# TWO CLASSES OF RINGS RELATED TO NILPOTENT–IDEMPOTENT EQUIVALENCE

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Abstract. A ring is said to be NUR if every nilpotent element is equivalent to an idempotent, and IUN if every idempotent different from 1 is equivalent to a nilpotent element. We study these two, generally distinct, classes of rings.

## 1. Introduction

In any unital ring $R$, the three sets $U(R)$, $Id(R)$, and $N(R)$, denoting respectively the group of units, the set of idempotents, and the set of nilpotent elements of $R$, play a fundamental role.

Two important binary relations used in ring theory (and extensively in matrix theory) are conjugation (called similarity in the matrix setting) and equivalence, defined as follows.

Elements $a, b \in R$ are said to be *conjugate* if there exists a unit $u \in U(R)$ such that $b = u^{-1}au$, and *equivalent* if there exist units $u, v \in U(R)$ such that $b = uav$.

It is well known that both relations are equivalence relations in the sense of set theory, that is, they are reflexive, symmetric, and transitive. Consequently, each induces a partition of $R$ into equivalence classes. Clearly, conjugation is a refinement of equivalence: conjugate elements are always equivalent.

As observed above, we use the word *equivalence* in two different senses. When referring to the set-theoretic notion, we will use quotation marks—"equivalence"—and omit them when referring to the ring-theoretic relation. The same convention applies to the word *class*: quotation marks are used for the subsets arising from an "equivalence" relation, and omitted when referring to classes of rings.

Clearly, units are equivalent only to units, and in fact any two units are equivalent. Hence the "equivalence" class of any unit in $R$ coincides with $U(R)$ itself. While conjugates of units are again units, it is easy to find examples showing that not all units are conjugate. For instance, the matrices $I_2$ and $E_{12} + E_{21}$ are not conjugate in $\mathbb{M}_2(R)$ for any unital ring $R$.

Moreover, the sets of idempotent elements and of nilpotent elements are each invariant under conjugation. Nevertheless, two idempotents (or two nilpotents) need not be conjugate.

A situation not addressed above concerns the following elementary question: *what is the behavior of idempotent and nilpotent elements with respect to equivalence*? It is well known that neither of these properties is invariant under equivalence. Moreover, idempotents and nilpotents may be equivalent to each other.

*Key words and phrases.* idempotent, unit, NUR-ring, IUN-ring; MSC 2020 Classification: 16U10, 16U40, 16U60, 16U99. Orchid 0000-0002-3353-6958.

As a simple illustration, consider the ring $\mathbb{M}_2(R)$ over any ring $R$. The idempotent matrix $E_{11}$ is equivalent to the nilpotent matrix $E_{12}$, since $I_2 E_{11} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_{12}$.

From the outset, we may disregard the trivial idempotents 0 and 1. Indeed, the "equivalence" class of 0 is the singleton $\{0\}$, whereas 1, being a unit, is equivalent to every unit; hence its "equivalence" class coincides with $U(R)$.

In [4], a nonunit element of a ring was called a *UN*-element if it can be written as a product of a unit and a nilpotent element. Symmetrically, one defines *NU*-elements. It is shown there that an element is a UN-element if and only if it is an NU-element, and it is observed that a nonunit element is UN precisely when it is equivalent to a nilpotent element. UN-rings were further studied in [16]. Recall also that an element $a$ of a ring $R$ is called *unit-regular* if there exists a unit $u \in U(R)$ such that $a = aua$, that is, if $a$ has a unit inner inverse.

The main goal of this paper is to study the following two classes of rings.

**Definitions**. A ring $R$ is called a *nilpotent-unit-regular* ring (or simply a *NUR*-ring) if every nilpotent element is equivalent to an idempotent, or equivalently (see Section 2), if every nilpotent element is unit-regular. Obvious examples of NUR-rings include unit-regular rings and reduced rings. A ring $R$ is called an *idempotent-UN* ring (or simply an *IUN*-ring) if every idempotent different from 1 is equivalent to a nilpotent element, or equivalently (see Section 2), if every non-identity idempotent is a UN-element. Obvious examples of IUN-rings include UN-rings and connected rings (recall that a ring is called *connected* if it has only the trivial idempotents).

