative point of view, two extreme behaviors may occur. First, there may exist clean elements whose product with every element remains clean; this happens if and only if the ring itself is clean. Second, there may exist non-clean elements whose nonzero products with every other nonzero element remain non-clean. Since the first situation characterizes clean rings and is therefore already well understood, we focus on the second, more rigid behavior. This leads to the following definitions.

An element is called *strongly not clean* (*sn-clean*) if all its products (on either side) with units are not clean. An element is called *very strongly not clean* (*vsn-clean*) if all its nonzero products (on either side) with arbitrary (nonzero) elements are not clean. As we work throughout with rings with identity, these notions define subclasses of the not clean elements.

Clearly,

$$\{\text{vsn-clean elements}\} \subset \{\text{sn-clean elements}\} \subset \{\text{not clean elements}\}.$$

Since our interest lies exclusively in non-clean elements, we may disregard those elements that are automatically clean. In particular, idempotents, nilpotents, and units (together with their conjugates) are clean and therefore irrelevant to our study. This excludes, in particular, the elements 0 and 1, as well as trivial clean elements of the form $1 + u$ with a unit u.

The motivation for introducing (very) strongly not clean elements is twofold. First, they measure an extreme form of multiplicative rigidity: their failure of cleanness is stable under multiplication, either by units or by arbitrary nonzero elements. In this sense, they represent the strongest possible obstruction to cleanness. Second, their existence and structure reflect subtle arithmetic properties of the underlying ring, linking additive decompositions (as sums of idempotents and units) with multiplicative and divisibility conditions.

In this note, we investigate (very) strongly not clean 2×2 matrices over commutative rings. The case of 2×2 matrices already displays a rich interplay between matrix factorizations and arithmetic properties of the entries, while remaining sufficiently tractable to allow a complete description. Our main theorem shows that a 2×2 matrix over a Bézout domain with almost stable range one is vsn-clean if and only if its entries are not coprime, with the single exception of $2I_2$. This characterization reveals that maximal multiplicative non-cleanness is governed purely by a divisibility condition on the entries.

The rings we consider are associative, nonzero, with identity. We consider only matrices over commutative rings; this assumption will henceforth remain implicit. The set of all units of a ring R is denoted by $U(R)$. A ring is called a *GCD* ring if any two elements admit a greatest common divisor.

2 Clean elements with only clean multiples

This case is straightforward for rings with identity.

Proposition 1 *Let R be a ring with identity. If every multiple of clean element of R is clean, then R is clean. The converse holds.*

Proof. Since 1 is clean, by hypothesis, all its multiples are clean. Hence, the ring is clean. The converse is obvious. ■

As it is well-known, "limit" clean examples are the division rings (all nonzero elements are units) and the Boolean rings (all elements are idempotent).

3 Vsn matrices over some Bézout domains

First, we list 2×2 matrices *over domains* that are clean and should be excepted in the remaining of the section.

a) Idempotents: 0_2, I_2 and matrices with trace 1 and zero determinant.
b) Nilpotents: matrices with zero trace and zero determinant.
c) Units: matrices with unit determinant.
d) Matrices $A = I_2 + U$ with unit U. If $\det(U) = u$, these have the property $Tr(A) + u = \det(A) + 1$.

The (c), (d) matrices are the trivial clean matrices.

The nontrivial clean 2×2 matrices over domains are characterized as follows.

Theorem 2 *Let R be a domain, $u \in U(R)$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$. Then A is nontrivial clean iff the system*

$$\begin{cases} x^2 - x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - a = u & (2) \end{cases}$$

with unknowns x, y, z, has at least one solution over R. If $b \neq 0$ and (2) holds, then (1) is equivalent to

$$bx^2 - (a-d)xy - cy^2 - bx + (a - \det(A) + u)y = 0 \quad (3).$$

Proof. For a nontrivial idempotent $E = \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ with (1), we write $\det(A - E) = u \in U(R)$ which gives (2). If $b \neq 0$, we multiply (1) by b and eliminate bz using (2). This gives (3). ■

Remark. The equations (3) always have the solutions $(0, 0)$ and $(1, 0)$.

Our first main result follows. To simplify the writing/wording, some finitely many elements in a GCD domain are called *coprime* if their greatest common divisor equals 1. If the entries of a 2×2 matrix A are coprime, we also use the term *unimodular* (for a more general definition, see [6]). In notation, $A \in Um(\mathbb{M}_2(R))$.

