BIPOTENTS: A SPECIAL CLASS OF QUASIREGULAR ELEMENTS IN RINGS

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Abstract. In the monoid $(R, \circ)$ associated to any general ring R, an element $b \in R$ is called *bipotent* if $b \circ b = 0$. The paper develops the theory of such elements, mainly in rings with identity.

1. Introduction

The following construction is well-known. In any (nonzero, associative) ring (with or without identity) R, define $a \circ b = a + b - ab$. Then this binary operation is associative, and $(R, \circ)$ is a monoid with zero as the identity element. An element $a \in R$ is called *left (resp. right) quasi-regular* if a has a left (resp. right) inverse in the monoid $(R, \circ)$ with identity. If a is both left and right quasi-regular, we say that a is *quasi-regular*.

The object of our study are some special (left and right) quasi-regular elements. An element b of a (general) ring R will be called *bipotent* if it is its own inverse in the monoid $(R, \circ)$, that is, if $b \circ b = 0$. Equivalently, an element b of a (general) ring R is called *bipotent* if $b^2 = b + b$. For a ring R, $Bip(R)$ will denote the set of all the bipotents of R. We shall discard rings of characteristics 2, as in such rings the bipotents are just the zero-square (nilpotent) elements. To simplify the wording, we say that an element a "*is a double*" if there is an element c such that $a = c + c$.

Clearly $0, 2$ are bipotents in any ring with identity and are called the *trivial* bipotents.

More general, *every double of idempotent is bipotent*. However, examples show that not every bipotent is the double of an idempotent.

A ring will be called *biconnected* if it has only the trivial bipotents. As an example, *every* (not necessarily commutative) *domain is biconnected*.

It follows from definitions that *the bipotent are quasi-regular*, and, as *nonzero idempotents are not* quasi-regular, these are also not *bipotents*.

If the ring R has identity (i.e., is unital), since $b^2 = 2b$ is equivalent to $(b-1)^2 = 1$, it follows that *the bipotents are precisely the sums* $1 + u$ with an *order two unit* u ($u^2 = 1$).

Recall that an element $r \in R$ in a ring with identity is *quasi-regular*, if $1 - r$ is a unit in R. Equivalently, r is quasi-regular if $r = 1 - u$ with $u \in U(R)$.

As every nilpotent element is quasi-regular and every element of the Jacobson radical of a (not necessarily commutative) ring is quasi-regular, a discussion of when such elements are bipotents is in order (and will be done in subsection 2.1).

We also recall an important condition in developing our theory: a ring R is called *2-torsionfree* if $2x = 0$ implies $x = 0$, for every $x \in R$. Equivalently, 2 is a non-zero-divisor (or is cancellable). Also equivalently, 2^k is a non-zero-divisor (or is cancellable) for some $k \geq 2$. In particular, any domain (not necessarily commutative) is 2-torsionfree.

The plan of the paper is as follows. In section 2 we gather some simple properties of the bipotent elements and give some examples and nonexamples. In section 3 we characterize the bipotents of $\mathbb{Z}_n$ for any positive integer n and in section 4 we characterize the bipotent 2×2 matrices over commutative rings.

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2. Simple properties

If b is bipotent, then $-b$ is not bipotent, unless $b = 0$ or $char(R) = 4$. As examples will show, *sums and products of bipotents need not be bipotents.*

The following properties (gathered here for easy reference) are straightforward, so we omit the proofs but provide the necessary examples.

Lemma 1. *(i) If b is bipotent, so is $2-b$ (hereafter called* complementary bipotent*) and these are orthogonal. As such, bipotents are zero-divisors.*

(ii) Bipotence is invariant to conjugation.

(iii) Bipotence is not invariant to equivalence.

(iv) For any idempotent e (incl. 0 and 1), $2e$ is a bipotent. There are bipotents that are not doubles of idempotents.

(v) If a bipotent b is a double and R is 2-torsionfree then b is the double of an idempotent. In particular, if 2 is a unit of R, every bipotent in R is the double of some idempotent. The converse fails.

(vi) If $ab = 2$ then ba is bipotent.

(vii) Equivalent bipotents may not be conjugate.

