MATRIX INVERTIBLE EXTENSIONS OVER COMMUTATIVE RINGS. PART II: DETERMINANT LIFTABILITY

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Abstract. A unimodular 2×2 matrix A with entries in a commutative ring R is called weakly determinant liftable if there exists a matrix B congruent to A modulo $R\det(A)$ and $\det(B) = 0$; if we can choose B to be unimodular, then A is called determinant liftable. If A is extendable to an invertible 3×3 matrix A^+, then A is weakly determinant liftable. If A is simply extendable (i.e., we can choose A^+ such that its $(3,3)$ entry is 0), then A is determinant liftable. We present necessary and/or sufficient criteria for A to be (weakly) determinant liftable and we use them to show that if R is a Π_2 ring in the sense of Part I (resp. is a pre-Schreier domain), then A is simply extendable (resp. extendable) iff it is determinant liftable (resp. weakly determinant liftable). As an application we show that each $J_{2,1}$ domain (as defined by Lorenzini) is an elementary divisor domain.

1. Introduction

Let R be a commutative ring with identity element 1. For $n \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathbb{M}_n(R)$ be the R-algebra of $n \times n$ matrices with entries in R. We say that $B, C \in \mathbb{M}_n(R)$ are congruent modulo an ideal $\mathfrak{i}$ of R if all entries of $B - C$ belong to $\mathfrak{i}$, i.e., $B - C \in \mathbb{M}_n(\mathfrak{i})$. Let $SL_n(R) := \{I \in \mathbb{M}_n(R) | \det(I) = 1\}$. For a free R-module F, let $Um(F)$ be the set of *unimodular* elements of F, i.e., of elements $v \in F$ for which there exists an R-linear map $L : F \to R$ such that $L(v) = 1$.

In this paper we study a unimodular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$. Recall that A is called *extendable* if there exists $A^+ = \begin{bmatrix} a & b & f \\ c & d & -e \\ -t & s & v \end{bmatrix} \in SL_3(R)$ (see [4], Def. 1.1); we call A^+ an *extension* of A. If we can choose A^+ such that $v = 0$, then A is called *simply extendable* and A^+ is called a *simple extension* of A.

Definition 1.1. *We say that $A \in Um\big(\mathbb{M}_2(R)\big)$ is weakly determinant liftable if there exists $B \in \mathbb{M}_2(R)$ congruent to A modulo $R\det(A)$ and $\det(B) = 0$. If there exists such a matrix B which is unimodular, then we remove the word 'weakly', i.e., we say that A is determinant liftable.*

If either A is invertible or $\det(A) = 0$, then A is determinant liftable. Also, if $A' \in \mathbb{M}_2(R)$ is equivalent to A, then A' is (weakly) determinant liftable iff A is so. The following characterizations of determinant liftability are proved in Section 4.

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Theorem 1.2. *For* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ *the following statements are equivalent.*

(1) The matrix A *is determinant liftable.*

(2) There exists $C \in Um\big(\mathbb{M}_2(R)\big)$ *such that* $A + \det(A)C \in Um\big(\mathbb{M}_2(R)\big)$ *and* $\det(C) = \det\big(A + \det(A)C\big) = 0$.

(3) There exists $(x, y, z, w) \in R^4$ *such that* $ax+by+cz+dw = 1$ *and* $xw-yz = 0$.

(4) There exists $C \in \mathbb{M}_2(R)$ *such that* $\det(C) = \det\big(A + \det(A)C\big) = 0$.

Definition 1.1 is motivated by the following implications proved in Section 5.

Theorem 1.3. *For* $A \in Um\big(\mathbb{M}_2(R)\big)$ *the following properties hold.*

(1) If A *is simply extendable, then* A *is determinant liftable.*

(2) If A *is extendable, then* A *is weakly determinant liftable.*

The converses of Theorem 1.3 do not hold in general (see Example 5.1).

Recall from [4], Def. 1.2(1) that R is called a Π_2 *ring* if each matrix in $Um\big(\mathbb{M}_2(R)\big)$ of zero determinant is extendable, equivalently it is simply extendable by [4], Lem. 4.1(1). We will use stable ranges and pre-Schreier domains as recalled in [4], Def. 1.5 and Sect. 2. Each pre-Schreier domain is a Π_2 ring (see [4], paragraph after Thm. 1.4). Recall from [4], Def. 1.2(3) that R is called an SE_2 *ring* if each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is simply extendable. In particular, an SE_2 ring is a Π_2 ring.

The next two theorems are proved in Section 7.

Theorem 1.4. *The ring* R *is a* Π_2 *ring iff the simply extendable and determinant liftable properties on a matrix in* $Um\big(\mathbb{M}_2(R)\big)$ *are equivalent.*

From Theorem 1.4 and [4], Thm. 1.6 we get directly the following result.

Corollary 1.5. *If* R *is a* Π_2 *ring with* $sr(R) \leq 2$*, then the simply extendable, extendable and determinant liftable properties on a matrix in* $Um\big(\mathbb{M}_2(R)\big)$ *are equivalent.*

As SE_2 rings are Π_2 rings, from Theorem 1.4 we get directly the following result.

Corollary 1.6. *The ring* R *is an* SE_2 *ring iff it is a* Π_2 *ring with the property that each matrix in* $Um\big(\mathbb{M}_2(R)\big)$ *with nonzero determinant is determinant liftable.*

A matrix $N \in \mathbb{M}_2(R)$ is called *non-full* if it is a product $\begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$ with $(l, m, o, q) \in R^4$; so $\det(N) = 0$. Recall that an integral domain R is a pre-Schreier domain iff each matrix in $\mathbb{M}_2(R)$ of zero determinant is non-full by [13], Lem. 1.

Theorem 1.7. *Assume* R *is such that each zero determinant matrix in* $\mathbb{M}_2(R)$ *is non-full (e.g.,* R *is a product of pre-Schreier domains). Then a unimodular matrix* $A \in Um\big(\mathbb{M}_2(R)\big)$ *is extendable iff it is weakly determinant liftable.*

Note that R of Theorem 1.7 is a Π_2 ring by [4], Thm. 1.4. Based on this, from Theorems 1.3 and 1.7 and Corollary 1.5 we get directly the following result.

Corollary 1.8. *Assume* R *is such that* $sr(R) \leq 2$ *and each zero determinant matrix in* $\mathbb{M}_2(R)$ *is non-full (e.g.,* R *is a product of pre-Schreier domains of stable range at most* 2*). Then the simply extendable, extendable, determinant liftable and weakly determinant liftable properties on a matrix in* $Um\big(\mathbb{M}_2(R)\big)$ *are equivalent.*

Example 1.9. Let R be a pre-Schreier domain such that $sr(R) = 3$ and there exists $A \in Um\big(\mathbb{M}_2(R)\big)$ which is extendable but not simply extendable (e.g., $R = K[X,Y]$ with K a subfield of $\mathbb{R}$, see [4], Ex. 6.1). Then A is weakly determinant liftable by Theorem 1.3(2) but it is not determinant liftable by Theorem 1.4. So the inequalities in Corollaries 1.5 and 1.8 are optimal.

Recall that R is called an *elementary divisor ring* if for each $(j,n) \in \mathbb{N}^2$ with $j \leq n$, every matrix of size either $j \times n$ or $n \times j$ with entries in R admits diagonal reduction, i.e., is equivalent to a matrix whose off diagonal entries are 0 and whose diagonal entries $a_{1,1}, \ldots, a_{j,j}$ are such that $a_{i,i}$ divides $a_{i+1,i+1}$ for all $i \in \{1, \ldots, j-1\}$. Also, recall that R is a *Hermite ring* in the sense of Kaplansky if $R^2 = RUm(R^2)$, equivalently if each 1×2 matrix with entries in R admits diagonal reduction. Clearly, each elementary divisor ring is a Hermite ring. Moreover, each elementary divisor ring is an SE_2 ring (as defined above) by [4], Prop. 1.3.

Lorenzini introduced 3 classes of rings that are 'between' elementary divisor rings and Hermite rings (see [12], Prop. 4.11). We define the first class, $J_{2,1}$, as follows.

Definition 1.10. *We say that R is:*

(1) *a $WJ_{2,1}$ ring if for each $(a,b,c,d) \in Um(R^4)$ and every $(\Psi, \Delta) \in R^2$, there exists $(x,y,z,w) \in R^4$ such that $ax + by + cx + dw = \Psi$ and $xw - yz = \Delta$;*

(2) *a $J_{2,1}$ ring if it is a Hermite ring and a $WJ_{2,1}$ ring.*

The above definition of a $J_{2,1}$ ring is equivalent to the one in [12], Def. 4.6 (see Proposition 8.1). For $J_{n,r}$ rings with $(n,r) \in \mathbb{N}^2$ and $n > r$ see [12], Def. 4.6. Lorenzini shows that an elementary divisor ring is a $J_{n,1}$ ring for every integer $n > 1$ (see [12], Prop. 4.8) and Fresnel shows that an Euclidean domain is a $J_{n,r}$ ring for all $(n,r) \in \mathbb{N}^2$ with $n > r$ (see [6], Thm. 1.1). Theorem 1.11 below, proved in Section 8 based on Theorem 1.4, solves a problem posed by Lorenzini (see [12], p. 618) and Fresnel (see [6], Subsect. 3.1) for the case of Π_2 rings.

Theorem 1.11. *Let R be a $WJ_{2,1}$ ring. Then the following properties hold.*

(1) *Each matrix $A \in Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable.*

(2) *Assume R is also a Hermite ring (i.e., R is a $J_{2,1}$ ring). Then R is an elementary divisor ring iff it is a Π_2 ring.*

(3) *Let R be a $J_{2,1}$ domain. Then R is an elementary divisor domain.*

Therefore the constructions for an arbitrary commutative ring performed in [12], Ex. 4.10 for $n = 2$ or in Ex. 3.5 always produce rings which are either elementary divisor rings or are not Π_2 rings (in particular, the integral domains produced are elementary divisor domains).

By combining Theorem 1.11(2) with [12], Prop. 4.8 we get directly the following implication between these classes of rings.

Corollary 1.12. *Let R be a $J_{2,1}$ ring. If R is a Π_2 ring, then R is a $J_{n,1}$ ring for each integer $n > 1$.*

The commutative R-algebras associated to A and required in the proofs of the above theorems are introduced in Section 2; their smoothness properties are presented in Theorem 3.1 of Section 3. Section 6 proves criteria for weakly determinant liftability. Section 9 studies rings R for which all $A \in Um\big(\mathbb{M}_2(R)\big)$ are (weakly) determinant liftable. Section 10 proves criteria for determinant liftability via completions. Section 11 uses Picard groups to refine Theorem 3.1(7) when $\det(A) = 0$. Part III of this series of articles [5] will contain applications to Hermite rings.

2. Five Algebras

For a commutative R-algebra S, let $N(S)$, $Z(S)$, $U(S)$, $\mathrm{Pic}\,(S)$ be the nilradical, the set of zero divisors, the multiplicative group of units and the Picard group (respectively) of S. Let $\mathrm{Spec}\, S$ be the spectrum of S. Let $\mathrm{Max}\, S$ be the set of maximal prime ideals of S. For $f \in S$, let $(f) := Sf$ and we often abuse the notation by denoting $\bar{f} := f + \mathfrak{i} \in S/\mathfrak{i}$ for several ideals $\mathfrak{i}$ of S.

To $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ we attach five commutative R-algebras as follows. Let X, Y, Z, W, V and T be variables. The first four R-algebras are

$$\begin{aligned} \mathcal{U} = \mathcal{U}_A &:= R[X,Y,Z,W]/(1 - aX - bY - cZ - dW), \\ \mathcal{E} = \mathcal{E}_A &:= R[X,Y,Z,W,V]/\big(1 - aXW - bXZ - cYW - dYZ - (ad - bc)V\big), \\ \mathcal{X} = \mathcal{X}_A &:= R[X,Y,Z,W]/(1 - aXW - bXZ - cYW - dYZ), \\ \mathcal{D} = \mathcal{D}_A &:= R[X,Y,Z,W]/(1 - aX - bY - cZ - dW, XW - YZ). \end{aligned}$$

The polynomial

$$\Phi = \Phi_A(X,Y,Z,W) := 1 - aX - bY - cZ - dW + (ad - bc)(XW - YZ) \in R[X,Y,Z,W]$$

admits several natural decompositions of the form $e_{1,1}e_{2,2} - e_{1,2}e_{2,1}$, such as $\Phi = (1 - aX - cZ)(1 - bY - dW) - (aY + cW)(bX + dZ)$. The fifth R-algebra is

$$\mathcal{W} = \mathcal{W}_A := R[X,Y,Z,W]/(\Phi).$$

If $c = 0$, then

$$\text{(I)} \qquad \begin{aligned} \mathcal{W} &= R[X,Y,Z,W]/\big((1 - aX)(1 - dW) - Y(b + adZ)\big) \\ &= R[X,Y,Z,W]/\big((1 - aX)(1 - bY - dW) - aY(bX + dZ)\big). \end{aligned}$$

We define an arrow diagram of R-algebra homomorphisms

$$\text{(II)} \qquad \begin{array}{ccccc} \mathcal{W} & \longrightarrow & \mathcal{D} & \longleftarrow & \mathcal{U} \\ & & \downarrow{\scriptstyle \rho} & & \\ \mathcal{E} & \longrightarrow & \mathcal{X} & & \end{array}$$

as follows. The horizontal homomorphisms are surjections defined by R-algebra isomorphisms $\mathcal{W}/(\bar{X}\bar{W} - \bar{Y}\bar{Z}) \cong \mathcal{D} \cong \mathcal{U}/(\bar{X}\bar{W} - \bar{Y}\bar{Z})$ and $\mathcal{X} \cong \mathcal{E}/(\bar{V})$ given by the Third Isomorphism Theorem, and ρ is defined by mapping $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$ to $\bar{X}\bar{W}$, $\bar{X}\bar{Z}$, $\bar{Y}\bar{W}$, $\bar{Y}\bar{Z}$ (respectively). Equation (I) and Diagram (II) encode many applications in this Part II and the sequel Part III [5].