The paper is organized as follows. In Section 2 we describe the motivating example for this research, namely matrix rings over fields, which are both NUR and IUN. Section 3 is devoted to the study of NUR-rings, while Section 4 deals with IUN-rings. Finally, in Section 5 we consider rings that are simultaneously NUR and IUN.

Throughout the paper, all rings are assumed to be associative with identity; accordingly, we henceforth omit the adjective unital.

## 2. Motivating example and simple facts

We start with a simple example which (actually) motivated this research.

**Proposition 2.1.** *Let $R = \mathbb{M}_n(k)$ be a matrix ring over a field $k$. Every nilpotent matrix is equivalent to an idempotent matrix ($\neq I_n$) and conversely, every idempotent different from $I_n$ is equivalent to a nilpotent matrix.*

*Proof.* First recall that every nilpotent matrix *over a field* is similar (i.e. conjugate, and so equivalent) to a *block diagonal* matrix $T = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}$, where each block $B_i$ is a *shift* matrix (possibly of different sizes). To simplify the writing (and editing!) we (sometimes) use the "direct sum" notation for such block matrices: $T = B_1 \oplus B_2 \oplus ... \oplus B_k$. A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix. When $n = 1$, $S = 0$.

Next, observe that using permutation matrices (which are units in $R$), we can interchange the columns of each shift as follows: $\mathrm{col}_1(S) \leftrightarrow \mathrm{col}_2(S)$, then $\mathrm{col}_2(S) \leftrightarrow \mathrm{col}_3(S)$ and so on. This way we transform every shift matrix into a diagonal matrix. Since the only nonzero entries are 1's, the resulting matrix is idempotent. The same can be done for $T$, so indeed every nilpotent matrix is equivalent to an idempotent matrix.

As for the converse, recall that *every idempotent matrix is equivalent to a diagonal matrix* (i.e. is diagonalizable), all whose entries are 1's and 0's. Since the idempotent is not the identity matrix $I_n$, the diagonal has at least one zero. By suitable interchange of rows and columns, this diagonal matrix can be transformed into a direct sum of shifts, which is indeed a nilpotent matrix. □

**Corollary 2.2.** *Matrix rings over fields are NUR and IUN.*

The commutativity of the base ring is not necessary. Indeed, in [7], it is proved that

**Theorem 2.3.** *The following are equivalent for a ring $R$:*

*1. Every nilpotent matrix over $R$ is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).*

*2. $R$ is a division ring.*

**Corollary 2.4.** *Matrix rings over division rings are NUR and IUN.*

**Remark**. In the proof of Proposition 2.1, for the IUN property, we used the fact that every idempotent matrix is equivalent to a diagonal matrix, which holds for matrices over division rings. However, the property is more general.

Following Steger [14], we say that a ring $R$ is an *ID* ring if every idempotent matrix over $R$ is *similar* to a diagonal one. Thus, by a result of Song and Guo ([13], Corollary 5), if every idempotent matrix over $R$ is *equivalent* to a diagonal matrix, then $R$ is an ID ring. Examples of ID rings include: Division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

**Proposition 2.5.** *Matrix rings over ID rings are IUN.*

The study of NUR and/or IUN rings is trivial for *commutative* rings.

**Lemma 2.6.** *Let $t \in N(R)$ and $e^2 = e \in R$ for a commutative ring $R$. If $e$ and $t$ are equivalent then $e = t = 0$.*

*Proof.* Suppose $ut = ev$ for $u, v \in U(R)$. Since the ring is commutative, $ut \in N(R)$ and so is $ev$. Hence $(ev)^n = ev^n = 0$ for some positive integer $n$. Therefore $e = 0$, $ut = 0$ and so $t = 0$. □

**Corollary 2.7.** *A commutative ring is NUR iff it is reduced, and is IUN iff it is connected.*

**Corollary 2.8.** *Let $n$ be any positive integer. The residue class ring $\mathbb{Z}_n$ is NUR iff $n$ is square-free and IUN iff $n = p^k$ for some prime $p$.*

The study is also trivial for *domains* (no nonzero nilpotent, no idempotent $\neq 0, 1$): *every domain is NUR and IUN.* In particular, division rings and fields are NUR and IUN.