Proof. (iii) Over any (unital, nonzero) ring, the unit-bipotent matrix product

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

is not bipotent. Indeed, $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 6 & 6 \\ 3 & 3 \end{bmatrix} \neq 2 \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

(iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ was above mentioned as a bipotent in $\mathbb{M}_2(R)$, which is not the double of any matrix unless 2 is a unit.

As another example, take $\mathbb{Z}_8$, a local ring with $Id(\mathbb{Z}_8) = \{0, 1\}$. We also have $U(\mathbb{Z}_8) = \{1, 3, 5, 7\}$ and all four are order 2 units. Then $Bip(\mathbb{Z}_8) = 1 - U(\mathbb{Z}_8) = \{0, 2, 4, 6\}$ and 4 and 6 are not doubles of (the only trivial) idempotents.

(v) If $b^2 = 2b$ and $b = 2a$ for some a, we get $4(a^2 - a) = 0$. Note that 4 is no zero divisor iff 2 is not.

In the special case, take $b^2 = 2b$ and consider $e = 2^{-1}b$. Then $e^2 = (2^{-1}b)^2 = 2^{-1}b^2 = 2^{-2}(2b) = 2^{-1}b = e$.

In $\mathbb{Z}_{12}$, the only nontrivial idempotents are 4 and 9 and the only nontrivial bipotents are 8 and 6, that is, precisely the doubles of idempotents. However, 2 is not a unit in $\mathbb{Z}_{12}$. Hence the condition is sufficient but not necessary.

(vii) For a negative example we use again the local ring $\mathbb{Z}_8$ (see the details in the proof of (iv), above). As $6 = 3 \cdot 2$, 6 and 2 are equivalent bipotents. However, as $\mathbb{Z}_8$ has only order two units, the conjugates of 2 equal 2 and the conjugates of 6 equal 6. Hence 2 and 6 are not conjugate. □

Remark. If R is 2-torsion-free and the bipotents are doubles (of idempotents, according to (v)) then equivalent bipotents are conjugate. Indeed, if e and f are idempotents and $p, q \in U(R)$ then $2f = p(2e)q$ implies $f = peq$ and we use the Song-Guo result (see [3]).

Example. Sums and products of bipotents need not be bipotents. Indeed, in $\mathbb{M}_2(R)$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a bipotent but, unless $4 = 0$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not bipotent

The bipotents of $\mathbb{Z}_n$ are described in the next section.

2.1. **Nilpotent bipotents and Jacobson radical bipotents.** As every nilpotent element is quasi-regular and every element of the Jacobson radical of a (not necessarily commutative) ring is quasi-regular, a discussion of when such elements are bipotents is in order.

Proposition 2. *If R is 2-torsion-free, 0 is the only bipotent nilpotent.*

Proof. Suppose $t^k = 0$ and $t^{k-1} \neq 0$ for some $k \geq 2$. If $t^2 = 2t$ by multiplication with t^{k-2} we get $2t^{k-1} = 0$ and so $t^{k-1} = 0$, a contradiction. □

However, if 2 is not cancellable, there are nonzero bipotent nilpotents. Examples abound in the next two sections.

It is well-known that the Jacobson radical $J(R)$ is the largest ideal I of R such that $1 + I \subseteq U(R)$. The bipotents of the Jacobson radical are only those $j \in J(R)$ such that $1 + j$ is an order two unit.

Example. Let D be a division ring and let $R = \mathbb{T}_n(D)$, the ring of the upper triangular $n \times n$ matrices over D. It is well-known that $J(R)$ consists in all the matrices with only zero entries on the main diagonal. As for bipotents there are several possibilities.

If $char(D) = 2$, all matrices E_{ij} with $i < j$ are bipotents in $J(R)$, but sums $E_{ij} + E_{jk}$ with $i < j < k$ are not.

If $char(D) \neq 2$, all matrices E_{ij} with $i < j$ are zerosquare but not bipotents in $J(R)$.

Bipotents which belong to the Jacobson radical appear in characterizations of SIT-rings (see [4], Theorem 3.6). A ring was called (strongly) *SIT* if every element is a sum of an idempotent and a tripotent (that commute).

Theorem 3. *The following are equivalent for a ring R.*

(i) R is a strongly SIT-ring with $2 \in J(R)$.