The R-algebra homomorphism $R[X,Y,Z,W] \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X}$ that maps X, Y, Z, and W to $T \otimes \bar{X}$, $T \otimes \bar{Y}$, $T^{-1} \otimes \bar{Z}$, and $T^{-1} \otimes \bar{W}$ (respectively) maps $1 - aXW - bXZ - cYW - dYZ$ to $1 \otimes (1 - a\bar{X}\bar{W} - b\bar{X}\bar{Z} - c\bar{Y}\bar{W} - d\bar{Y}\bar{Z}) = 1 \otimes 0 = 0$ and hence it induces an R-algebra homomorphism

$$\chi : \mathcal{X} \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X}$$

that maps $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{W}$ to $T \otimes \bar{X}$, $T \otimes \bar{Y}$, $T^{-1} \otimes \bar{Z}$, and $T^{-1} \otimes \bar{W}$ (respectively). The composite R-algebra homomorphism $\chi \circ \rho : \mathcal{D} \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X}$ maps $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{W}$ to $1 \otimes \bar{X}\bar{W} = \bar{X} \otimes 1$, $1 \otimes \bar{X}\bar{Z} = \bar{Y} \otimes 1$, $1 \otimes \bar{Y}\bar{W} = \bar{Z} \otimes 1$, $1 \otimes \bar{Y}\bar{Z} = \bar{W} \otimes 1$ (respectively) and hence χ is a $\mathcal{D}$-algebra homomorphism.

For two R-algebras S and S', we consider the set

$$\mathrm{Hom}_R(S,S') := \{\varrho : S \to S' | \varrho \text{ is an } R\text{-algebra homomorphism}\}.$$

The next two lemmas form the first justification for introducing these R-algebras.

Lemma 2.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$ *and the R-algebras $\mathcal{U}$, $\mathcal{E}$, $\mathcal{X}$, $\mathcal{D}$ and $\mathcal{W}$ be as above. Then the following properties hold.*

(1) *For a commutative R-algebra S we have functorial bijections of sets*

(III) $\mathrm{Hom}_R(\mathcal{U},S) \cong \{(x,y,z,w) \in S^4 | ax+by+cz+dw=1\}$,

(IV) $\mathrm{Hom}_R(\mathcal{E},S) \cong \{(x,y,z,w,v) \in S^5 | axw+bxz+cyw+dyz+(ad-bc)v=1\}$,

(V) $\mathrm{Hom}_R(\mathcal{X},S) \cong \{(x,y,z,w) \in S^4 | axw+bxz+cyw+dyz=1\}$,

(VI) $\mathrm{Hom}_R(\mathcal{D},S) \cong \{(x,y,z,w) \in S^4 | ax+by+cz+dw=1 \text{ and } xw=yz\}$,

(VII) $\mathrm{Hom}_R(\mathcal{W},S) \cong \{(x,y,z,w) \in S^4 | \Phi(x,y,z,w)=0\}$.

(2) *For a commutative R-algebra S we have a functorial diagram of sets*

$$\begin{array}{ccc} \{(x,y,z,w) \in S^4 | axw+bxz+cyw+dyz=1\} & \longrightarrow & \mathrm{Hom}_R(\mathcal{X},S) \\ \varrho(S) \downarrow & & \downarrow \rho(S) \\ \{(x,y,z,w) \in S^4 | ax+by+cz+dw=1,\ xw=yz\} & \longrightarrow & \mathrm{Hom}_R(\mathcal{D},S), \end{array}$$

where the horizontal arrows are identifications given by Equations (V) and (VI), $\rho(S)$ maps $h \in \mathrm{Hom}_R(\mathcal{X},S)$ to $h \circ \rho \in \mathrm{Hom}_R(\mathcal{D},S)$, and $\varrho(S)$ is defined by the rule $(x,y,z,w) \mapsto (xw,xz,yw,yz)$.[1]

(3) *Let S be a commutative R-algebra. Let $\varrho(S)$ be as in part (2). Let $(x',y',z',w') \in S^4$ be such that $ax'+by'+cz'+dw'=1$, $x'w'=y'z'$, and $\{x',y',z',w'\} \cap U(S) \neq \emptyset$. Then there exists a quadruple $\upsilon=(x,y,z,w) \in S^4$ such that $axw+bxz+cyw+dyz=1$ and $\varrho(S)(\upsilon)=(x',y',z',w')$. Moreover, for each such quadruple υ, the function $\varrho(S)_\upsilon : U(S) \times \{\upsilon\} \to \varrho(S)^{-1}(x',y',z',w')$ defined by $\varrho(S)_\upsilon(u,\upsilon) := (ux,uy,u^{-1}z,u^{-1}w)$ is a bijection.*

Proof. Let $\mathrm{H}_S := \{\varrho \in \mathrm{Hom}_R(R[X,Y,Z,W],S) | 1-aX-bY-cZ-dW \in \mathrm{Ker}(\varrho)\}$. Factorization of homomorphisms theorem gives a functorial bijection $\mathrm{Hom}_R(\mathcal{U},S) \cong \mathrm{H}_S$. Similarly, the universal property of polynomial rings gives a functorial bijection $\mathrm{Hom}_R(R[X,Y,Z,W],S) \cong S^4$ defined by the rule $\varrho \mapsto (\varrho(X),\varrho(Y),\varrho(Z),\varrho(W))$; this induces a functorial bijection $\mathrm{H}_S \cong \{(x,y,z,w) \in S^4 | ax+by+cz+dw=1\}$. The last two sentences imply that Equation (III) holds. Equations (IV) to (VI) are proved similarly. So part (1) holds.

Part (2) follows from the definition of $\rho : \mathcal{D} \to \mathcal{X}$.

For part (3), to fix the ideas we can assume that $x' \in U(S)$; so $w'=(x')^{-1}y'z'$. As υ we can take $(1,(x')^{-1}z',y',x')$. If $\varrho(S)_\upsilon(x_1,y_1,z_1,w_1)=(x',y',z',w')$, then $x_1w_1=xw=x' \in U(S)$, $x_1z_1=xz=y'$, $y_1w_1=yw=z'$ and $y_1z_1=yz=w'$.

[1]Thus the R-algebra homomorphism $\rho : \mathcal{D} \to \mathcal{X}$, as a morphism of functors from the category of R-algebras to the category of sets, represents decompositions of the unimodular matrix $\begin{bmatrix} \bar{X} & \bar{Y} \\ \bar{W} & \bar{Z} \end{bmatrix} \in Um(\mathbb{M}_2(\mathcal{D}))$ as a product $\begin{bmatrix} \bar{X}_{\mathcal{X}} \\ \bar{Y}_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} \bar{W}_{\mathcal{X}} & \bar{Z}_{\mathcal{X}} \end{bmatrix}$, the lower right index $\mathcal{X}$ emphasizing variables for $\mathcal{X}$.

For $u \in U(S)$, $\varrho(S)_v(u,v)$ is equal to (x_1,y_1,z_1,w_1) iff $u := ww_1^{-1}$. So $\varrho(S)_v$ is a bijection and part (3) holds. $\square$

Recall that a retraction of an R-algebra S is an R-algebra homomorphism $S \to R$.

Lemma 2.2. *For* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ *the following statements are equivalent.*

(1) *The matrix A is extendable (resp. simply extendable).*

(2) *There exists $(x,y,z,w,v) \in R^5$ (resp. $(x,y,z,w) \in R^4$) such that we have $axw + bxz + cyw + dyz + (ad-bc)v = 1$ (resp. $axw + bxz + cyw + dyz = 1$).*

(3) *The R-algebra $\mathcal{E}$ (resp. $\mathcal{X}$) has a retraction.*

Proof. The equivalence (3) $\Leftrightarrow$ (2) follows from Lemma 2.1, Equation (IV) (resp. (V)) applied to the R-algebra R. By very definition, the matrix A is extendable (resp. simply extendable) iff there exists $(x,y,z,w,v) \in R^5$ (resp. $(x,y,z,w) \in R^4$) such that the matrix $\begin{bmatrix} a & b & y \\ c & d & -x \\ -z & w & v \end{bmatrix}$ (resp. $\begin{bmatrix} a & b & y \\ c & d & -x \\ -z & w & 0 \end{bmatrix}$) has determinant 1, i.e., the identity $axw+bxz+cywdyz+(ad-bc)v = 1$ (resp. $axw+bxz+cyw+dyz = 1$) holds; hence (1) $\Leftrightarrow$ (2). Thus the lemma holds. $\square$

3. Smoothness properties

The analogs of Lemma 2.2 for determinant liftability (see Theorem 1.2 and Lemma 2.1, Equation (VI)) and for weakly determinant liftability (see Theorem 6.1(2) to (4) below and Lemma 2.1, Equation (VII)) require substantial extra work and even some algebraic geometry reviews which are carried out in this section.

For an R-algebra homomorphism $h : S \to S'$ between commutative R-algebras and $f \in R$, let $h_f : S_f \to S'_f$ be the localization of h with respect to the multiplicative set $\mathcal{M}_f := \{f^i | i \in \mathbb{N} \cup \{0\}\}$ (with $f^0 := 1$ by convention). In particular, $R_f := \mathcal{M}_f^{-1}R$ and $S_f \cong R_f \otimes_R S$ as R_f-algebras. Similarly, for an S-linear map $\lambda : M \to M'$ between S-modules, let $\lambda_f : M_f \to M'_f$ be its localization with respect to $\mathcal{M}_f$; we have a functorial S_f-linear isomorphism $M_f \cong R_f \otimes_R M$, where $R_f \otimes_R M$ is viewed as an S_f-module via the R_f-algebra isomorphism $S_f \cong R_f \otimes_R S$. If $f \in N(R)$, then M_f is the zero module over the zero ring S_f.

Given $n \in \mathbb{N} \cup \{0\}$, for the definition of a smooth morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ (equivalently, a smooth homomorphism $R \to S$) of relative dimension n we refer to [2], Ch. 2, Sect. 2.2, Def. 3. The main example of a smooth homomorphism of relative dimension n is the polynomial R-algebra homomorphism $R \to R[X_1, \ldots, X_n]$. Smoothness of relative dimension n is a local notion in the Zariski topologies of either $\operatorname{Spec} S$ or $\operatorname{Spec} R$ and hence (i) the homomorphism $R \to S$ is smooth of relative dimension n if S is a symmetric R-algebra of a projective R-module of rank n and (ii) the localization homomorphism $R \to R_f$ is smooth of relative dimension 0. If $r \in \mathbb{N} \cup \{0\}$ and the homomorphisms $R \to S$ and $S \to S'$ are smooth of relative dimensions n and r (respectively), then the composite homomorphism $R \to S'$ is smooth of relative dimension $n + r$.