As already mentioned in the Introduction

**Proposition 2.9.** *An element in a ring is unit-regular iff it is equivalent to an idempotent.*

*Proof.* Suppose $a = uev$ with $e^2 = e \in R$ and $u, v \in U(R)$. Then

$$a = (uev)(v^{-1}u^{-1})(uev) = av^{-1}u^{-1}a,$$

so $a$ is unit-regular. Conversely, it is well-known that every unit-regular element is a product $ue$ (or $eu$) with idempotent $e$ and unit $u$, so is equivalent to an idempotent. $\square$

With respect to NUR, only a specific class of nilpotent elements is involved.

**Corollary 2.10.** *A nilpotent is equivalent to an idempotent iff it is unit-regular.*

**Remarks**. 1) If $R$ is NUR, all nilpotents are unit-regular and conversely.
2) Notice that the *nilpotent* and the *unit-regular* properties *are independent* (in general): $2 \in \mathbb{Z}_4$ is a example of nilpotent which is not unit-regular, and any nonzero idempotent is an example of unit-regular element which is not nilpotent.
3) If a nilpotent is equivalent to an idempotent, it has *stable range 1* (indeed, unit-regular elements have stable range one). The converse fails. As an example, take $T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$. Since $\det(T) = 0$, $T$ has stable range one (see [6]), but $T$ is not equivalent to any (nontrivial) idempotent. Indeed, for any two units $U, V \in \mathbb{M}_2(\mathbb{Z})$, the trace $\mathrm{Tr}(UTV)$ is even.

Next, observe that

**Proposition 2.11.** *A nonunit in a ring is UN iff it is equivalent to a nilpotent.*

*Proof.* A unit multiple of a nilpotent is equivalent to the nilpotent. Conversely, suppose $a = utv \in R$ with units $u$, $v$ and nilpotent $t$. Then $a = (uv)(v^{-1}tv)$ is UN. $\square$

**Remark**. One way in the previous proposition holds more generally. In [17] it is proved that, in any ring, *an element equivalent to a UN-element is a UN-element* (nilpotents are obviously UN). Indeed, if $a = ut$ then $paq = (puq)(q^{-1}tq)$ is UN.

In the context of NUR, only a distinguished class of idempotent elements plays a role.

**Corollary 2.12.** *An idempotent $\neq 1$ is equivalent to a nilpotent iff it is a UN-element.*

In [4], it was pointed out that if an idempotent is a UN-element, then the corresponding nilpotent is unit-regular.

**Remarks**. 1) If $R$ is IUN, all idempotents $\neq 1$ are UN and conversely.

2) Notice that the *idempotent* and the *UN* properties *are independent* (in general): 1 is a example of idempotent which is not UN (a less trivial example is $-1$ in $\mathbb{Z}$; however, idempotents $\neq I_2$ are UN in $\mathbb{M}_2(\mathbb{Z})$-see [4]), and, $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \in \mathbb{M}_2(R)$ over any ring $R$ with $2^2 \neq 2$, is an example of UN element which is not idempotent.

## 3. NUR

In this section $R$ denotes a (unital) ring, such that each (nonzero) nilpotent is equivalent to some idempotent. By Proposition 2.10, these are precisely *the rings all whose nilpotents of are unit-regular*. As special cases, *unit-regular rings have this property*, and trivially, *reduced* rings are NUR.

Moreover, obviously

**Lemma 3.1.** *Let $R$ be connected. Then $R$ is NUR iff $R$ is reduced.*

Thus, *connected not reduced rings* (e.g., local rings) *are not NUR.*

Hence, in order to describe the NUR rings, we have to discuss only *rings which are not connected nor reduced,* and we have to determine which nilpotents are unit-regular.

**Lemma 3.2.** *Let $I$ be a nil ideal of $R$.*

*(1) If $R$ is NUR, then $R/I$ is NUR.*

*(2) If idempotents lift modulo $I$, then $R$ is NUR iff $R/I$ is NUR.*

*Proof.* (1) If $\overline{a} \in N(R/I)$ and $I$ is nil then $a \in N(R)$. By hypothesis, $a$ is equivalent to an idempotent $e \in R$. Hence $\overline{a}$ is equivalent to $\overline{e}$.