(ii) R is a strongly SIT-ring with $2^3 = 0$.

(iii) R has the identity $x^6 = x^4$ and $2 \in J(R)$.

(iv) $R/J(R)$ is Boolean and $j^2 = 2j$ for all $j \in J(R)$.

(v) $R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2.

Consequently

Corollary 4. *Suppose $R/J(R)$ is Boolean. All elements in the Jacobson radical are bipotents iff $U(R)$ is a group of exponent 2.*

3. The bipotents of $\mathbb{Z}_n$

In order to determine the bipotents of $\mathbb{Z}_n$ we first prove some preparative results.

As doubles of idempotents are bipotents, we recall the construction of idempotents of $\mathbb{Z}_n$.

Assume $n = p_1^{\alpha_1}...p_l^{\alpha_l}$. Then $|Id(\mathbb{Z}_n)| = |\mathcal{P}\{p_1, ..., p_l\}| = 2^l$, including the empty subset and $\{p_1, ..., p_l\}$.

The following straightforward result describes the behavior of bipotents with respect to direct products.

Proposition 5. *Let $\{R_i : i \in I\}$ be a set of rings. Then $(b_i) \in \prod_i R_i$ is bipotent iff b_i is bipotent in R_i, for each $i \in I$. If I is finite, we can replace the direct product by the direct sum.*

In $\mathbb{Z}_n$ (a finite commutative ring) 2 is a zero divisor iff $2 \mid n$ iff n is even iff 2 is not a unit.

Lemma 6. *Let $n = 2^k p_1^{\alpha_1}...p_l^{\alpha_l}$ for some odd primes $p_1, ..., p_l$ and some nonnegative integers $k, \alpha_1, ..., \alpha_l$.*

(i) If $k = 0$, $\mathbb{Z}_n$ has 2^l bipotents.

(ii) If $k = 1$, $\mathbb{Z}_n$ has 2^l bipotents.

(iii) If $k = 2$, $\mathbb{Z}_n$ has 2^{l+1} bipotents.

(iv) If $k \geq 3$, $\mathbb{Z}_n$ has 2^{l+2} bipotents.

Proof. (i) If $k = 0$ then n is odd and so 2 is a unit. We use Lemma 1, (vi) together with the well-known fact that 2^l is the number of idempotents.

(ii), (iii) We use the previous proposition and (i), together with the facts: $\mathbb{Z}_2$ has only one bipotent, $\mathbb{Z}_4$ has only (the trivial) two bipotents.

(iv) We show that for $k \geq 3$, $\mathbb{Z}_{2^k}$ has only 4 bipotents: 0, 2, 2^{k-1} and $2^{k-1} + 2$ and we use Proposition 5. As observed in the Introduction, the bipotents are $1 - u$ with $u \in U(\mathbb{Z}_{2^k})$ and $u^2 = 1$. Clearly, $1 - 1 = 0$ and $1 - (-1) = 2$ are the trivial bipotents and as all odd classes in $\mathbb{Z}_{2^k}$ are units, it remains to check which are order two in $\mathbb{Z}_{2^k}$. These are only $2^{k-1} \pm 1$ (on the general results about the units in $\mathbb{Z}_n$ see e.g., [2], Theorems 4.19, 4.20). □

More precisely, if p is an odd prime, $U(\mathbb{Z}(p^k))$ is cyclic of order $p^{k-1}(p-1)$, $U(\mathbb{Z}_2)$, $U(\mathbb{Z}_4)$ are cyclic and, for $k \geq 3$, $U(\mathbb{Z}(2^k))$ is the direct product of a cyclic group of order 2 and a cyclic group of order 2^{k-2}. However, in the above proof we only need to count the order two units.

We are now ready to describe the bipotents in the rings of integers modulo some positive integer n.