Recall that $\mathbb{G}_{m,R}$ is the commutative affine group scheme over $\operatorname{Spec} R$ defined by the commutative Hopf R-algebra $R[T, T^{-1}]$ whose R-algebra comultiplication homomorphism $\mathrm{co}_R : R[T,T^{-1}] \to R[T,T^{-1}] \otimes_R R[T,T^{-1}]$ is defined by $\mathrm{co}_R(T) =$

$T \otimes T$ (the R-algebra homomorphisms that are the counit $R[T,T^{-1}] \to R$ and the coinverse $R[T,T^{-1}] \to R[T,T^{-1}]$ are uniquely determined by the comultiplication and they map T to 1 and respectively T^{-1}); so for each commutative R-algebra S, we have a functorial isomorphism of abstract commutative groups

$$\{\hbar : \operatorname{Spec} S \to \mathbb{G}_{m,R} | \hbar \text{ is a morphism of schemes over } \operatorname{Spec} R\} \cong U(S).$$

Recall that a morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ with a (left) action of $\mathbb{G}_{m,R}$ on it defined by a morphism $\mathcal{A} : \mathbb{G}_{m,R} \times_{\operatorname{Spec} R} \operatorname{Spec} S \to \operatorname{Spec} S$ is a $\operatorname{Spec} R$*-torsor* (in the Zariski topology) *under* $\mathbb{G}_{m,R}$ if there exists $n \in \mathbb{N}$ and $(f_1, \ldots, f_n) \in Um(R^n)$ (so $\operatorname{Spec} R = \cup_{i=1}^n \operatorname{Spec} R_{f_i}$) such that for each $i \in \{1, \ldots, n\}$ there exists a $\operatorname{Spec} R_{f_i}$-isomorphism $\operatorname{Spec} S_{f_i} \to \mathbb{G}_{m,R_{f_i}}$ under which the action $\mathbb{G}_{m,R_{f_i}} \times_{\operatorname{Spec} R_{f_i}} \operatorname{Spec} S_{f_i} \to \operatorname{Spec} S_{f_i}$ becomes isomorphic to the product morphism $\mathbb{G}_{m,R_{f_i}} \times_{\operatorname{Spec} R_{f_i}} \mathbb{G}_{m,R_{f_i}} \to \mathbb{G}_{m,R_{f_i}}$ defined by the comultiplication $\mathrm{co}_{R_{f_i}}$.

If we can take $n = 1$, then $\operatorname{Spec} S \to \operatorname{Spec} R$ with the action of $\mathbb{G}_{m,R}$ on it is called a *trivial* $\operatorname{Spec} R$*-torsor under* $\mathbb{G}_{m,R}$. If $n = 1$, then $f_1 \in U(R)$ and it follows that there exist R-algebra isomorphisms $S \cong S_{f_1} \cong R_{f_1}[T,T^{-1}] \cong R[T,T^{-1}]$ and in particular that the R-algebra S has a retraction. Conversely, if the R-algebra S has a retraction $h : S \to R$, then the natural composite morphism

$$\mathbb{G}_{m,R} \cong \mathbb{G}_{m,R} \times_{\operatorname{Spec} R} \operatorname{Spec} R \to \mathbb{G}_{m,R} \times_{\operatorname{Spec} R} \operatorname{Spec} S \to \operatorname{Spec} S$$

defined by restricting $\mathcal{A}$ to the closed subscheme $\operatorname{Spec} h : \operatorname{Spec} R \to \operatorname{Spec} S$ of $\operatorname{Spec} S$ is an isomorphism between $\operatorname{Spec} R$-schemes and its inverse is a $\operatorname{Spec} R$-isomorphism $\operatorname{Spec} S \to \mathbb{G}_{m,R}$ under which $\mathcal{A}$ becomes isomorphic to the product morphism $\mathbb{G}_{m,R} \times_{\operatorname{Spec} R} \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ and hence we can take $n = 1$ with $f_1 := 1$.

If S is a commutative R-algebra, the pullback of a $\operatorname{Spec} R$-torsor under $\mathbb{G}_{m,R}$ via the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is a $\operatorname{Spec} S$-torsor under $\mathbb{G}_{m,S}$.

For more details related to the last two paragraphs see [2], Ch. 6, Sect. 5.4 or [14], Ch. III, Sect. 4 which use the fppq and the flat (respectively) topology; see [14], Ch. III, Prop. 4.9 for why we can restrict to the Zariski topology.

As $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$, the R-linear map $\Lambda_A : R^4 \to R$ defined by the rule $\Lambda_A(x,y,z,w) = ax + by + cz + dw$ is surjective. Thus

$$\text{(VIII)} \qquad P = P_A := \operatorname{Ker}(\Lambda_A) = \{(x,y,z,w) \in R^4 | ax + by + cz + dw = 0\}$$

and its dual $P^* := \{f : P \to R | f \text{ is } R\text{-linear}\}$ are projective R-modules of rank 3 with $P \oplus R \cong P^* \oplus R \cong R^4$. Also, $P \cong P^*$ by [11], Ch. III, Sect. 6, Thm. 6.7 (1).

Theorem 3.1. *For* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$*, let* P *be as above. The following properties hold for the* R*-algebras* $\mathcal{U}$*,* $\mathcal{E}$*,* $\mathcal{X}$*,* $\mathcal{D}$*, and* $\mathcal{W}$ *of Section 2.*

(1) For $f \in \{a,b,c,d\}$*, the* R_f*-algebra* $\mathcal{U}_f$ *is isomorphic to a polynomial* R_f*-algebra in* 3 *variables. In particular, the* R*-algebra* $\mathcal{U}$ *is smooth of relative dimension* 3.

(2) The R*-algebra* $\mathcal{U}$ *is isomorphic to the symmetric* R*-algebra of* P*.*

(3) The R*-algebra* $\mathcal{D}$ *is smooth of relative dimension* 2.

(4) The R*-algebra* $\mathcal{E}$ *is smooth of relative dimension* 4.

(5) *The localization* $\mathcal{W}_{\big(1-(ad-bc)(\bar{X}\bar{W}-\bar{Y}\bar{Z})\big)}$ *of* $\mathcal{W}$ *is a smooth* R*-algebra of relative dimension* 3*. In particular, if* $ad-bc\in N(R)$*, then the* R*-algebra* $\mathcal{W}$ *is smooth of relative dimension* 3*.*

(6) *We consider the morphism of* $\mathrm{Spec}\, R$*-schemes* $\mathrm{Spec}\,\rho:\mathrm{Spec}\,\mathcal{X}\to\mathrm{Spec}\,\mathcal{D}$ *defined by the* R*-algebra homomorphism* $\rho:\mathcal{D}\to\mathcal{X}$ *of Section 2 and the morphism*

$$\mathrm{Spec}\,\chi:\mathbb{G}_{m,\mathcal{D}}\times_{\mathrm{Spec}\,\mathcal{D}}\mathrm{Spec}\,\mathcal{X}\to\mathrm{Spec}\,\mathcal{X}$$

of $\mathrm{Spec}\, D$*-schemes defined by the* $\mathcal{D}$*-algebra homomorphism* χ *of Section 2. Then this morphism of* $\mathrm{Spec}\,\mathcal{D}$*-schemes is a* $\mathbb{G}_{m,\mathcal{D}}$*-action on* $\mathrm{Spec}\,\rho$*.*

(7) *Referring to part (6),* $\mathrm{Spec}\,\rho$ *with its* $\mathbb{G}_{m,\mathcal{D}}$*-action is a* $\mathrm{Spec}\,\mathcal{D}$*-torsor under* $\mathbb{G}_{m,\mathcal{D}}$*. In particular, the morphism* $\mathrm{Spec}\,\rho$ *is smooth of relative dimension* 1 *and the* R*-algebra* $\mathcal{X}$ *is smooth of relative dimension* 3*.*

(8) *Assume* $\det(A)=0$*. Then for* $f\in\{a,b,c,d\}$ *the* R_f*-algebra* $\mathcal{D}_f$ *is a polynomial* R_f*-algebra in* 2 *variables. Moreover, there exists a self-dual projective* R*-module* Q *of rank* 2 *such that the* R*-algebra* $\mathcal{D}$ *is isomorphic to the symmetric* R*-algebra of* Q *and* $Q\oplus R\cong P$ *(thus* $Q\oplus R^2\cong R^4$*).*

Proof. (1) As $A\in Um\big(\mathbb{M}_2(R)\big)$, we have $\mathrm{Spec}\,R=\cup_{f\in\{a,b,c,d\}}\mathrm{Spec}\,R_f$. For $f=a$, as a is invertible in R_a, the R_a-algebra $\mathcal{U}_a$ is easily seen to be isomorphic to $R_a[Y,Z,W]$. The other three cases $f\in\{b,c,d\}$ are similar.

(2) We consider the R-algebra

$$\mathcal{U}':=R[X',Y',Z',W']/(aX'+bY'+cZ'+dW').$$

Let $(a',b',c',d')\in R^4$ be such that $\Lambda_A(a',b',c',d')=1$. The R-algebra isomorphism $R[X,Y,Z,W]\to R[X',Y',Z',W']$ that maps X to $a'-X'$, Y to $b'-Y'$, Z to $c'-Z'$, and W to $d'-W'$ also maps $1-aX-bY-cZ-dW$ to $aX'+bY'+cZ'+dW'$ and hence it induces an R-algebra isomorphism $\mathcal{U}\cong\mathcal{U}'$. We consider the R-module

$$M_3:=(X',Y',Z',W')/\big(aX'+bY'+cZ'+dW'+(X',Y',Z',W')^2\big)$$

of quotient of ideals of $R[X',Y',Z',W']$; it is isomorphic to the quotient R-module

$$[RX'\oplus RY'\oplus RZ'\oplus RW']/R(aX',bY',cZ',dW')$$

whose dual is naturally isomorphic to P. Thus the R-module M_3 is isomorphic to P^* and hence also to P. We consider the R-linear map $l:M_3\to\mathcal{U}'$ that maps $\star+\mathfrak{I}$ to $\star+\mathfrak{J}$ for each $\star\in\{X',Y',Z',W'\}$, where $\mathfrak{I}$ and $\mathfrak{J}$ are the ideals $\big(aX'+bY'+cZ'+dW'+(X',Y',Z',W')^2\big)$ and $(aX'+bY'+cZ'+dW')$ of $R[X',Y',Z',W']$ (respectively). If $\mathfrak{S}$ is the symmetric algebra of the R-module M_3, then there exists a unique R-algebra homomorphism $\mathfrak{l}:\mathfrak{S}\to\mathcal{U}'$ that extends l. To check that $\mathfrak{l}$ is an isomorphism it suffices to show that the R_f-algebra homomorphism $\mathfrak{l}_f:\mathfrak{S}_f\to\mathcal{U}'_f$ is an isomorphism for each $f\in\{a,b,c,d\}$. For $f=a$, as a is invertible in R_a, the R_a-algebra homomorphism $\mathfrak{l}_a$ is naturally identified with the identity automorphism of $R_a[Y,Z,W]$. The other three cases $f\in\{b,c,d\}$ are similar. We conclude that there exists an R-algebra isomorphism $\mathcal{U}\to\mathfrak{S}$ with $M_3\cong P$, thus part (2) holds.

(3) We consider the matrix

$$N_A:=\begin{bmatrix} a & b & c & d\\ W & -Z & -Y & X\end{bmatrix}\in\mathbb{M}_2(R[X,Y,Z,W]).$$

For $\mathfrak{n}\in\mathrm{Max}\,(R[X,Y,Z,W])$ such that $\{XW-YZ,1-aX-bY-cZ-dW\}\subset\mathfrak{n}$. Let $\mathrm{r}_{\mathfrak{n}}\in\{0,1,2\}$ be the rank of N_A modulo $\mathfrak{n}$ and let $\kappa:=R[X,Y,Z,W]/\mathfrak{n}$. We show that the assumption that $\mathrm{r}_{\mathfrak{n}} \le 1$ leads to a contradiction. As N_A modulo $\mathfrak{n}$ has unimodular rows, we have $\mathrm{r}_{\mathfrak{n}} = 1$ and thus there exists $\alpha \in R[X,Y,Z,W] \setminus \mathfrak{n}$ such that $(a,b,c,d) + \alpha(W,-Z,-Y,X) \in \mathfrak{n}^4$. So $2\alpha(XW - YZ)$ is congruent to $(aX+bY+cZ+dW)$ modulo $\mathfrak{n}$, and thus, as $\{XW-YZ, 1-aX-bY-cZ-dW\} \subset \mathfrak{n}$, it is congruent to both 0 and -1 modulo $\mathfrak{n}$, a contradiction. Therefore $\mathrm{r}_{\mathfrak{n}} = 2$.

We denote the differential forms operator by δ in order to avoid confusion with the element $d \in R$. As $\mathrm{r}_{\mathfrak{n}} = 2$, the two differential forms $\delta(XW - YZ) = W\delta X - Z\delta Y - Y\delta Z + X\delta W$ and $\delta(1 - aX - bY - cZ - dW) = -a\delta X - b\delta Y - c\delta Z - d\delta W$ of the free $R[X,Y,Z,W]$-module $\Omega^1_{R[X,Y,Z,W]/R} = \oplus_{\star\in\{X,Y,Z,W\}} R[X,Y,Z,W]\delta\star$ of relative differential forms (of degree 1) of $R[X,Y,Z,W]$ over R (see [2], Ch. 2, Sect. 2.1, p. 31) are linearly independent in $\Omega^1_{R[X,Y,Z,W]/R} \otimes_{R[X,Y,Z,W]} \kappa$. From this and Jacobian Criterion (see the implication $(a) \Leftrightarrow (d)$ in [2], Ch. 2, Sect. 2.2, Prop. 7) it follows that the morphism $\operatorname{Spec}\mathcal{D} \to \operatorname{Spec} R$ is smooth of relative dimension $4-2=2$ at $\mathfrak{n}/(XW - YZ, 1 - aX - bY - cZ - dW) \in \operatorname{Spec}\mathcal{D}$. From this and [2], Ch. 2, Sect. 2.2, Prop. 11 it follows that $\operatorname{Spec}\mathcal{D} \to \operatorname{Spec} R$ is smooth of relative dimension 2 at all points of $\operatorname{Spec} D$ contained in an open subset of $\operatorname{Spec}\mathcal{D}$ that contains $\operatorname{Max}\mathcal{D}$ and thus at all points of $\operatorname{Spec}\mathcal{D}$. Thus part (3) holds by the very definitions (see [2], Ch. 2, Sect. 2.2, Def. 3).