(2) If $a \in N(R)$ then $\overline{a} \in N(R/I)$ and by hypothesis it is equivalent to an idempotent $\overline{e} \in R/I$. Since idempotents lift modulo $I$, $a$ is equivalent to the idempotent $e$. □

**Remarks**. 1) For Lemma 3.2, (1), the assumption that idempotents lift modulo $I$ is not superfluous (even for $I$ being the Jacobson radical of the ring). If $R = \{\frac{m}{n} \in \mathbb{Q} : 2 \nmid n, 3 \nmid n\}$, then $R$ is NUR (being reduced), but $R/J(R) = \mathbb{Z}_2 \times \mathbb{Z}_3$ is not NUR (not connected; both $(1,0)$ and $(0,1)$ are nontrivial idempotents).

2) The converse of Lemma 3.2, (1) above, fails. By Corollary 2.7, the (commutative) domain $\mathbb{Z}$ is both NUR and IUN, but $\mathbb{Z}/12\mathbb{Z}$ is not NUR nor IUN.

**Question**. Does a ring $R$ exist such that idempotents lift modulo $J(R)$, $R/J(R)$ is NUR but $R$ is not NUR ?

A well-known result of Ara (see [1]) says that strongly $\pi$-regular rings have stable range 1. A key step in the proof is the observation that *any (von Neumann) regular, nilpotent element of an exchange ring is a unit-regular element* (by [1], Theorem 2, but also see [9]). Hence, since regular rings are exchange, we have the following result.

**Proposition 3.3.** *(Von Neumann) regular rings are NUR.*

Nielsen and Šter in [12] (see also [2]) have shown that *a regular nilpotent element in general, may not be unit-regular.*

In [2], *an example of a ring $S$ and a regular element $a \in S$ such that $a^3 = 0$, but $a$ is not unit-regular in $S$*, is given. Consequently, $S$ is not NUR.

The Proposition above, can be also obtained from [3], Corollary 3.7: *any strongly $\pi$-regular element (in particular, nilpotent) which has all powers, regular, is unit-regular (together with all its powers).*

A necessary condition for NUR rings is

**Proposition 3.4.** *If $R$ is NUR and $I$ is an ideal which contains no nonzero idempotents then $N(R) \cap I = 0$. In particular, $N(R) \cap J(R) = 0$ and so $R$ has no nonzero central nilpotents.*

*Proof.* Let $t \in N(R) \cap I$ and $t = ue$ with $u \in U(R)$, $e^2 = e \in R$. Since $I$ is an ideal, $u^{-1}t = e \in I$ and so $e = 0$ and $t = 0$.

Since the central nilpotents belong to the radical (which contains no nonzero idempotents), the claim follows. □

Notice that $N(R) \cap J(R) = 0$ obviously holds for $J(R) = 0$, that is, for *semiprimitive* rings (in particular, for regular rings), or for $N(R) = 0$ i.e., for *reduced* rings.

**Remarks**. 1) In [15], a ring was termed *central reduced* if all nilpotents are central. So our above condition (*no nonzero central nilpotents*) should be termed differently: *weakly reduced*, for instance.

2) Since for any ring $R$, the center of $\mathbb{M}_n(R)$ consists of the *scalar* matrices with central diagonal entries, it follows that $\mathbb{M}_n(R)$ has central nilpotents (scalar matrices $t \cdot I_n$ with $t \in N(R) \cap Z(R)$) iff $R$ has central nilpotents.

3) Since $N(R) \cap U(R) = \varnothing = J(R) \cap U(R)$, in any NUR ring these three subsets are disjoint (excepting 0 in $N(R) \cap J(R)$).

We can easily decide *in the negative*, two natural converses.

*If $N(R) \cap J(R) = 0$, is $R$ a NUR-ring* ? **No**, and

*Are matrix rings over NUR-rings, also NUR* ? **No.**

**Example**. Over $\mathbb{Z}$ (every domain is trivially NUR), $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$ is nilpotent but not unit-regular (see [10], Proposition 2.1), so $\mathbb{M}_2(\mathbb{Z})$ is not NUR. As well-known, $J(\mathbb{M}_2(\mathbb{Z})) = \mathbb{M}_2(J(\mathbb{Z})) = 0$ and so is $N(\mathbb{M}_2(\mathbb{Z})) \cap J(\mathbb{M}_2(\mathbb{Z})) = 0$.