Theorem 7. *In $\mathbb{Z}_n$ all bipotents are doubles of idempotents iff n is odd or else if $n = 2^2(2m + 1)$ for some nonnegative integer m. In the remaining cases, $2\,|Bip(\mathbb{Z}_n)| = |Id(\mathbb{Z}_n)|$ iff $n = 2(2m + 1)$ for some nonnegative integer m and $|Bip(\mathbb{Z}_n)| = 2\,|Id(\mathbb{Z}_n)|$ iff n is divisible by 2^3.*

Proof. Write $n = 2^k p_1^{\alpha_1}...p_l^{\alpha_l}$ for some nonnegative integers $k, \alpha_1, ..., \alpha_l$. As already mentioned $|Id(\mathbb{Z}_n)| = 2^{l+1} = |Bip(\mathbb{Z}_n)|$, precisely in the (two) cases stated, according to the previous lemma. The remaining cases also follow from the previous lemma. □

Remarks. 1) As well-known for idempotents, the number of bipotents in any $\mathbb{Z}_n$ is also a power of 2.

2) If n is even, the bipotents are also even.

3) If $b = 2e$ for some idempotent e then the complementary bipotent $2 - b = 2(1 - e)$ is also the double of an idempotent.

4) Clearly, the trivial bipotents are always doubles of the trivial idempotents.

5) A bipotent can be the double of more than one idempotent.

Examples. 1) In $\mathbb{Z}_6$, 3 is idempotent and $2 \cdot 3 = 2 \cdot 0 = 0$, i.e., 0 is also the double of a nontrivial idempotent.

2) Note that $\mathbb{Z}_6$ has four idempotents but only two (the trivial) bipotents (this happens if $2e = 0$; then $2(1 - e) = 2$).

3) $\mathbb{Z}_6$ and $\mathbb{Z}_{18}$ are biconnected but $\mathbb{Z}_{12}$ is not: $Bip(\mathbb{Z}_{12}) = \{0, 2, 6, 8\}$. All decompose in the primes 2 and 3.

4) More: $Bip(\mathbb{Z}_{15}) = \{0, 2, 5, 12\}$ with $Id(\mathbb{Z}_{15}) = \{0, 1, 6, 10\}$.

5) $Bip(\mathbb{Z}_{30}) = \{0, 2, 12, 20\}$ with $Id(\mathbb{Z}_{30}) = \{0, 1, 6, 10, 15, 16, 21, 25\}$. So also for $n = 2^1 \cdot 3 \cdot 5$, $\mathbb{Z}_n$ may **not** be biconnected.

Question. *Find the rings all whose bipotents are doubles of idempotents* (that is, $Bip(R) = 2Id(R)$).

Hint. Theorem 7 suggests that some property of $2 \in R$ is essential for the characterization.

Two partial results are already Lemma 1, (iv) and (v). More precisely we have

Proposition 8. *For any ring R, $2Id(R) \subseteq Bip(R) \cap 2R$. If R is 2-torsion-free, the equality holds.*

As mentioned in this section, $\mathbb{Z}_n$ has bipotents that are not doubles of idempotents iff 2^3 divides n. Thus, the equality above holds also in rings that are not 2-torsion-free. For example, in $\mathbb{Z}_6$, $2 \cdot 3 = 2 \cdot 0 = 0$, so $\mathbb{Z}_6$ is not 2-torsion-free, but $\mathbb{Z}_6$ is biconnected.

As another example, in $\mathbb{Z}_8$, which is a connected ring, 4 and 6 *are* bipotents and *doubles*, but not doubles of idempotents. This is possible since 2 is a zero divisor.

Remarks. 1) If $|Bip(R)| = |Id(R)|$ then $Bip(R)$ may not be equal to $2Id(R)$. This is because $|2Id(R)| < |Id(R)|$ may happen (again $2 \cdot 3 = 2 \cdot 0$ in $\mathbb{Z}_6$).

2) $Bip(R) = 2Id(R)$ obviously implies $Bip(R) \cap 2R = 2Id(R)$, but not conversely, as shows the above mentioned example in $\mathbb{Z}_8$.

The question above amounts to the converse of the property: if $e^2 = e$ then $(2e - 1)^2 = 1$, i.e., $2e = 1 + u$ with $u^2 = 1$.

Q: *In which rings, for every $u^2 = 1$, there exist an idempotent e such that $u = 2e - 1$?*

In the example discussed in this section, this holds whenever 2^3 does nor divide n.

4. Bipotent 2×2 matrices

By Cayley-Hamilton's theorem, over any *commutative* ring R, zero determinant and trace $= 2$ matrices in $\mathbb{M}_2(R)$ are bipotent, but the converse fails.