(4) Similar to the last two paragraphs, it suffices to show that for

$$\Theta(X,Y,Z,W,V) := 1 - aXW - bXZ - cYW - dYZ - (ad-bc)V \in R[X,Y,Z,W,V],$$

and its five partial derivatives Θ_X, Θ_Y, Θ_Z, Θ_W, and Θ_V, the differential form $\delta\Theta = \Theta_X\delta X + \Theta_Y\delta Y + \Theta_Z\delta_Z + \Theta_W\delta W + \Theta_V\delta V \in \Omega^1_{R[X,Y,Z,W,V]/R}$ is nonzero modulo each maximal ideal of $R[X,Y,Z,W,V]$ that contains Θ. So it suffices to show that Θ and its five partial derivatives generate $R[X,Y,Z,W,V]$, which follows from the identity $1 = \Theta - V\Theta_V - X\Theta_X - Y\Theta_Y$. So part (4) holds.

(5) Similarly to the last paragraph, part (5) follows from the fact that we have an identity $1-(ad-bc)(XW-YZ) = \Phi - X\Phi_X - Y\Phi_Y - Z\Phi_Z - W\Phi_W$. Note that if $ad - bc \in N(R)$, then $1 - (ad-bc)(\bar{X}\bar{W} - \bar{Y}\bar{Z}) \in U(\mathcal{W})$ and thus the R-algebra localization homomorphism $\mathcal{W} \to \mathcal{W}_{\left(1-(ad-bc)(\bar{X}\bar{W}-\bar{Y}\bar{Z})\right)}$ is an isomorphism.[2]

(6) It is easy to see that we have an identity of $\mathcal{D}$-algebra homomorphisms

$$(\mathrm{co}_{\mathcal{D}} \otimes 1_{\mathcal{X}}) \circ \chi = (1_{\mathcal{D}[T,T^{-1}]} \otimes \chi) \circ \chi : \mathcal{X} \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D}} \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X},$$

with $1_\bigstar$ denoting the identity automorphism of the R-algebra $\bigstar$ and the comultiplication $\mathrm{co}_{\mathcal{D}} : \mathcal{D}[T,T^{-1}] \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D}} \mathcal{D}[T,T^{-1}]$ as in the beginning of this section. For instance, $(\mathrm{co}_{\mathcal{D}} \otimes 1_{\mathcal{X}}) \circ \chi(\bar{Z}) = (\mathrm{co}_{\mathcal{D}} \otimes 1_{\mathcal{X}})(T^{-1} \otimes \bar{Z}) = T^{-1} \otimes T^{-1} \otimes \bar{Z}$ is equal to $(1_{\mathcal{D}[T,T^{-1}]} \otimes \chi) \circ \chi(\bar{Z}) = (1_{\mathcal{D}[T,T^{-1}]} \otimes \chi)(T^{-1} \otimes \bar{Z}) = T^{-1} \otimes T^{-1} \otimes \bar{Z}$.

Also, $\chi : \mathcal{X} \to \mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X}$ has a retraction defined by the $\mathcal{X}$-algebra homomorphism $\mathcal{D}[T,T^{-1}] \otimes_{\mathcal{D},\rho} \mathcal{X} \to \mathcal{X}$ that maps $T \otimes 1$ to 1.

Part (6) follows from the last two paragraphs.

(7) As $1 = a\bar{X} + b\bar{Y} + c\bar{Z} + d\bar{W}$ in $\mathcal{D}$, $\operatorname{Spec}\mathcal{D} = \cup_{\bar{f}\in\{\bar{X},\bar{Y},\bar{Z},\bar{W}\}} \operatorname{Spec}\mathcal{D}_{\bar{f}}$. So it suffices to show that for $\bar{f} \in \{\bar{X},\bar{Y},\bar{Z},\bar{W}\}$, the pullback $\mathcal{D}_{\bar{f}}$-torsor under $\mathbb{G}_{m,\mathcal{D}_{\bar{f}}}$, i.e., the morphism $\operatorname{Spec}\mathcal{X}_{\bar{f}} \to \operatorname{Spec}\mathcal{D}_{\bar{f}}$ defined by $\rho_{\bar{f}}$ with its induced action, is trivial. We only show this for $\bar{f} = \bar{X}$ as the other three cases are similar; based on

[2] The morphism $\operatorname{Spec}\mathcal{W} \to \operatorname{Spec} R$ is not smooth of relative dimension 3 at precisely the points of the closed subscheme of $\operatorname{Spec}\mathcal{W}$ which is the closed subscheme of $\operatorname{Spec}\mathcal{W}_{ad-bc}$ defined by the zero locus $\bar{X} - (ad-bc)^{-1}d = \bar{Y} - (ad-bc)^{-1}c = \bar{Z} - (ad-bc)^{-1}b = \bar{W} - (ad-bc)^{-1}a = 0$. In particular, if A is invertible, then this closed subscheme is isomorphic to $\operatorname{Spec}\big(R/N(R)\big)$.

the diagram of Lemma 2.1(2), this case follows from the last part of Lemma 2.1(3) applied to $S = \mathcal{D}_{\bar{X}}$.

(8) For the polynomial R-algebra part we only consider the case $f = a$ as the other three cases are similar. By eliminating $\bar{X} = a^{-1}(1 - b\bar{Y} - c\bar{Z} - d\bar{W}) \in \mathcal{D}_a$, we obtain an isomorphism

$$\mathcal{D}_a \cong R_a[Y, Z, W]/\big(YZ - a^{-1}W(1 - bY - cZ - dW)\big),$$

which via the change of variables $(Y_1, Z_1, W_1) := (aY + cW, Z + a^{-1}bW, W)$ over R_a is isomorphic, as $ad = bc$, to $R_a[Y_1, Z_1, W_1]/a^{-2}(Y_1Z_1 - W_1) \cong R_a[Y_1, Z_1]$ and hence to the symmetric R_a-algebra of R_a^2. From [1], Thm. (4.4) it follows that $\mathcal{D}$ is isomorphic to the symmetric R-algebra of a projective R-module Q. From [1], Lem. (4.6) applied to R and R_f with $f \in \{a, b, c, d\}$ it follows that Q is uniquely determined up to isomorphism and that $Q_f \cong R_f^2$ for each $f \in \{a, b, c, d\}$. Thus Q has rank 2. So there exists an R-algebra homomorphism $\mathcal{D} \to R$; let $\zeta := (a', b', c', d') \in R^4$ be such that $\Lambda_A(\zeta) = 1$ and $a'd' = b'c'$ by Lemma 2.1, Equation (VI). The R-algebra $\mathcal{D}$ is isomorphic to

$$R[X', Y', Z', W']/(aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W' - X'W' + Y'Z')$$

via the change of variables $(X', Y', Z', W') := (a', b', c', d') + (X, Y, Z, W)$.

We consider the R-submodule $M_2 := R(a, b, c, d) + R(d', -c', -b', a')$ of R^4. To study M_2 we introduce the nondegenerate bilinear form $\langle\, ,\rangle : R^4 \times R^4 \to R$ on the R-module R^4 defined by the rule

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\rangle := \sum_{i=1}^{4} x_i y_i.$$

We have identities $\langle \zeta, (a, b, c, d)\rangle = 1$ and $\langle \zeta, (d', -c', -b', a')\rangle = 0$. Let $\mathfrak{m} \in \operatorname{Max} R$; so $R/\mathfrak{m}$ is a field and $V_4 := (R/\mathfrak{m})^4$ is a vector space over it of dimension 4. The image V_2 of M_2 in V_4 is a vector subspace of dimension $d_\mathfrak{m} \in \{0, 1, 2\}$. The images of $(a, b, c, d) \in Um(R^4)$ and $(d', -c', -b', a') \in Um(R^4)$ in V_2 are two nonzero vectors v and v' (respectively), hence $d_\mathfrak{m} \geq 1$. We show that the assumption that $d_\mathfrak{m} = 1$ leads to a contradiction. This assumption implies that there exists a nonzero scalar $\beta \in R/\mathfrak{m}$ such that $v = \beta v'$. If $\zeta_\mathfrak{m}$ is the image of ζ in V_4, by denoting also by $\langle\, ,\rangle$ its reduction modulo $\mathfrak{m}$, we compute $0 = \langle \zeta_\mathfrak{m}, v'\rangle = \beta\langle \zeta_\mathfrak{m}, v\rangle = \beta$, a contradiction to $\beta \neq 0$. Thus $d_\mathfrak{m} \neq 1$, hence $d_\mathfrak{m} = 2$. As $d_\mathfrak{m} = 2$ for each $\mathfrak{m} \in \operatorname{Max} R$, we have $M_2 \cong R^2$ and R^4/M_2 is a projective R-module of rank 2 by Lemma 3.2 below. Hence M_2 is a direct summand of R^4.

Let

$$\mathcal{D}' := R[X', Y', Z', W']/(aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W').$$

As $\Lambda_A(\zeta) = aa' + bb' + cc' + dd' = 1$, we have $\operatorname{Spec} R = \cup_{f\in\{a,b,c,d\}} \operatorname{Spec} R_{ff'}$ (if $f = a$ then $f' = a'$, if $f = b$ then $f' = b'$, etc.). For $f \in \{a, d\}$ (resp. $f \in \{b, c\}$), the $R_{ff'}$-algebra $\mathcal{D}'_{ff'}$ is isomorphic to $R_{ff'}[Y', Z']$ (resp. $R_{ff'}[X', W']$). Let J and J' be the ideals of $\mathcal{D}$ and $\mathcal{D}'$ (respectively) generated by the images of X', Y', Z', W'. We view $\mathcal{D}$ and $\mathcal{D}'$ as augmented R-algebras (in the terminology of [1], Sect. 4, Cor. (4.3)) with the augmentations $\mathcal{D} \to R$ and $\mathcal{D}' \to R$ which are the R-algebra homomorphisms having J and J' (respectively) as their kernels. The R-modules $Q := J/J^2$ and $Q' = J'/(J')^2$ are identified via the Third Isomorphism Theorem to the following R-module quotient of ideals

$$(X', Y', Z', W')/\big((aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W') + (X', Y', Z', W')^2\big)$$

of $R[X',Y',Z',W']$, thus they are isomorphic to R^4/M_2. From [1], Cor. (4.3) and the isomorphisms of localizations $\mathcal{D}_a \cong R_a[Y_1,Z_1]$ and $\mathcal{D}'_{aa'} \cong R_a[Y',Z']$ that involve variables that are linear polynomials in X_1,Y_1,Z_1,W_1 and X',Y',Z',W' (respectively) and their analogs with a replaced by b, c, or d, it follows that the augmented R-algebras $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic to the symmetric R-algebras of Q and Q' (respectively) endowed with their natural augmentations, and so they are isomorphic. So the R-algebra $\mathcal{D}$ is isomorphic to the symmetric R-algebra of M_2.

As M_2 is a direct summand of R^4, there exists a short exact sequence $0 \to R \to P^* \to Q' \to 0$ of R-modules, so $Q \oplus R \cong Q' \oplus R \cong P^* \cong P$. Thus $Q \oplus R^2 \cong P \oplus R \cong R^4$. As the R-module Q is stably free of rank 2, it is self-dual by [11], Ch. III, Sect. 6, Thm. 6.8. □

For reader's convenience we include the following lemma whose statement and proof are probably well-known, but for which we could not find a reference.