Therefore, the necessary condition in Proposition 3.4 is **not** sufficient. This also shows that:

(i) *extensions of NUR rings may not be NUR*;

(ii) *semiprimitive rings may not be NUR.*

Related to the second question above we have

**Proposition 3.5.** *Let $R$ be any commutative ring. If $\mathbb{M}_n(R)$ is NUR then so is $R$ (i.e., it is reduced).*

*Proof.* According to Corollary 2.7, a commutative ring is NUR iff it is reduced. We show that assuming $\mathbb{M}_n(R)$ is NUR while $R$ is not reduced leads to a contradiction.

Suppose $t \in N(R)$, $t^k = 0 \neq t^{k-1}$. As $(tE_{11})^k = 0$, $tE_{11}$ is nilpotent, so (unit-)regular. Hence $(tE_{11})U(tE_{11}) = t^2E_{11}UE_{11} = tE_{11}$ for some unit $U \in U(\mathbb{M}_n(R))$. Multiplying by $t^{k-2}$ gives $t^{k-1}E_{11} = 0$, a contradiction. □

Since the $n \times n$ matrix ring over any regular ring is again regular, using Proposition 3.3, we get

**Proposition 3.6.** *Matrix rings over regular rings are NUR.*

**Question**. Does the NUR property pass to corners ? That is, let $e^2 = e \in R$ be a NUR-ring. Is $eRe$ also a NUR-ring ?

**Remark**. In [5], it is proved that the $2 \times 2$ matrix rings over Bézout domains are NUR (i.e., the nilpotent matrices are unit-regular).

## 4. IUN

In this section $R$ denotes a (unital) ring, such that each idempotent different from 1 is equivalent to some nilpotent. By Proposition 2.12, these are precisely *the rings all whose idempotents* $\neq 1$ *are UN*. As a special case, *UN rings have this property.*

However, (nonzero) *reduced rings are not IUN*, unless these are connected.

Note that the question of IUN-rings was already raised as an open problem in [4], where it was shown that *all idempotents* $\neq I_2$ *in* $\mathbb{M}_2(\mathbb{Z})$ *are UN-matrices.* Hence

**Proposition 4.1.** $\mathbb{M}_2(\mathbb{Z})$ *is IUN (but is not NUR).*

More generally,

**Lemma 4.2.** *Let $R$ be an IUN-ring. Then*

*(1)* $eR(1-e) \neq 0$ *for any non-trivial idempotent $e$ in $R$. In particular, $R$ is indecomposable.*

*(2) The center $Z(R)$ is a connected ring.*

*Proof.* (1) From $eR(1-e) = 0$, it follows that ($er = ere$, for every $r \in R$ and so) $exy = exe \cdot eye$ and $ex^n = (exe)^n$ for all $x, y \in R$ and all $n \geq 1$. Write $e = ut$ where $u \in U(R)$ and $t^n = 0$. Then $e = eut = eue \cdot ete$ and $0 = et^n = (ete)^n$. Moreover, $e = euu^{-1} = eue \cdot eu^{-1}e$ and, similarly, $e = eu^{-1}e \cdot eue$. So $e$ is a UN-element in $eRe$, a contradiction (the identity is no UN-element). In particular, $R$ has no non-trivial central idempotents, so it is indecomposable.

(2) It follows from (1). Indeed, if a non-trivial idempotent is central then $eR(1-e) = e(1-e)R = 0$. □

**Corollary 4.3.** *A commutative ring is IUN iff it is connected.*

**Remarks**. 1) Replacing $e$ by its complementary, from (1) we also get $(1-e)Re \neq 0$.

2) The condition (1) in Lemma 4.2 is only necessary, but not sufficient.

**Example.** An indecomposable ring which is not IUN.

$\mathbb{T}_n(k)$ over any indecomposable ring $k$, is indecomposable (see **Ex. 22.1** [11]), so in particular, $\mathbb{T}_2(\mathbb{F}_2)$ is indecomposable. It has only $E_{12}$ nonzero nilpotent, and four nontrivial idempotents: $E_{11}$, $E_{22}$, $E_{11} + E_{12}$ and $E_{12} + E_{22}$. Moreover, it has only one unit $\neq I_2$: $I_2 + E_{12}$. For instance, $E_{11}$ is not equivalent to $E_{12}$ (the only nonzero nilpotent): $(I_2 + E_{12})E_{11} = E_{11}$ and $E_{11}(I_2 + E_{12}) = E_{11} + E_{12}$. So $\mathbb{T}_2(\mathbb{F}_2)$ is not IUN.