Theorem 3 *Let R be a (commutative) GCD domain and* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$ *with* $\gcd(a,b,c,d) \neq 1$. *Then A is vsn-clean, excepting* $2I_2$ *.*

Proof. Recall that statements about divisibility are well-defined only up to association (i.e., up to multiplication by a unit). Thus, $\gcd(a,b,c,d) \neq 1$ means that this gcd is not a unit. Note that if $\gcd(a,b,c,d) \neq 1$, then the equations (2) from Theorem 2, have no solution (in any commutative ring, any divisor of a unit must be a unit). If $\gcd(a,b,c,d) \neq 1$, the entries of A are not coprime, and so are the entries of AB and BA for any (nonzero) 2×2 matrix B, such that $AB \neq 0_2 \neq BA$. Therefore all (nonzero) products are not clean. ■

Next, for our second main result, some prerequisites are necessary.

A commutative ring R is called *pre-Schreier*, if every nonzero element $r \in R$ is *primal*, i.e., if r divides xy, there are r_1, r_2 elements in R such that $r = r_1 r_2$, r_1 divides x and r_2 divides y.

The following chain of implications is well known: Bézout domain ⇒ GCD-domain ⇒ pre-Schreier domain.

The *inner rank* of an $m \times n$ matrix A over a ring is defined as the least integer k such that A can be expressed as a product of an $m \times k$ matrix and an $k \times n$ matrix. For example, over a division ring this notion coincides with the usual notion of rank. A square matrix is called *full* if its inner rank equals its order, and *non-full* otherwise.

From [5], we recall

Theorem 4 *Let R be any ring (with identity). Consider the following conditions:*

(i) every 2×2 zero determinant matrix over R is non-full;

(ii) R is pre-Schreier.

Then (i) implies (ii) and, if R is a domain, (ii) implies (i).

Further, a ring R has *almost stable range 1*, denoted $asr(R) = 1$, if for each ideal I of R not contained in $J(R)$, $sr(R/I) = 1$. Also recall the notation $J_3(R) := \{(a,b,c) \in R^3 | aR + bR = R \text{ and } c \notin J(R)\}$, where $J(R)$ denotes the Jacobson radical of R.

We will need the following key general result.

Lemma 5 *Let (A,B,C,D) be a unimodular element of the R-module R^4. If R is pre-Schreier domain, then the following two statements are equivalent:*

(1) There exist $E, F \in R$ such that $(AE + CF, BE + DF) \in R^2$ is unimodular.

(2) There exist $x, y, z, u \in R$ such that $Ax + By + Cz + Du = 1$ and the matrix $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ has zero determinant (equivalently, it has rank 1 or also equivalently, it is non-full).

Proof. To prove that (1) $\Rightarrow$ (2) let $E, F, s, t \in R$ be such that we have an identity $(AE + CF)s + (BE + DF)t = 1$. We take $\begin{bmatrix} x & y \end{bmatrix} := E\begin{bmatrix} s & t \end{bmatrix}$ and $\begin{bmatrix} z & u \end{bmatrix} := F\begin{bmatrix} s & t \end{bmatrix}$. Thus $\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix}\begin{bmatrix} s & t \end{bmatrix}$ is non-full of zero determinant and from the identity

$$Ax+By+Cz+Du = AEs+BEt+CFs+DFt = (AE+CF)s+(BE+DF)t = 1$$

it follows that statement (2) holds.

To prove that (2) $\Rightarrow$ (1) let x, y, z, u be as in the statement (2). According to Theorem 4, there exist $E, F, s, t \in R$ such that we have $\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix}\begin{bmatrix} s & t \end{bmatrix}$. From this and the relation $Ax + By + Cz + Du = 1$ we get that $(AE + BF)s + (CE + DF)t = 1$, which implies that statement (1) holds. ■

Proposition 6 *Let A, B, C, D be coprime elements of a Bézout domain R that has almost stable range 1. Then the two equivalent statements of Lemma 5 hold.*

Proof. Let $g := \gcd(A, C)$ and $h := \gcd(B, D)$. So $\gcd(g, h) = 1$ and we can write $A = ga$, $C = gc$, $B = hb$ and $D = hd$ with $a, b, c, d \in R$ such that we have $\gcd(a, c) = \gcd(b, d) = 1$. Let $e, f \in R$ be such that $ae + cf = 1$. As $\gcd(g, h) = 1$, by the symmetry between the pairs (A, C) and (B, D), we can assume that $h \notin J(R)$.

For some $j, v \in R$, take $E := je + cv$ and $F := jf - av$. Thus

$$AE + CF = g(ae + cf) = gj(ae + cf) + g(acv - cav) = gj.$$

For $l := bc-ad$ and $w := be+df$ we can write $\begin{bmatrix} l & w \end{bmatrix} = \begin{bmatrix} b & d \end{bmatrix}\begin{bmatrix} c & e \\ -a & f \end{bmatrix}$. As the matrix $\begin{bmatrix} c & e \\ -a & f \end{bmatrix}$ has determinant 1, we get $\gcd(l, w) = \gcd(b, d) = 1$. Let $p, q \in R$ be such that $1 = pl + qw$.