Lemma 3.2. *Let $(i,j) \in [\mathbb{N} \cup \{0\}]^2$. Let M_i be an R-submodule of R^{i+j} generated by i elements and such that for each $\mathfrak{m} \in \operatorname{Max} R$ the image of M_i in $R^{i+j}/\mathfrak{m}R^{i+j}$ is a vector space of dimension i over the field $R/\mathfrak{m}$. Then $M_i \cong R^i$ and R^{i+j}/M_i is a projective R-module of rank j.*

Proof. Let $l : R^i \to M_i$ be a surjective R-linear map. For $\mathfrak{m} \in \operatorname{Max} R$, let $l^{\perp}_{\mathfrak{m}} : R^j \to R^{i+j}$ be an R-linear map such that the rule $(x_1,\ldots,x_{i+j}) \mapsto l(x_1,\ldots,x_i) + l^{\perp}_{\mathfrak{m}}(x_{i+1},\ldots,x_{i+j})$ defines an R-linear map $\ell_{\mathfrak{m}} : R^{i+j} \to R^{i+j}$ whose reduction modulo $\mathfrak{m}$ is an isomorphism. Let $f_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ be the determinant of $\ell_{\mathfrak{m}}$. As the $R_{f_{\mathfrak{m}}}$-linear map $(\ell_{\mathfrak{m}})_{f_{\mathfrak{m}}}$ is an isomorphism, it follows that $l_{f_{\mathfrak{m}}}$ is an isomorphism and $(R^{i+j}/M_i)_{f_{\mathfrak{m}}} \cong R^j_{f_{\mathfrak{m}}}$. As $\operatorname{Spec} R = \cup_{\mathfrak{m} \in \operatorname{Max} R} \operatorname{Spec} R_{f_{\mathfrak{m}}}$, we conclude that l is an isomorphism and R^{i+j}/M_i is a projective R-module of rank j. □

Corollary 3.3. *Assume there exist two ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of R such that $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \{0\}$ and $\det(A) \in \mathfrak{i}_2$. Then for $\mathcal{C} \in \{\mathcal{D},\mathcal{W}\}$, each R-algebra homomorphism $\mathcal{C} \to R/\mathfrak{i}_1$ lifts to an R-algebra homomorphism $\mathcal{C} \to R$.*

Proof. Let $h_{1,2} : \mathcal{C} \to R/(\mathfrak{i}_1 + \mathfrak{i}_2)$ be induced by an R-algebra homomorphism $h_1 : \mathcal{C} \to R/\mathfrak{i}_1$. As A modulo $\mathfrak{i}_2$ has zero determinant, $\mathcal{C}/\mathfrak{i}_2\mathcal{C}$ is the symmetric algebra of a projective $R/\mathfrak{i}_2$-module Q_2 of rank 2 if $\mathcal{C} = \mathcal{D}$ and of rank 3 if $\mathcal{C} = \mathcal{W}$ (see Theorem 3.1(1) and (8)). The R-algebra homomorphism $h_{1,2}$ is uniquely determined by an R-linear map $l_{1,2} : Q_2 \to R/(\mathfrak{i}_1 + \mathfrak{i}_2)$. If $l_2 : Q_2 \to R/\mathfrak{i}_2$ is an R-linear map that lifts $l_{1,2}$ and if $h_2 : \mathcal{C} \to R/\mathfrak{i}_2$ is the R-algebra homomorphism uniquely determined by l_2, then h_2 lifts $h_{1,2}$. As $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \{0\}$, we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{i}_1 \\ \downarrow & & \downarrow \\ R/\mathfrak{i}_2 & \longrightarrow & R/(\mathfrak{i}_1 + \mathfrak{i}_2), \end{array}$$

so there exists a unique R-algebra homomorphism $\mathcal{C} \to R$ that lifts h_1 and h_2. □

Corollary 3.4. *Assume R is a Hermite ring. With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$, let P be as in Equation (VIII). Then the following properties hold.*

(1) *We have $P \cong R^3$ and the R-algebra $\mathcal{U}$ is a polynomial R-algebra in 3 variables.*

(2) *Assume* $\det(A) = 0$ *and let the projective* R*-module* Q *be as in Theorem 3.1(8). Then* $Q \cong R^2$ *and the* R*-algebra* $\mathcal{D}$ *is a polynomial* R*-algebra in* 2 *variables.*

Proof. The stably free R-module P is free by [17], Cor. 3.2. As $P \cong R^3$, part (1) follows from Theorem 3.1(1).

If $\det(A) = 0$, then we similarly argue that the stably free R-module Q is free and that part (2) follows from Theorem 3.1(8). □

4. Proof of Theorem 1.2

For $E \in \mathbb{M}_2(R)$ let $\mathrm{Tr}(E)$ be its trace, let $\mathrm{adj}(E) \in \mathbb{M}_2(R)$ be its adjugate, and let Ker_E be the kernel and Im_E be the image of the R-linear map $L_E : R^2 \to R^2$ defined by E. Let $I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{M}_2(R)$.

We first prove the following general lemma.

Lemma 4.1. *Let* $G, H, E \in \mathbb{M}_2(R)$*. Then the following properties hold.*

(1) *There exists a matrix* $O \in \mathbb{M}_2(R)$ *such that* $H = G(I_2 + \mathrm{adj}(G)O)$ *iff* G *and* H *are congruent modulo* $R\det(G)$*.*

(2) *If* GE *is unimodular, then* G *and* E *are unimodular.*

(3) *If* G *is unimodular and* G *and* GE *are congruent modulo* $R\det(G)$*, then* GE *is unimodular iff* E *is unimodular.*

Proof. As $G\mathrm{adj}(G) = \det(G)I_2$, for $O \in \mathbb{M}_2(R)$ we have $H = G + \det(G)O$ iff $H = G(I_2 + \mathrm{adj}(G)O)$. So part (1) holds.

The only nontrivial implication of parts (2) and (3) is the 'if' of part (3). It suffices to show that the ideal $\mathfrak{h}$ of R generated by the entries of GE is not contained in any $\mathfrak{m} \in \mathrm{Max}\, R$. This holds if $\det(G) \in \mathfrak{m}$ as G and GE are congruent modulo $R\det(G)$. If $\det(G) \notin \mathfrak{m}$, then G modulo $\mathfrak{m}$ is invertible, thus GE modulo $\mathfrak{m}$ is nonzero as this is so for E modulo $\mathfrak{m}$, so $\mathfrak{h} \not\subseteq \mathfrak{m}$. □

We are now ready to prove Theorem 1.2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ be as in Theorem 1.2. To A and a quadruple $\upsilon = (x, y, z, w) \in R^4$, we associate matrices in $\mathbb{M}_2(R)$ as follows:

$$C_\upsilon := \begin{bmatrix} -w & z \\ y & -x \end{bmatrix},$$

$$D_{A,\upsilon} := I_2 + \mathrm{adj}(A)C_\upsilon = \begin{bmatrix} 1 - by - dw & bx + dz \\ ay + cw & 1 - ax - cz \end{bmatrix},$$

$$B_{A,\upsilon} := AD_{A,\upsilon} = A + \det(A)C_\upsilon = \begin{bmatrix} a - \det(A)w & b + \det(A)z \\ c + \det(A)y & d - \det(A)x \end{bmatrix}.$$

We have $\mathrm{Tr}(D_{A,\upsilon}) = 2 - ax - by - cz - dw$ and one computes

$$\det(D_{A,\upsilon}) = \Phi_A(x, y, z, w), \tag{IX}$$

$$\det(B_{A,\upsilon}) = \det(A)\Phi_A(x, y, z, w), \tag{X}$$

$$1 - \det(A)\det(C_\upsilon) = \mathrm{Tr}(D_{A,\upsilon}) - \det(D_{A,\upsilon}). \tag{XI}$$

To prove Theorem 1.2, we first remark that clearly (2) ⇒ (1) and (2) ⇒ (4).

To prove that (3) $\Rightarrow$ (2), let the quadruple $\upsilon = (x, y, z, w) \in R^4$ be such that $ax + by + cz + dw = 1$ and $xw - yz = 0$. We claim that the matrix $C := C_\upsilon$ satisfies the conditions in (2). We have $\Phi(x, y, z, w) = 0$, $\det(C) = 0$, $C \in Um\big(\mathbb{M}_2(R)\big)$ and for $B_{A,\upsilon} = A + \det(A)C$ we have $\det(B_{A,\upsilon}) = 0$ by Equation (X). As $\mathrm{Tr}(D_{A,\upsilon}) - \det(D_{A,\upsilon}) = 1$ by Equation (XI), $D_{A,\upsilon}$ is unimodular, so $B_{A,\upsilon} = AD_{A,\upsilon}$ is unimodular by Lemma 4.1(3). Hence (3) $\Rightarrow$ (2).

To show that (4) $\Rightarrow$ (3), let $C \in \mathbb{M}_2(R)$ be such that $\det(C) = \det\big(A + \det(A)C\big) = 0$. There exists a unique $\upsilon = (x, y, z, w) \in R^4$ such that $C = C_\upsilon$; so $A + \det(A)C = B_{A,\upsilon}$. We have $\det(B_{A,\upsilon}) = \det(C_\upsilon) = 0$. Thus $xw - zy = 0$ and $\det(A)\Phi(x, y, z, w) = \det(B_{A,\upsilon}) = 0$ by Equation (X). If $\det(A) \notin Z(R)$, then $\Phi(x, y, z, w) = 0$; from this and $xw - yz = 0$ it follows that $1 - ax - by - cz - dw = 0$. Hence (4) $\Rightarrow$ (3) if $\det(A) \notin Z(R)$.

In general, (3) holds iff there exists an R-algebra homomorphism $h : \mathcal{D} \to R$ by Lemma 2.1, Equation (VI) applied to $S = R$. By replacing R with a finitely generated $\mathbb{Z}$-subalgebra S of R such that $A, B_{A,\upsilon} \in Um\big(\mathbb{M}_2(S)\big)$ and $C = C_\upsilon \in \mathbb{M}_2(S)$, to show that h exists we can assume that R is noetherian. Thus $N(R)$ is nilpotent and there exists $j \in \mathbb{N}$ such that the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\}$ of minimal prime ideals of R has j elements. As the R-algebra $\mathcal{D}$ is smooth (see Theorem 3.1(3)) and $N(R)$ is nilpotent, each R-algebra homomorphism $\mathcal{D} \to R/N(R)$ lifts to an R-algebra homomorphism $\mathcal{D} \to R$ (e.g., see [2], Ch. 2, Sect. 2.2, Prop. 6). Thus, by replacing R with $R/N(R)$, we can assume also that $N(R) = \cap_{i=1}^{j}\mathfrak{p}_i = \{0\}$. If $\det(A) = 0$, then the R-algebra $\mathcal{D}$ is isomorphic to a symmetric R-algebra by Theorem 3.1(8) and thus h exists. So we can assume also that $\det(A) \neq 0$. Let $\det(A)_i := \det(A) + \mathfrak{p}_i \in R/\mathfrak{p}_i$. As $\det(A) \neq 0$, there exists an index $i \in \{1, \ldots, j\}$ such that $\det(A) \notin \mathfrak{p}_i$, i.e., $\det(A)_i \neq 0$. We can assume that the minimal prime ideals are indexed such that there exists $j' \in \{1, \ldots, j\}$ for which $\det(A)_i \neq 0$ if $i \in \{1, \ldots, j'\}$ and $\det(A)_i = 0$ if $i \in \{j' + 1, \ldots, j\}$. If $\mathfrak{i}_1 := \cap_{i=1}^{j'}\mathfrak{p}_i$ and $\mathfrak{i}_2 := \cap_{i=j'+1}^{j}\mathfrak{p}_i$, we have $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \{0\}$ and $\det(A) \in \mathfrak{i}_2$. As $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$, from the prior paragraph applied to $R/\mathfrak{i}_1$ and Lemma 2.1, Equation (VI) applied to $S = R/\mathfrak{i}_1$ it follows that there exists an R-algebra homomorphism $h_1 : \mathcal{D} \to R/\mathfrak{i}_1$. So there exists $h : \mathcal{D} \to R$ that lifts h_1 by Corollary 3.3. Hence (4) $\Rightarrow$ (3).

We conclude that (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

We prove that (1) $\Rightarrow$ (2). As (2) $\Leftrightarrow$ (3), as above we argue that it suffices to prove that (1) $\Rightarrow$ (2) when R is noetherian and $N(R) = \{0\}$. Let the ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of R be as above. Let $B \in Um\big(\mathbb{M}_2(R)\big)$ be congruent to A modulo $R\det(A)$ and $\det(B) = 0$. Let $\upsilon = (x, y, z, w) \in R^4$ be such that $B = B_{A,\upsilon}$ (see Lemma 4.1(1)). With $C := C_\upsilon$ and $D := D_{A,\upsilon}$, as $B = AD \in Um\big(\mathbb{M}_2(R)\big)$ we have $D \in Um\big(\mathbb{M}_2(R)\big)$ (see Lemma 4.1(2)). As $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$, from the identity $\det(B) = \det(A)\det(D) = 0$ and Equation (X) it follows that $\Phi(x, y, z, w) \in \mathfrak{i}_1$, hence there exists an R-algebra homomorphism $g_1 : \mathcal{W} \to R/\mathfrak{i}_1$ that maps the elements $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ of $\mathcal{W}$ to $x+\mathfrak{i}_1, y+\mathfrak{i}_1, z+\mathfrak{i}_1, w+\mathfrak{i}_1$ (respectively). As $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \{0\}$, let $g : \mathcal{W} \to R$ be an R-algebra homomorphism that lifts g_1 (see Corollary 3.3).