Indeed, over any commutative ring R, a matrix is (left or right) zero-divisor in $\mathbb{M}_n(R)$ iff its determinant is a zero divisor (see Theorem 9.1, [1]). So only zero determinant occurs iff the base ring is a commutative domain.

Since b is bipotent iff $b = 1 + u$ with $u^2 = 1$, in order to describe the bipotent matrices we just have to describe the order two matrix units.

Theorem 9. *Let R be a commutative ring. A (nonzero) matrix $U \in \mathbb{M}_2(R)$ is an order two unit (i.e., $U^2 = I_2$) iff any of the following holds:*

(i) $Tr(U) = 0$ and $\det(U) = -1$, i.e. $U = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 1$.

If $Tr(U) \neq 0$, a necessary condition is $Tr(U)$ divides $1 + \det(U)$.

(ii) If $Tr(U)$ is not a zerodivisor (or R is a domain) and $Tr(U)$ divides $1 + \det(U)$ then $U = u_{11}I_2$ is a scalar matrix with $u_{11} = u_{22}$, $u_{11}^2 = 1$, $2u_{11} \neq 0$ and $Tr(U)u_{11} = 1 + \det(U)$.

(iii) If $Tr(U)$ is a zerodivisor and $Tr(U)$ divides $1 + \det(U)$, then the entries of U are determined by $Tr(U)u_{11} = Tr(U)u_{22} = 1 + \det(U)$ and $Tr(U)u_{12} = Tr(U)u_{21} = 0$.

Proof. First observe that if $U^2 = I_2$, by Cayley-Hamilton's theorem, $Tr(U)U = (1 + \det(U)I_2$, where the RHS is a scalar matrix. Thus, denoting $U = [u_{ij}]$, $Tr(U)u_{11} = Tr(U)u_{22} = 1 + \det(U)$, and $Tr(U)u_{12} = Tr(U)u_{21} = 0$.

(i) If $Tr(U) = 0$ it follows that $\det(U) = -1$ and so U has the form stated. Conversely, for $U = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 1$, we get $U^2 = I_2$.

(ii) If $Tr(U)$ is not a zerodivisor, then $u_{12} = u_{21} = 0$, $u_{11} = u_{22}$ and $U = u_{11}I_2 = u_{22}I_2$ iff $Tr(U) \mid (1 + \det(U)$ with $Tr(U)u_{11} = 1 + \det(U) = Tr(U)u_{22}$. Conversely, if $U = u_{11}I_2$ then $U^2 = u_{11}^2 I_2 = I_2$, since $Tr(U)u_{11} = 1 + \det(U)$ with $u_{12} = u_{21} = 0$ imply $u_{11}^2 = 1$.

(iii) If $Tr(U)$ is a zerodivisor, the entries of U are determined by the divisibility $Tr(U) \mid (1 + \det(U)$ and by $Tr(U)u_{12} = Tr(U)u_{21} = 0$ □

Corollary 10. *Let R be a commutative domain. A nonzero matrix $B \in \mathbb{M}_2(R)$ is a bipotent iff $B = (1 + a)I_2$ is diagonal with $a^2 = 1$ and $2a \neq 0$, or $B = \begin{bmatrix} 1+a & b \\ c & 1-a \end{bmatrix}$ with $a, b, c \in r$ and $a^2 + bc = 1$.*

Proof. As R is a domain, we just discard case (iii) from the previous theorem. □

Examples. 1) $2I_n$ is bipotent in $\mathbb{M}_n(R)$ for any ring R and any $n \geq 1$.

2) Over any unital ring R, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a (zero determinant and trace 2) bipotent in $\mathbb{M}_2(R)$, which is not the double of any matrix, unless 2 is a unit.

3) Over any ring with $4 \neq 0$, $B = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not a bipotent. Indeed (see (ii) with $B = I_2 + U$, $U = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$), $Tr(U) = 2$, $\det(U) = 1 - 4$ and so U is not unit.

Actually, $B^2 = 8\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2B$, unless $4 = 0$. If $4 = 0$, $B^2 = 0_2 = 2B$ so B is bipotent. Following (i), for $B = I_2 + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, with RHS an order two unit U we have $Tr(U) = 2 = 1 + \det(U)$.