We compute

$$BE+DF = h(bE+dF) = h(bje+bcv+djf-adv) = hj(be+df)+hv(bc-ad)$$
$$= h(jw + vl).$$

We now take $j = q + lm$ and $v = p - wm$, with $m \in R$. Thus

$$BE + DF = h(qw + pl + lwm - wlm) = h(qw + pl) = h.$$

As $(q, l, m) \in J_3(R)$ and R has almost stable range 1.5, there exists $r \in R$ such that $\gcd(q + lr, h) = 1$. Therefore

$$\gcd(AE + CF, BE + DF) = \gcd(gj, h) = \gcd(j, h) = \gcd(q + lr, h) = 1.$$

Thus the statement (1) holds. ∎

Finally, here is our second main result.

Theorem 7 *Over any Bézout domain with almost stable range 1, every* 2×2 *matrix with coprime entries has an nontrivial idempotent multiple.*

Proof. By Proposition 6, there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and the (zero determinant) matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is non-full.

Note that as $\det(X) = 0$, then $\det(AX) = 0$ as well. Moreover, $Tr(AX) = 1$ and so AX is a (nontrivial) idempotent. ∎

Theorem 8 *Let* R *be a Bézout domain with almost stable range 1 and let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$ *with* $\gcd(a, b, c, d) = 1$*. Then* A *is not vsn-clean.*

Proof. Choose X as in the previous proposition. Then AX has zero determinant and trace $= 1$, so it is idempotent, and therefore clean. ∎

Here is our third main result.

Corollary 9 *Over any Bézout domain with almost stable range 1, a nonzero* 2×2 *matrix is vsn-clean iff its entries are not coprime, with only one exception:* $2I_2$.

Corollary 10 *Over any Bézout domain with almost stable range 1, the classes of* 2×2 *sn-clean matrices and* 2×2 *vsn-clean matrices coincide.*

Remarks. 1) Using results from [6], slightly more general results of this type may be proved for the so-called SE_2 domains, class which includes elementary divisors domains and semilocal rings.

2) Related to our above hypothesis, it is worth mentioning the following characterization (see [7] Theorem 3.7): Suppose R has almost stable range 1. Then R is a Bézout ring iff R is an elementary divisor ring.

A different result is available for nonzero nilpotent multiples.

Proposition 11 *Over any commutative pre-Schreier ring, every nonzero* 2×2 *matrix with zero divisor determinant has an nonzero nilpotent multiple.*

Proof. Let $a, b, c, d \in R$ not all zero. Then we show that there exist $x, y, z, t \in R$, not all zero, such that $xt - yz = 0$ and $ax + bz + cy + dt = 0$.

The condition $xt - yz = 0$ means that the matrix $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ has determinant zero. According to Theorem 4, there exist $r, s, u, v \in R$, not all zero, such that $\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}$, that is, $x = ru, \quad y = rv, \quad z = su, \quad t = sv$. Substituting into the second condition, we obtain

$$ax + bz + cy + dt = a(ru) + b(su) + c(rv) + d(sv) = u(ar + bs) + v(cr + ds).$$

Set

$$\alpha = ar + bs, \qquad \beta = cr + ds.$$

Then the required condition becomes $u\alpha + v\beta = 0$, which has always the solution $u = -\beta$, $v = \alpha$.

It remains to deal with the case when both $\alpha = \beta = 0$. Then we can choose arbitrary nonzero u and v. For the homogeneous linear system $ar + bs = 0 = cr + ds$, the determinant is precisely $\det(A)$. As this was supposed a zero divisor, the rank of A is < 2. Using McCoy's theorem (see [3], Theorem 5.3), the system has a nontrivial solution, and this completes the proof: AX has zero determinant and zero trace, so, by Cayley-Hamilton's theorem, is a (nonzero) nilpotent. ■

Example. Take again $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Most of the computation in the (idempotent) previous example holds, including the choice $r = -1$, $s = 1$ which gives $\alpha = \beta = 1$. Now we choose $u = -\beta = -1$, $v = \alpha = 1$ and indeed, $AX = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ is a nonzero nilpotent.

4 Not clean matrices with clean unit multiples

The examples in this section show that the right inclusion mentioned in the Introduction is proper.

The following result is particularly useful when working with clean and non-clean elements.