Let $\upsilon' = (x', y', z', w') := g^4(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}) \in R^4$. For the matrices $C' := C_{\upsilon'}$ and $D' := D_{A,\upsilon'}$ we have (see Equation (IX)) $\det(D') = \Phi(x', y', z', w') = 0$ and C' and C are congruent modulo $\mathfrak{i}_1$. Hence D' and D are congruent modulo $\mathfrak{i}_1$. As D is unimodular, it follows that the ideal $\mathfrak{d}'$ of R generated by the entries of D' satisfies $\mathfrak{d}' + \mathfrak{i}_1 = R$ and thus $\mathfrak{d}'$ is not contained in any $\mathfrak{m} \in \mathrm{Max}\, R$ with $\det(A) \notin \mathfrak{m}$. As $\mathrm{Tr}(D') - \det(D') = 1 - \det(A)\det(C') \in \mathfrak{d}'$ by Equation (XI), $\mathfrak{d}'$ is not contained in any maximal ideal which does not contain $1 - \det(A)\det(C')$. The last two sentences imply that $\mathfrak{d}'$ is not contained in any $\mathfrak{m} \in \operatorname{Max} R$, thus $\mathfrak{d}' = R$, i.e., D' is unimodular. From Lemma 4.1(3) it follows that $B' := AD'$ is unimodular.

By replacing the triple (C, B, D) with (C', B', D'), we can assume that $\det(D) = 0$. As $D = I_2 + \operatorname{adj}(A)C$ has zero determinant, it follows that $C \in Um\big(\mathbb{M}_2(R)\big)$.

To complete the proof that (1) $\Rightarrow$ (2), it suffices to show that we can replace C by a matrix $C_1 \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(C_1) = 0$ and such that for $D_1 := I_2 + \operatorname{adj}(A)C_1$ we have $\det(D_1) = 0$ and $D_1 \in Um\big(\mathbb{M}_2(R)\big)$: so $B_1 := AD_1$ is congruent to A modulo $R\det(A)$ and unimodular by Lemma 4.1(1) and (3) with $\det(B_1) = 0$. As Ker_D and Im_D are projective R-modules of rank 1 (see [4], Lem. 3.1), the short exact sequence $0 \to \mathrm{Ker}_D \to R^2 \to \mathrm{Im}_D \to 0$ splits, i.e., it has a section $\sigma : \mathrm{Im}_D \to R^2$. Let $C_1 \in \mathbb{M}_2(R)$ be the unique matrix such that $\mathrm{Ker}_D \subseteq \mathrm{Ker}_{C_1 - C}$ and $\sigma(\mathrm{Im}_D) \subseteq \mathrm{Ker}_{C_1}$. As Ker_D is a direct summand of R^2 of rank 1 and for $t \in \mathrm{Ker}_D$ we have $\operatorname{adj}(A)C_1(t) = \operatorname{adj}(A)C(t) = -t$, it follows first that $\mathrm{Ker}_D \subseteq \mathrm{Ker}_{D_1}$, second that $\mathrm{Im}_{C_1} = C_1(\mathrm{Ker}_D) = C(\mathrm{Ker}_D)$ is a direct summand of R^2 of rank 1 isomorphic to Ker_D, and third that $\mathrm{Ker}_{C_1} = \sigma(\mathrm{Im}_D)$ is also a direct summand of R^2 of rank 1. As $\mathrm{Ker}_D \subset \mathrm{Ker}_{D_1}$, $R^2 = \mathrm{Ker}_D \oplus \sigma(\mathrm{Im}_D)$ and $\sigma(\mathrm{Im}_D) \subset \mathrm{Ker}_{C_1}$ we compute

$$\mathrm{Im}_{D_1} = D_1\big(\sigma(\mathrm{Im}_D)\big) = \{x + \operatorname{adj}(A)C_1(x) | x \in \sigma(\mathrm{Im}_D)\} = \sigma(\mathrm{Im}_D).$$

We conclude that $C_1, D_1 \in Um\big(\mathbb{M}_2(R)\big)$ and $\det(C_1) = \det(D_1) = 0$. Hence (1) $\Rightarrow$ (2), thus Theorem 1.2 holds.

5. Proof of Theorem 1.3

Let $A \in Um\big(\mathbb{M}_2(R)\big)$. We include two proofs of Theorem 1.3(1). First, part (1) holds as clearly statement (4) of [4], Thm. 4.3 implies statement (3) of Theorem 1.2. Second, recall the R-algebra homomorphism $\rho : \mathcal{D} \to \mathcal{X}$ of Equation (II); as A is simply extendable there exists an R-algebra homomorphism $\mathcal{X} \to R$ by Lemma 2.2, (1) implies (3), which composed with ρ gives an R-algebra homomorphism $\mathcal{D} \to R$, hence statement (3) of Theorem 1.2 holds by Lemma 2.1, Equation (VI) and thus A is determinant liftable.

To prove part (2), assume that A is extendable. Then A modulo $R\det(A)$ is non-full by [4], Prop. 5.1(2). Let $(\bar{l}, \bar{m}, \bar{o}, \bar{q}) \in [R/R\det(A)]^4$ be such that A modulo $R\det(A)$ is $\begin{bmatrix} \bar{l} \\ \bar{m} \end{bmatrix} \begin{bmatrix} \bar{o} & \bar{q} \end{bmatrix}$. If $(l, m, o, q) \in R^4$ lifts $(\bar{l}, \bar{m}, \bar{o}, \bar{q})$, then $B := \begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$ is congruent to A modulo $R\det(A)$ and $\det(B) = 0$, hence A is weakly determinant liftable. Thus Theorem 1.3 holds.

Example 5.1. Let R be a Dedekind domain which is not a PID. So R is not a Π_2 ring by [4], Thm. 1.7(4). Hence there exist a matrix $A \in Um\big(\mathbb{M}_2(R)\big)$ of zero determinant which is not (simply) extendable. As A is (weakly) determinant liftable, it follows that the converses of Theorem 1.4 do not hold.

6. A criterion for weakly determinant liftability

As the R-algebra $\mathcal{W}$ of Section 2 is not smooth in general (see Theorem 3.1(5) and its proof), the analog of the equivalence (1) $\Leftrightarrow$ (3) in Theorem 1.2 for weakly determinant liftability has the following more complex form.

Theorem 6.1. *For* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$ *let* $\Phi = \Phi_A$ *be as in Section 2. Then the following properties hold.*

(1) *If* A *is determinant liftable, then there exists* $(x, y, z, w) \in R^4$ *such that* $\Phi(x, y, z, w) = 0$.

(2) *If there exists* $(x, y, z, w) \in R^4$ *such that* $\Phi(x, y, z, w) = 0$*, then* A *is weakly determinant liftable.*

(3) *If either* $N(R) = \{0\}$ *or* $\det(A) \notin Z(R)$*, then the converse of part (2) holds.*

(4) *If* $\det(A) \in Z(R)$ *and* A *is weakly determinant liftable, then there exists* $(x, y, z, w) \in R^4$ *such that* $\Phi(x, y, z, w) \in N(R)$.

Proof. If A is determinant liftable, then there exists $(x, y, z, w) \in R^4$ with $xw-yz = 0$ and $ax+by+cz+dw = 1$ by Theorem 1.2, so $\Phi(x, y, z, w) = 0$. So part (1) holds.

If $v = (x, y, z, w) \in R^4$ is as in part (2), then for $B := B_{A,v}$ we have $\det(B) = 0$ by Equation (X). Thus, as A and B are congruent modulo $R\det(A)$, A is weakly determinant liftable. So part (2) holds.

To prove part (3) let $B \in \mathbb{M}_2(R)$ be congruent to A modulo $R\det(A)$ with $\det(B) = 0$. Let $v = (x, y, z, w) \in R^4$ be such that $B = B_{A,v} = A\big(I_2 + \mathrm{adj}(A)C_v\big)$ (see Lemma 4.1(1)). Thus $\det(A)\Phi(x, y, z, w) = \det(B) = 0$ by Equation (X). If $\det(A) \notin Z(R)$, then from $\det(A)\Phi(x, y, z, w) = 0$ it follows that $\Phi(x, y, z, w) = 0$. So we can assume that $N(R) = \{0\}$. As in Section 4 we argue that we can assume that R is noetherian; if the ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of R are as in Section 4, then $\det(A) \in \mathfrak{i}_2$ and $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$. As $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$ and $\det(A)\Phi(x, y, z, w) = 0$, it follows that $\Phi(x, y, z, w) \in \mathfrak{i}_1$. So as in the first paragraph of the proof of the implication (1) $\Rightarrow$ (2) of Section 4 we argue based on Corollary 3.3 that there exists an R-algebra homomorphism $\mathcal{W} \to R$. From this and Lemma 2.1, Equation (VII) it follows that there exists $(x, y, z, w) \in R^4$ with $\Phi(x, y, z, w) = 0$. So part (3) holds.

Part (4) follows from part (3) applied to $R/N(R)$. $\square$

Example 6.2. If R is such that $N(R) = \{0\}$ and there exists $A \in Um(\mathbb{M}_2(R))$ which is not determinant liftable but is weakly determinant liftable (see Example 1.9), then there exists $(x, y, z, w) \in R^4$ such that $\Phi(x, y, z, w) = 0$ by Theorem 6.1(3). Hence the converse of Theorem 6.1(1) does not hold in general.

Remark 6.3. If $v = (x, y, z, w) \in R^4$ is such that $\Phi(x, y, z, w) \neq 0 = \Phi(x, y, z, w)^2$ and the matrix $B_{A,v}$ is not unimodular, i.e., the ideal $\mathfrak{b}$ generated by its entries is not R, then there exists $(x', y', z', w') \in v + [R\Phi(x, y, z, w)]^4$ such that $\Phi(x', y', z', w') = 0$ iff the annihilator of $\Phi(x, y, z, w)$ in R contains an element of $1 - \mathfrak{b}$ (i.e., iff we have $\Phi(x, y, z, w) \in \Phi(x, y, z, w)\mathfrak{b}$).

7. Proofs of Theorems 1.4 and 1.7

Assume the ring R is such that the simply extendable and determinant liftable properties on a matrix in $Um(\mathbb{M}_2(R))$ are equivalent. As unimodular matrices in $\mathbb{M}_2(R)$ of zero determinant are determinant liftable, they are simply extendable, so R is a Π_2 ring by definition. Thus the 'if' part of Theorem 1.4 holds.

Based on Theorem 1.3(1), to prove the 'only if' part of Theorem 1.4 it suffices to show that if R is a Π_2 ring and if for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$ there exists

$B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in Um(\mathbb{M}_2(R))$ congruent to A modulo $R\det(A)$ and $\det(B) = 0$, then A is simply extendable. As R is a Π_2 ring, B is simply extendable. From this and [4], Thm. 4.3, (3) applied to B, it follows that there exists $(e, f) \in Um(R^2)$ such that $(a_1e+c_1f, b_1e+d_1f) \in Um(R^2)$ and so $(a_1e+c_1f, b_1e+d_1f, ad-bc) \in Um(R^3)$. As $B - A \in \mathbb{M}_2(R\det(A))$, it follows that $(ae + cf, be + df, ad - bc) \in Um(R^3)$. Thus A is simply extendable by [4], Cor. 4.7(2). So Theorem 1.4 holds.

The 'only if' of Theorem 1.7 holds by Theorem 1.3(2).

To prove the 'if' part of Theorem 1.7, we assume that A is weakly determinant liftable. Let $B \in \mathbb{M}_2(R)$ be congruent to A modulo $R\det(A)$ and $\det(B) = 0$. As B is non-full by hypothesis, A modulo $R\det(A)$ is non-full and thus A is extendable by [4], Prop. 5.1(2). So Theorem 1.7 holds.

8. On $WJ_{2,1}$ and $J_{2,1}$ rings

We first prove Theorem 1.11. Let R be a $WJ_{2,1}$ ring. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$. By taking $(\Psi, \Delta) = (1, 0)$ in Definition 1.10(1), it follows that there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and $xw - yz = 0$, and hence A is determinant liftable by Theorem 1.2. So part (1) holds.

We assume now that R is also a Hermite ring. The 'only if' of part (2) follows from the fact that each elementary divisor ring is an SE_2 ring (see [4], Prop. 1.3) and hence a Π_2 ring. For the 'if' of part (2), if R is also a Π_2 ring, then from part (1) and Corollary 1.6 we get that R is an SE_2 ring and thus an elementary divisor ring by [4], Cor. 1.8. So part (2) holds.

Each Hermite domain is a Bézout domain and hence a pre-Schreier domain (see [4], Sect. 2) and a Π_2 domain (see [4], paragraph after Thm. 1.4). Based on this, part (3) follows from part (2). Thus Theorem 1.11 holds.