**Lemma 4.4.** *Let $I$ be an ideal of $R$ such that idempotents lift modulo $I$.*

*(1) If $R$ is IUN, then $R/I$ is IUN.*

*(2) If $I$ is nil, then $R$ is IUN iff $R/I$ is IUN.*

*Proof.* (1) Let $\overline{a}^2 = \overline{a} \neq \overline{1}$ in $R/I$. Then, there exists $e^2 = e \neq 1$ in $R$ such that $a - e \in I$. Write $e = ut$ where $u \in U(R)$ and $t \in N(R)$. Then $\overline{a} = \overline{u}\overline{t}$, a UN-element in $R/I$.

(2) The necessity follows from (1). For the sufficiency, let $e^2 = e \neq 1$ in $R$. Then $\overline{e}$ is a non-identity idempotent in $R/I$, so we can write $\overline{e} = \overline{u}\overline{t}$ where $\overline{u} \in U(R/I)$ and $\overline{t} \in N(R/I)$. As $I$ is nil, $u \in U(R)$ and $t \in N(R)$. Moreover, $e = ut + c$ for some $c \in I$. Thus, $a = u(t + u^{-1}c)$ with $t + u^{-1}c \in N(R)$. □

**Remarks**. (1) For Lemma 4.4, (1), the assumption that idempotents lift modulo $I$ is not superfluous even for $I$ being the Jacobson radical of the ring. If $R = \{\frac{m}{n} \in \mathbb{Q} : 2 \nmid n, 3 \nmid n\}$, then $R$ is IUN, but $R/J(R) = \mathbb{Z}_2 \times \mathbb{Z}_3$ is not IUN (since it is not indecomposable).

(2) The converse of Lemma 4.4, (1) does not hold: Let $R = \mathbb{Z}_2 \times S$ be a direct product of rings where $S$ is a non-trivial ring and $I = \{(0, y) \in R : y \in S\}$. Then $I$ is an ideal of $R$ such that idempotents lift modulo $I$. Here $R/I = \mathbb{Z}_2$ is IUN but $R$ is not (since it is not indecomposable).

**Question**. Does a ring $R$ exists, such that idempotents lift modulo $J(R)$, $R/J(R)$ is IUN but $R$ is not IUN ?

For a ring $R$ and $n \geq 1$, denote $B_n(R) =: \{[a_{ij}] \in \mathbb{T}_n(R) : a_{11} = a_{22} = ... = a_{nn}\}$. Then $B_n(R)$ is a subring of $\mathbb{T}_n(R)$.

The next examples follow from Lemma 4.4, (1).

**Proposition 4.5.** *Let $R$ be a ring, $M$ be a bimodule over $R$ and $n \geq 1$.*

*(1) $R$ is IUN iff $R[x]/(x^{n+1})$ is IUN.*

*(2) $R$ is IUN iff $B_n(R)$ is IUN.*

*(3) $R$ is IUN iff the trivial extension $R \ltimes M$ is IUN.*

Recall from [8] the following definitions. Let $S$ be a commutative ring with identity. If all $1 \times 2$ and all $2 \times 1$ matrices over $S$ admit diagonal reduction, then every matrix over $S$ admits *triangular reduction*; in this case, $S$ is called an *Hermite* ring. If, in addition, all $2 \times 2$ matrices over $S$ admit diagonal reduction, then every matrix over $S$ admits *diagonal reduction*; in this case, $S$ is called an *elementary divisor* ring.

Y. Zhou (see [17]) has proved the following extension (removing the commutativity) of the matrix rings results in [4]: *matrix rings over an elementary divisor UN-ring are UN.* So

**Proposition 4.6.** *Matrix rings over elementary divisor UN-rings are IUN.*

Since in order to determine IUN rings, we have to determine which idempotents are UN, the following result is useful.