Lemma 12 *Let R be a ring, $a \in R$ and $u \in U(R)$. Then ua is (not) clean iff au is (not) clean.*

Proof. As the clean property is invariant under conjugations, the statement follows from $au = u^{-1}(ua)u$ and $ua = u(au)u^{-1}$. ■

Next, for $R = \mathbb{Z}$, in Theorem 2 we replace the unit u by ± 1 and obtain

$$(a-d)x + cy + bz + \det(A) - a = \pm 1,$$

which we denote (± 2) and

$$bx^2 - (a-d)xy - cy^2 - bx + (a - \det(A) \pm 1)y = 0,$$

which we denote (± 3).

To solve quadratic Diophantine equations we use [2] or [9].

1. Consider $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ (the example in [1]). To check that A is not clean, we use Theorem 2. For $a = 3$, $b = 9$, $c = -7$ and $d = -2$ we have

$$(\pm 3) : 9x^2 - 5xy + 7y^2 - 9x + (-54 \pm 1)y = 0,$$

with the solutions $(0,0), (1,0)$ for (+3), respectively $(0,0), (1,0), (2,9), (4,9)$ for (-3). Now

$$(\pm 2) : 5x - 7y + 9z + 54 = \pm 1$$

and one can check that the solutions of (+3) do not verify (+2), and the solutions of (-3) do not verify (-2). Hence A is not clean.

A **multiplied by some units.**

According to Lemma 12, if AU is clean, so is UA. Hence we check for cleanness only products AU with invertible U.

(i) For $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $AU = \begin{bmatrix} 9 & 3 \\ -2 & -7 \end{bmatrix}$, with equations

$$(\pm 3) : 3x^2 - 16xy + 2y^2 - 3x + (66 \pm 1)y = 0$$

and

$$(\pm 2) : 16x - 2y + 3z - 66 = \pm 1.$$

(+3) has infinitely many solutions; among these $(0,0)$, $(1,0)$, $(6,10)$, $(19,114)$.

(-3) has infinitely many solutions; among these $(0,0)$, $(1,0)$.

Obviously, $(1,0)$ satisfies (+2) which gives $z = 17$ and the clean decomposition $AU = \begin{bmatrix} 1 & 0 \\ 17 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 3 \\ -19 & -7 \end{bmatrix}$. Hence AU is clean.

In the remaining of this section, since the use of the equations (± 3) and (± 2), is analogous, we present just the clean decompositions.

(ii) For $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have $AU = \begin{bmatrix} 3 & -9 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 6 & -9 \end{bmatrix} + \begin{bmatrix} -7 & 6 \\ -13 & 11 \end{bmatrix}$.

As seen above, $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ is not clean, and also *not sn-clean.*

2. Consider $B = \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix}$, which, as shown in [8], is known to be not clean.

B **multiplied by some units**.

(i) For $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $BU = \begin{bmatrix} 5 & 12 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ -2 & -5 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -6 & -14 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 11 & 26 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -4 & -10 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 22 \\ -2 & -5 \end{bmatrix}$.

So BU has clean index 3.

(ii) For $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have $BU = \begin{bmatrix} 12 & -5 \\ 0 & 0 \end{bmatrix}$. However, BU is not clean.

These examples show that *not clean matrices may have clean unit multiples.* Moreover, some unit multiples may be clean, other multiples may not.

Just reversing the rôles, we get *clean matrices with not clean unit multiples.*

To show that the left inclusion mentioned in the Introduction is proper, an **example** follows.

Take $R = \mathbb{Z} \times F$ for any field F, and consider $a = (3, 1)$. The units of R are $u = (\pm 1, f)$ for every nonzero $f \in F$. Hence $au = (\pm 3, f)$ are not clean because ± 3 is not clean in $\mathbb{Z}$. Thus, a is sn-clean.

Now take $r = (0, 1) \in R$. Since $ar = (0, 1)$ is clean in R, it follows that a is not vsn-clean.

In closing, we provide some diagonal (non)examples of vsn-clean 2×2 matrices (see also [4], for diagonal examples).

Proposition 13 *Let R be any commutative ring.*

(i) For every $n \geq 3$ or $n \leq -2$, nE_{11} is vsn-clean.

(ii) For every $n \geq 3$ or $n \leq -2$, nI_2 is vsn-clean.

(iii) For every $n \geq 3$, $\begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}$ is not clean and also not sn-clean (and so not vsn-clean).

Remark. 1) $2E_{11} = I_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is trivial clean, but $-2E_{11}$ is not clean.
2) $2I_2$ is trivial clean, but $-2I_2$ is not clean.

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Received: March 15, 2026; Published: April 3, 2026