Proposition 8.1. *A ring R is a $J_{2,1}$ ring in the sense of Definition 1.10(2) iff it is a $J_{2,1}$ ring in the sense of* [12]*, Def. 4.6.*

Proof. Assume R is a $J_{2,1}$ ring in the sense of Definition 1.10(2). Let $(\alpha, \beta, \gamma, \delta) \in R^4$. As R is a Hermite ring, there exist $e \in R$ and $(a, b, c, d) \in Um(R^4)$ such that $(\alpha, \beta, \gamma, \delta) = e(a, b, c, d)$. For $\psi \in R$, the equation $\alpha X + \beta Y + \gamma Z + \delta W = \psi$ has a solution $(x, y, z, w) \in R^4$ iff $\Psi \in Re$. Let $(\Psi, \Delta) \in Re \times R$. Let $f \in R$ be such that $\Psi = ef$. From Definition 1.10 applied to $(\Psi, \Delta) = (f, 0)$ it follows that there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = f$ and $xw - yz = \Delta$. So $\alpha x + \beta y + \gamma z + \delta w = ef = \Psi$ and $xw - yz = \Delta$. Thus the 'only if' part holds.

Assume R is a $J_{2,1}$ ring in the sense of [12], Def. 4.6. Clearly, R is a $WJ_{2,1}$ ring. As R is a Hermite ring by [12], Prop. 4.11, the 'if' part holds. □

9. Rings with universal (weakly) determinant liftability

Let $GL_2(R)$ be the group of units of $\mathbb{M}_2(R)$. For a matrix $E \in \mathbb{M}_2(R)$, let $[E] \in GL_2(R)\backslash\mathbb{M}_2(R)/GL_2(R)$ be its double coset. For a projective R-module M of rank 1, let $[M] \in \mathrm{Pic}\,(R)$ be its class.

Proposition 9.1. *We consider the following statements on R.*

(1) *For each $a \in R$, the map of sets*

$$\{B \in Um(\mathbb{M}_2(R)) \,|\, \det(B) = 0\} \to \{\bar{B} \in Um(\mathbb{M}_2(R/Ra)) \,|\, \det(\bar{B}) = 0\},$$

defined by the reduction modulo Ra, is surjective.

(2) *For each $a \in R$, the map of sets of double coset*

$$\{[B] | B \in Um\big(\mathbb{M}_2(R)\big),\ \det(B) = 0\} \to \{[\bar{B}] | \bar{B} \in Um\big(\mathbb{M}_2(R/Ra)\big),\ \det(\bar{B}) = 0\},$$

defined by the reduction modulo Ra, is surjective.

(3) *For each $a \in R$, every projective R/Ra-module of rank 1 generated by 2 elements is isomorphic to the reduction modulo Ra of a projective R-module of rank 1 generated by 2 elements.*

(4) *Each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) and (1) $\Rightarrow$ (4). If $sr(R) \leq 4$, then (1) $\Leftrightarrow$ (4).

Proof. For a pair $\mu := (M, M')$ of projective R-submodules of R^2 of rank 1 and generated by 2 elements such that we have a direct sum decomposition $R^2 = M \oplus M'$, let $E_\mu \in Um\big(\mathbb{M}_2(R)\big)$ be the projection on M along M'; so $\det(E_\mu) = 0$, M and M' are dual to each other (i.e., $[M'] = -[M]$, with $\mathrm{Pic}\,(R)$ viewed additively), and $\mu_{[M]} := [E_\mu]$ depends only on $[M]$. Each projective R-module of rank 1 generated by 2 elements is isomorphic to such an M. For $G \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(G) = 0$, Ker_G and Im_G are projective R-module of rank 1 generated by 2 elements and the short exact $0 \to \mathrm{Ker}_G \to R^2 \to \mathrm{Im}_G \to 0$ has a section $\varphi : \mathrm{Im}_G \to R^2$ (see [4], Lem. 3.1); if $\tau_G := \big(\mathrm{Im}(\varphi), \mathrm{Ker}_G\big)$, then $[G] = [E_{\tau_G}] = \mu_{[\mathrm{Im}_G]}$. Thus

$$\{[B] | B \in Um\big(\mathbb{M}_2(R)\big),\ \det(B) = 0\} = \{\mu_{[M]} | M \oplus M' = R^2,\ M \text{ has rank } 1\}.$$

From this and its analog over R/Ra, it follows that (2) $\Leftrightarrow$ (3).

Clearly, (1) $\Rightarrow$ (2).

For (1) $\Rightarrow$ (4), let $A \in Um\big(\mathbb{M}_2(R)\big)$. By applying (1) to $a = \det(A)$ and the reduction $\bar{B}$ of A modulo Ra, it follows that there exists $B \in Um\big(\mathbb{M}_2(R)\big)$ congruent to A modulo $R\det(A)$ and $\det(B) = 0$, so A is determinant liftable. So (1) $\Rightarrow$ (4).

Assume $sr(R) \leq 4$. To prove (4) $\Rightarrow$ (1), let $a \in R$. Let $\bar{B} \in Um\big(\mathbb{M}_2(R/Ra)\big)$ with $\det(\bar{B}) = 0$. Let $C \in Um\big(\mathbb{M}_2(R)\big)$ be such that its reduction modulo Ra is $\bar{B}$ by [4], Prop. 2.4(1); we have $\det(C) \in Ra$. As C is determinant liftable, there exists $B \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(B) = 0$ and congruent to C modulo $R\det(C)$ and hence also modulo Ra; so the map of statement (1) is surjective, hence (4) $\Rightarrow$ (1). $\square$

Example 9.2. If R is an integral domain of dimension 1, then each matrix $A \in Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable. To check this we can assume that $\det(A) \neq 0$ and this case follows from [4], Thm. 1.7(1) and Theorem 1.3(1). Recall that $sr(R) \leq 2$ (cf. [4], Sect. 1). So parts (1) to (4) of Proposition 9.1 hold.

Proposition 9.3. *We consider the following two statements on R.*

(1) *For each $a \in R$, $Um\big(\mathbb{M}_2(R/Ra)\big)$ is contained in the image of the modulo Ra reduction map $\{B \in \mathbb{M}_2(R) | \det(B) = 0\} \to \{\bar{B} \in \mathbb{M}_2(R/Ra) | \det(\bar{B}) = 0\}$.*

(2) *Each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is weakly determinant liftable.*

Then (1) $\Rightarrow$ (2), and the converse holds if $sr(R) \leq 4$.

Proof. It is the same as the last two paragraphs of the proof of Proposition 9.1, with determinant and $B \in Um\big(\mathbb{M}_2(R)\big)$ replaced by weakly determinant and $B \in \mathbb{M}_2(R)$ (respectively). $\square$

10. A criterion for determinant liftability via completions

The following proposition is probably well-known.

Proposition 10.1. *Let $A \in Um(\mathbb{M}_2(R))$, let $t \in R$ be such that* $\det(A) \in Rt$ *and let $\hat{R}$ be the t-adic completion of R. Then there exists $B \in Um(\mathbb{M}_2(\hat{R}))$ whose reduction modulo* $\mathrm{Ker}(\hat{R} \to R/Rt)$ *is the reduction of A modulo Rt and* $\det(B) = 0$.

Proof. Let $B_0 := A$. By induction on $n \in \mathbb{N}$, we show that there exists $B_n \in \mathbb{M}_2(R)$ congruent to B_{n-1} modulo $Rt^{2^{n-1}}$ and $\det(B_n) \in Rt^{2^n}$. For $n = 1$, let $s \in R$ be such that $\det(A) = st$. With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for $B_1 := A + t \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbb{M}_2(R)$, $\det(B_1)$ is congruent modulo Rt^2 to $(dx+cy+bz+aw+s)t$. As $A \in Um(\mathbb{M}_2(R))$, we can choose $(x, y, z, w) \in R^4$ such that $dx+cy+bz+aw = -s$, hence $\det(B_1) \in Rt^2$. The passage from n to $n+1$ follows from the case $n = 1$ applied to $(B_n, Rt^{2^{n+1}})$ instead of (A, Rt^2). This completes the induction. Let $B \in \mathbb{M}_2(\hat{R})$ be the limit of the sequence $(B_n)_{n\geq 1}$. Clearly, $\det(B) = 0$. As $\mathrm{Ker}(\hat{R} \to R/Rt)$ is contained in the Jacobson radical of $\hat{R}$, we have $B \in Um(\mathbb{M}_2(\hat{R}))$. □

Proposition 10.1 also follows from the smoothness part of Theorem 3.1(3) via a standard limit lifting argument. Proposition 10.1 gives directly the following result.

Corollary 10.2. *Let $A \in Um(\mathbb{M}_2(R))$. If R is complete in the* $\det(A)$*-adic topology, then A is determinant liftable.*

11. Applications of Picard groups

To refine Theorem 3.1(7) in Theorem 11.4, we first recall that for a commutative R-algebra S we have functorial (cocycle) isomorphisms (part of Hilbert's Theorem 90 for $\mathrm{Spec}\, R$)

(XII) $$\mathrm{Pic}\,(S) \cong H^1(\mathrm{Spec}\, S, \mathcal{O}^*_{\mathrm{Spec}\, S}) \cong H^1(\mathrm{Spec}\, S, \mathbb{G}_{m,S})$$

(e.g., see [14], Ch. III, Prop. 4.9), where $\mathcal{O}_{\mathrm{Spec}\, S}$ is the structure ringed sheaf on $\mathrm{Spec}\, S$, $\mathcal{O}^*_{\mathrm{Spec}\, S}$ is its set subsheaf of units and $H^1(\mathrm{Spec}\, S, \mathbb{G}_{m,S})$ is the group of equivalence classes of $\mathrm{Spec}\, S$-torsors under $\mathbb{G}_{m,S}$.

The following result which was first proved in [4], (paragraph after) Thm. 1.4 is also a consequence of Theorem 3.1(8).

Corollary 11.1. *If* $\mathrm{Pic}\,(R)$ *is trivial then R is a Π_2 ring.*

Proof. We have to show each $A \in Um(\mathbb{M}_2(R))$ with $\det(A) = 0$ is simply extendable. Considering an R-algebra homomorphism $\varrho : \mathcal{D} \to R$ by Theorem 3.1(8), it defines a morphism $\mathrm{Spec}\, R \to \mathrm{Spec}\, \mathcal{D}$ of schemes and we pullback via it the $\mathrm{Spec}\, \mathcal{D}$-torsor under $\mathbb{G}_{m,\mathcal{D}}$ of Theorem 3.1(7). The resulting $\mathrm{Spec}\, R$-torsor under $\mathbb{G}_{m,R}$ is trivial as so is $\mathrm{Pic}\,(R)$ by Isomorphisms (XII) and it is defined by a suitable action of $\mathbb{G}_{m,R}$ on $\mathrm{Spec}\, R \otimes_{\varrho,\mathcal{D},\rho} \mathcal{X} \to \mathrm{Spec}\, R$. Thus there exists an R-algebra homomorphism $R \otimes_{\varrho,\mathcal{D},\rho} \mathcal{X} \to R$ and therefore there exists also an R-algebra homomorphism $\mathcal{X} \to R$. From this and Lemma 2.2 we get that A is simply extendable. □

For a commutative R-algebra S defined by a homomorphism $\mathrm{I}math : R \to S$, let $\mathrm{Pic}\,(\mathrm{I}math) : \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(S)$ be the functorial homomorphism: for a projective R-module M of rank 1 we have $\mathrm{Pic}\,(\mathrm{I}math)([M]) := [S \otimes_R M]$. If $\jmath : S \to R$ is an R-algebra homomorphism, then as $\jmath \circ \mathrm{I}math$ is the identity automorphism of R, Imath and Pic (Imath) are injective and $\jmath$ and Pic ($\jmath$) are surjective. In particular, if S is a symmetric R-algebra then Pic (Imath) is injective.

For $A \in Um\big(\mathbb{M}_2(R)\big)$ and the R-algebra $\mathcal{D} = \mathcal{D}_A$ of Section 2, let

$$\iota_A : \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(\mathcal{D})$$

be the functorial homomorphism. If $A \in Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable, then the R-algebra $\mathcal{D}$ has a retraction by Theorem 1.2 and Lemma 2.1, Equation (VI) and ι_A is injective. The $\mathrm{Spec}\,\mathcal{D}$-torsor under $\mathbb{G}_{m,\mathcal{D}}$ of Theorem 3.1(7) corresponds, under the isomorphism $\mathrm{Pic}\,(\mathcal{D}) \cong H^1(\mathrm{Spec}\,\mathcal{D}, \mathbb{G}_{m,\mathcal{D}})$, to a class $[\mathcal{P}] \in \mathrm{Pic}\,(\mathcal{D})$, where $\mathcal{P} = \mathcal{P}_A$ is a projective $\mathcal{D}$-module of rank 1; one would like to describe it and determine when it belongs to $\mathrm{Im}(\iota_A)$.

We include a proof of the following well-known result[3] as below it is essential.