**Proposition 4.7.** *If $R$ is an IUN-ring and $0 \neq e = ut$ with $u \in U(R)$ and $t \in N(R)$, then $t \notin J(R)$.*

*Proof.* Analogous with the proof of Proposition 3.4. □

The following block decomposition was mentioned in [4]: if $T = \mathrm{diag}(t_1, ..., t_m)$ then $\begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_k \\ I_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & T \\ I_k & \mathbf{0} \end{bmatrix}$. If $T = 0$, this is a UT-decomposition and we immediately obtain

**Theorem 4.8.** *Let $n$ be a positive integer. Over any ring $R$, the diagonal idempotents $E_{11} + ... + E_{ii}$, $i \in \{1, ..., n-1\}$, are UT in the matrix ring $\mathbb{M}_n(R)$.*

**Remark**. As $E_{11} + ... + E_{ii}$ ($i \in \{1, ..., n-1\}$) are in $\mathbb{M}_n(\mathbb{Z} \cdot 1_R)$, a subring of $\mathbb{M}_n(R)$, it suffices to show that $E_{11} + ... + E_{ii}$ ($i \in \{1, ..., n-1\}$) are UT in $\mathbb{M}_n(\mathbb{Z} \cdot 1_R)$. Thus, without loss of generality, we can assume that $R$ is commutative (and, if necessary, use determinants).

An **alternative proof** proceeds as follows. We first show that $E_{11}+...+E_{i-1,i-1}$ is UT in $\mathbb{M}_i(R)$, using the following UT-decomposition

$$U_{ii}T_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix},$$

then the UT block decomposition $\begin{bmatrix} U_{ii} & \mathbf{0} \\ \mathbf{0} & I_{n-i} \end{bmatrix} \begin{bmatrix} T_{ii} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ in $\mathbb{M}_n(R)$.

**Corollary 4.9.** *A semiperfect ring $R$ is IUN iff $R$ is isomorphic to a matrix ring over a local ring.*

*Proof.* The sufficiency follows from Proposition 2.5. For the necessity, since $R$ is semiperfect, idempotents lift modulo $J(R)$ in $R$. So, by Lemma 4.4 (1), $R/J(R)$ is IUN, and hence is indecomposable. But, as $R/J(R)$ is semisimple, we deduce that $R/J(R) = \mathbb{M}_n(D)$ where $n \geq 1$ and $D$ is a division ring. Because idempotents lift modulo $J(R)$, any set of matrix units of $R/J(R)$ can be lifted to a set of matrix units of $R$ (for details, see [11], Theorem 23.10), and so $R \cong \mathbb{M}_n(S)$ with $S/J(S) \cong D$. □

**Questions**. 1) Are matrix rings over IUN-rings, also IUN ? Probably no.

As already mentioned, for an example, $\mathbb{Z}$ (every domain is trivially IUN), is not suitable: as already mentioned, *all idempotents $\neq I_2$ in $\mathbb{M}_2(\mathbb{Z})$ are UN-matrices* (see [4]).

Moreover, as *matrix rings over elementary divisor UN-rings are also UN-rings*, for an example we need a ring which is *not* elementary divisor ring (see [8]).

2) Does the IUN property pass to corners ? That is, let $e^2 = e \in R$ be an IUN-ring. Is $eRe$ also IUN ?

In closing, here are just a few remarks on rings that are both NUR and IUN.

In section 2 we mentioned that *matrix rings over division rings are both NUR and IUN.*

As for commutative rings, by Corollary 2.7, we have the following characterization.

**Corollary 4.10.** *A commutative ring is NUR and IUN iff it is reduced and connected.*

In particular, domains are (trivially) both NUR and IUN. However, the class of commutative NUR + IUN rings is quite large. The reduced and connected conditions, generally do not imply domain or local or irreducible.

**Example**. For any field $k$, the factor ring $R = k[x, y]/(xy)$ is reduced and connected but not domain, nor irreducible nor local (having two minimal primes $(x)$ and $(y)$).

Also recall that by Corollary 2.7, the (integral) domain $\mathbb{Z}$ is both NUR and IUN, but $\mathbb{Z}/12\mathbb{Z}$ is not NUR nor IUN. Hence , *the combined property NUR + IUN does not pass to factor rings.*

In Section 3 we mentioned that $\mathbb{M}_2(\mathbb{Z})$ is IUN but not NUR. Hence, *the combined property NUR + IUN does not pass to matrix rings.*

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