4) $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = I_2 + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ with RHS idempotent not unit. Hence it is not bipotent (again, $B^2 = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \neq 2B$).

5) $B = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ over $\mathbb{Z}_6$. We have $B = I_2 + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $U^2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^2 = I_2$. Hence it is a bipotent (actually $B^2 = 4I_2 = 2B$). Following (iii), $Tr(U) = 2$ is a zero divisor in $\mathbb{Z}_6$, 2 divides $1 + 1 = 1 + \det(U)$, $2 \cdot 0 = 2 \cdot 3 = 0$ and $2 \cdot 1 = 1 + 1$.

6) $B = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} = I_2 + \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ is a bipotent as the RHS is an order two unit of type (i) (i.e., $-2^2 + 3 \cdot 1 = -1$).

Over commutative domains, we can prove the following

Theorem 11. *Let R be a commutative domain. A matrix $B \in \mathbb{M}_2(R)$ is a bipotent iff*

(i) $\det(B) = 0$ *and* $(Tr(B) - 2)B = 0_2$*, or else*

(ii) $\det(B) = 2(Tr(B) - 2)$ *and* $(2 - Tr(B)(2I_2 - B) = 0_2$.

Proof. Suppose $B^2 = 2B$ for an $n \times n$ matrix B. Equivalently, $B(2I_n - B) = 0_n$ and taking determinants, $\det(B)\det(2I_n - B) = 0$. As R is a domain, for $n = 2$, we distinguish two cases.

(i) $\det(B) = 0$ and then by Cayley-Hamilton's theorem, $(Tr(B) - 2)B = 0_2$, or else

(ii) $\det(2I_2 - B) = 0$. In this case, $\det(B) = 2(Tr(B) - 2)$ and by Cayley-Hamilton's theorem, $(2I_2 - B)^2 = Tr(2I_2 - B))(2I_2 - B)$ which amounts to $(2 - Tr(B))(2I_2 - B) = 0_2$.

Conversely, (i) if $\det(B) = 0$ and $(2 - Tr(B)B = 0_2$ then by Cayley-Hamilton's theorem $B^2 - Tr(B)B = 0$, and so $B^2 = 2B$. In the (ii) case, $\det(B) = 2(Tr(B) - 2)$ gives $\det(2I_2 - B) = 0$ and as in case (i) it follows $(2I_2 - B)^2 = 2(2I_2 - B)$, that is, $2I_2 - B$ is bipotent. So is its complementary bipotent $2I_2 - (2I_2 - B) = B$. □

Over commutative domains, we have a specific form of Lemma 1, (vi) for nontrivial bipotent matrices that are doubles of (nontrivial) idempotent matrices.

Proposition 12. *A nontrivial bipotent matrix B over a commutative domain R is the double of a (nontrivial) idempotent iff* $\det(B) = 0$, $Tr(B) = 2$ *and B is a double of some matrix (equivalently, 2 divides all the entries of B).*

Proof. A nontrivial idempotent matrix E has zero determinant and trace $= 1$, that is, $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$. The double $2E = \begin{bmatrix} 2a & 2b \\ 2c & 2(1-a) \end{bmatrix}$, still has zero determinant and trace $= 2$. According to Theorem 11, (i), a nontrivial bipotent matrix is a double of an idempotent *only if* it has zero determinant and has trace $= 2$. It just remains to use the lemma mentioned above. □

Remarks. 1) For a bipotent B to be the double of an idempotent, the conditions $\det(B) = 0$, $Tr(B) = 2$ are necessary but not sufficient, as witnessed by example 2 above.

2) Browsing the list of examples above, (1) $2I_2$ obviously is the double of a (nontrivial) idempotent, (6) $\det(B) = 0$, $Tr(B) = 2$, but (unless 2 is a unit), 2 does not divide the entries of B. Thus, this bipotent is not a double of an idempotent.

We cannot apply the proposition to example (5), as $\mathbb{Z}_6$ is not a domain. However, this bipotent is not a double since the equation $2x = 3$ is not solvable over $\mathbb{Z}_6$.

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