Lemma 11.2. *For an ideal $\mathfrak{i}$ of R let $\pi : R \to R/\mathfrak{i} =: S$ be the quotient homomorphism. The following properties hold for $\iota := \mathrm{Pic}\,(\pi) : \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(S)$.*

(1) *If $\mathfrak{i}$ is contained in the Jacobson radical of R, then ι is injective.*

(2) *If $\mathfrak{i} \subset N(R)$, then ι is an isomorphism.*

Proof. For part (1), let M be a projective R-module of rank 1 such that there exists an S-linear isomorphism $\bar{\ell} : S \to M/\mathfrak{i}M$. Let $m \in M$ be such that $\bar{\ell}(1+\mathfrak{i}) = m+\mathfrak{i}M$. Let $\ell : R \to M$ be the R-linear map such that $\ell(1) = m$. Then $M = \mathfrak{i}M + \ell(R)$. Thus $M = \ell(R)$ by Nakayama's Lemma (see [3], Sect. 9, Subsect. 3, Thm. 2). So ℓ is surjective. As M is a projective R-module, ℓ has an R-linear section, thus $R \cong M \oplus \mathrm{Ker}(\ell)$. From this and the fact that M has rank 1 it follows that $\mathrm{Ker}(\ell)$ is the zero R-module, so $\ell : R \to M$ is an isomorphism. Thus ι is injective.

Based on part (1), to prove part (2) it suffices to show that ι is surjective. We endow the set $\mathcal{P}_{\mathrm{f}}(\mathfrak{i})$ of finite subsets Γ of $\mathfrak{i}$ with the inclusion relation. For $\Gamma \in \mathcal{P}_{\mathrm{f}}(\mathfrak{i})$, let $\mathfrak{i}_\Gamma$ be the ideal of R generated by Γ. We get a direct system of R-algebras indexed by $\Gamma \in \mathcal{P}_{\mathrm{f}}(\mathfrak{i})$ whose transition R-algebra homomorphisms for inclusions $\Gamma \subset \nabla$ with $\nabla \in \mathcal{P}_{\mathrm{f}}(\mathfrak{i})$ are the natural surjections $R/\mathfrak{i}_\Gamma \to R/\mathfrak{i}_\nabla$. We have an R-algebra isomorphism $\varinjlim\{R/\mathfrak{i}_\gamma\} \to S$ and hence an isomorphism $\varinjlim\{\mathrm{Pic}\,(R/\mathfrak{i}_\gamma)\} \to \mathrm{Pic}\,(S)$ by [8], Thm. 1.3. Thus, by replacing $\mathfrak{i}$ with $\mathfrak{i}_\Gamma$s, we can assume that $\mathfrak{i}$ is finitely generated. So $\mathfrak{i}$ is nilpotent. For a projective S-module $\bar{M}$ of rank 1, let M be a projective R-module such that we have a surjective R-linear map $M \to \bar{M}$ whose kernel is $\mathfrak{i}M$ by [3], Sect. 9, Subsect. 5, Prop. 11. Thus we have an S-linear isomorphism $M/\mathfrak{i}M \cong \bar{M}$. As $\bar{M}$ is finitely generated and $\mathfrak{i}$ is nilpotent, a similar argument based on Nakayama's Lemma gives that M is finitely generated. As $\bar{M}$ has rank 1 and $\mathrm{Spec}\,S \to \mathrm{Spec}\,R$ is a homeomorphism, M has also rank 1. As $\iota([M]) = \bar{M}$, ι is surjective and part (2) holds. □

For seminormal rings we refer to [16] and [8]. If $N(R) = \{0\}$ and we view R as a subring of its total quotient ring $Q(R) := [R \setminus Z(R)]^{-1}R$, then (see [8], Thm. 1.1) R is seminormal iff it contains each $f \in Q(R)$ which is a root of a monic polynomial in $R[X]$ and for which there exists $n \in \mathbb{N}$ such that $f^r \in R$ for all integers $r \geq n$. A normal domain, i.e., an integral domain integrally closed in its field of fractions, is a seminormal domain. The next theorem only puts together several known results.

[3]For instance, Lemma 11.2(2) is stated in [8], Sect. 0. Also, if $\mathfrak{i}$ is a nilradical ideal of R, then the pair $(R, \mathfrak{i})$ is a henselian pair (e.g., see `https://stacks.math.columbia.edu/tag/09XD`) and hence Lemma 11.2(2) is a particular case of [7], Cor. 5.4.42 applied to $t = 1$.

Theorem 11.3. *Let S be a polynomial R-algebra in $n \in \mathbb{N}$ variables. The following properties hold for the injective functorial homomorphism $\iota : \mathrm{Pic}\,(R) \to \mathrm{Pic}\,(S)$.*

(1) *If $R/N(R)$ is not seminormal, then ι is not surjective.*

(2) *Assume that for each $\mathfrak{m} \in \mathrm{Spec}\, R$, the local ring $R_{\mathfrak{m}}/N(R_{\mathfrak{m}}) \cong \big(R/N(R)\big)_{\mathfrak{m}}$ is a seminormal domain. Then ι is an isomorphism*

(3) *If R is a Hermite ring, then ι is an isomorphism.*

Proof. Based on Lemma 11.2, we can assume that $N(R) = \{0\}$.

See [8], Thm. 1.5 for part (1).

To prove parts (2) and (3), by replacing R with $R_{\mathfrak{m}}$ with $\mathfrak{m} \in \mathrm{Max}\,(R)$, we can assume that R is a local ring by [15], Thm. 1.1. Thus part (2) holds by [8], Thm. 1.6. For part (3), the local Hermite ring R is a valuation ring by [9], Thms. 1 and 2 and thus a valuation domain (this is stated in [10], Sect. 10).[4] Thus R is a local normal domain and hence part (3) holds by part (2). □

Theorem 11.4. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$. If $\det(A) \in N(R)$, then the following properties hold.*

(1) *For $f \in \{a, b, c, d\}$, the morphism $\mathrm{Spec}\,\mathcal{X}_f \to \mathrm{Spec}\,\mathcal{D}_f$ with the action of $\mathbb{G}_{m,\mathcal{D}_f}$ on it induced by the action of $\mathbb{G}_{m,\mathcal{D}}$ on $\mathrm{Spec}\,\rho : \mathrm{Spec}\,\mathcal{X} \to \mathrm{Spec}\,\mathcal{D}$ of Theorem 3.1(6) is a trivial $\mathrm{Spec}\,\mathcal{D}_f$-torsor under $\mathbb{G}_{m,\mathcal{D}_f}$.*

(2) *The $\mathrm{Spec}\,\mathcal{D}$-torsor under $\mathbb{G}_{m,\mathcal{D}}$ of Theorem 3.1(7) is the pullback of a $\mathrm{Spec}\,R$-torsor under $\mathbb{G}_{m,R}$, i.e., we have $[\mathcal{P}] \in \mathrm{Im}(\iota_A)$.*

Proof. Based on Lemma 11.2(2) and Isomorphisms (XII), to prove parts (1) and (2) we can assume that $N(R) = \{0\}$; so $\det(A) = 0$. For part (1) we only consider the case $f = a$ as the other three cases are similar. We have three R_a-algebra isomorphisms $\mathcal{D}_a \cong R_a[Y_1, Z_1]$ where $(Y_1, Z_1) := (aY + cW, Z + a^{-1}bW)$ (see proof of Theorem 3.1(8)) and (as $d = a^{-1}bc$ in R_a)

$$\mathcal{X}_a \cong R_a[X, Y, Z, W]/\big(1 - (aX + cY)(W + a^{-1}bZ)\big) \cong R_a[X_2, Y, Z, W_2]/(1 - X_2W_2)$$

for $(X_2, W_2) := (aX + cY, W + a^{-1}bZ)$. The action of $\mathbb{G}_{m,\mathcal{D}_a}$ on $\mathrm{Spec}\,\mathcal{X}_a$ defined by the $\mathcal{D}_a$-algebra homomorphism $\mathcal{X}_a \to \mathcal{D}_a[T, T^{-1}] \otimes_{\mathcal{D}_a, \rho_a} \mathcal{X}_a \cong \mathcal{X}_a[T, T^{-1}]$ that maps $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{W}$ to $T\bar{X}$, $T\bar{Y}$, $T^{-1}\bar{Z}$, $T^{-1}\bar{W}$ (see Section 2), maps $\bar{X}_2$ to $T\bar{X}_2$ and $\bar{W}_2$ to $T^{-1}\bar{W}_2$, i.e., the substitution (X_2, W_2) is compatible with the action. As $\rho_a(\bar{Y}_1) = \bar{Z}(a\bar{X} + c\bar{Y})$ and $\rho_a(\bar{Z}_1) = \bar{Y}(\bar{W} + a^{-1}b\bar{Z})$, under the three R_a-algebra isomorphisms, $\rho_a : \mathcal{D}_a \to \mathcal{X}_a$ gets identified to the R_a-algebra homomorphism $\varrho_a : R_a[Y_1, Z_1] \to R_a[X_2, Y, Z, W_2]/(1 - X_2W_2)$ defined by $\varrho_a(Y_1) := \bar{Z}\bar{X}_2$ and $\varrho_a(Z_1) := \bar{Y}\bar{W}_2$. Clearly, $R_a[X_2, Y, Z, W_2]/(1 - X_2W_2) \cong R_a[X_2, ZX_2, YW_2, W_2]/(1 - X_2W_2)$. From either this and the compatibility of (X_2, W_2) with the action or simply the existence of the $R_a[Y_1, Z_1]$-algebra homomorphism (retraction of ϱ_a) $R_a[X_2, Y, Z, W_2]/(1 - X_2W_2) \to R_a[Y_1, Z_1]$ that maps $\bar{X}_2, \bar{Y}, \bar{Z}$ and $\bar{W}_2$ to $1, Z_1, Y_1$ and 1 (respectively), it follows that part (1) holds.

[4] Recall from [10], Sect. 10, Def. that a ring R is called a valuation ring if for each $(a, b) \in R^2$, either $Ra \subset Rb$ or $Rb \subset Ra$, equivalently, if the ideals of R are totally ordered by set inclusion. Valuation rings S with $N(S) = \{0\}$ are integral domains. This is so as the assumption that there exists $(a, b) \in (S \setminus \{0\})^2$ such that $ab = 0$ implies first that the nilpotent ideal $Sa \cap Sb$ is $\{0\}$ and second that the finitely generated ideal $Sa + Sb \cong Sa \oplus Sb$ is not principal, a contraction to S being a valuation ring.

For each $(f,g) \in \{a,b,c,d\}^2$, the functorial homomorphism $U(R_{fg}) \to U(\mathcal{D}_{fg})$ is an isomorphism as $N(R_{fg}) = \{0\}$ and $\mathcal{D}_{fg}$ is a polynomial R_{fg}-algebra by Theorem 3.1(8). For a projective $\mathcal{D}$-module M of rank 1 equipped with a $\mathcal{D}_f$-linear isomorphism $\lambda_f : M_f \cong \mathcal{D}_f$ for each $f \in \{a,b,c,d\}$, for every $(f,g) \in \{a,b,c,d\}^2$, the $\mathcal{D}_{fg}$-linear isomorphism $(\lambda_f)_g \circ (\lambda_g)_g^{-1} : \mathcal{D}_{fg} \to \mathcal{D}_{fg}$ is the multiplication by a unit $u_{f,g} \in U(R_{fg})$. Thus the cocyle $(u_{f,g})_{(f,g)\in\{a,b,c,d\}^2}$ that defines the class $[M] \in \mathrm{Pic}\,(\mathcal{D})$ (see Isomorphisms (XII)), defines also a class in $\mathrm{Pic}\,(R)$ whose image under the functorial homomorphism $\mathrm{Pic}\,(R) \to \mathrm{Pic}\,(\mathcal{D})$ is $[M]$. As the $\mathrm{Spec}\,\mathcal{D}$-torsor $\mathrm{Spec}\,\mathcal{X} \to \mathcal{D}$ under $\mathbb{G}_{m,\mathcal{D}}$ is defined via Isomorphisms (XII) applied to $\mathcal{D}$ by such a class $[M] \in \mathrm{Pic}\,(\mathcal{D})$ by part (1), part (2) holds. □

Remark 11.5. **(1)** If $\det(A) \notin N(R)$, then one can check based on the proof of Theorem 3.1(8) that $\mathcal{D}$ is not a symmetric R-algebra.

(2) Assume $\det(A) = 0$. Let $\mathcal{P}_R = \mathcal{P}_{R,A}$ be a projective R-module of rank 1 such that $[\mathcal{P}] = \iota_A([\mathcal{P}_R])$ by Theorem 11.4(2). One would like to describe all relations between the projective R-modules P and Q of Theorem 3.1(8) and the projective R-modules Im_A and $\mathcal{P}_R$ of rank 1; recall from [4], Lem. 3.1 that the image Im_A and the kernel Ker_A of the R-linear map $L_A : R^2 \to R^2$ are projective R-modules of rank 1 dual to each other.

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