Research Article

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1-Sylvester matrices

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Abstract: A nonzero element a is called *1-Sylvester* in a ring R, if there exist $b, c \in R$ such that $1 = ab + ca$. In this article, we study such elements, mainly in matrix rings over commutative rings. In particular, we study the case when $b = c$, when b is called an *anticommutator inverse* for a.

Keywords: Sylvester equation, idempotent, unit, nilpotent, matrix, 1-Sylvester element, anticommutator inverse

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1 Introduction

The Sylvester equation is a classic matrix equation of the form $AX + XB = C$, where A, B, and C are known matrices, and X is the unknown. Named after James Joseph Sylvester, this equation became central in areas like control theory, linear differential equations, and numerical linear algebra.

This article focuses on a special case of the Sylvester equation when $C = I_n$ and particularly when $B = C$, in the context of matrices over commutative rings. We introduce the notion of 1-Sylvester elements in a ring R with identity: a nonzero $a \in R$ is *1-Sylvester* if there exist $b, c \in R$ such that $1 = ab + ca$.

Although (the so-called) weakly fadelian rings (those where every nonzero element is 1-Sylvester, see [4]) are simple domains, matrix rings are not domains. Thus, studying 1-Sylvester elements in matrix rings requires different approaches. A special case involves anticommutators: for $a, b \in R$, $[a, b]_+ = ab + ba$. If $[a, b]_+ = 1$, then b will be called an *anticommutator inverse* (ACI) of a.

Our study of 1-Sylvester elements and ACIs is motivated in part by their relevance in quantum mechanics, where anticommutation relations govern fermionic operators. Not all elements have an ACI; some have none, others have one or many.

Finally, we note that while weakly fadelian rings are domains, individual 1-Sylvester elements may be zero divisors or nilpotent. For example, $E_{12} \in \mathbb{M}_2(R)$ satisfies $E_{12}E_{21} + E_{21}E_{12} = I_2$ but is nilpotent. Here, E_{ij} denotes the $n \times n$ matrix with all entries zero, excepting the (i, j)-entry, which is 1.

Throughout this article, all rings are assumed to be associative, unital, and nonzero (i.e., $1 \neq 0$). For matrix rings, unless otherwise stated, we assume that the base ring is commutative. For a ring $R, U(R)$ denotes the set of all units of R. Two elements a, b in a commutative ring R are called *coprime* if there exist $c, d \in R$ such that $ca + db = 1$. As is customary, $|$ denotes the binary relation of divisibility.

This article is structured as follows: Section 2 begins by discussing general properties of 1-Sylvester elements in arbitrary rings including the fact that in any ring, the only 1-Sylvester idempotent is the identity. We then prove that diagonal 1-Sylvester $n \times n$ matrices are precisely those that are invertible. In addition, we show that numerous nilpotent 1-Sylvester $n \times n$ matrices also exist.

Section 3 focuses on characterizing upper triangular 1-Sylvester 2×2 matrices over commutative domains, including special cases such as ACI matrices and integral matrices. We also explore the potential

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uniqueness of the ACI for 2×2 matrices over commutative rings. Simplifications are provided using the Kronecker product of matrices together with matrix vectorization.

2 1-Sylvester $n \times n$ matrices

We begin by collecting some straightforward yet useful properties of 1-Sylvester elements in any (unital) ring. Two elements a, b in a ring R are called *equivalent* if there exist units p, q in R such that $b = paq$.

Proposition 2.1.

(*i*) *The one-sided invertible elements are 1-Sylvester in any ring. However, units may not have ACIs.*

(*ii*) *In a ring R, if 2 is a unit, then all units have ACI, while if 2 is not a unit, no central units have ACI. However, a noncentral unit can have an ACI.*

(*iii*) *The central 1-Sylvester elements of a ring R are precisely the units of R. In particular, the only 1-Sylvester elements of a commutative ring are the units.*

(*iv*) *The only 1-Sylvester idempotent is 1.*

(*v*) *Zero-square 1-Sylvester elements are (von Neumann) regular.*

(*vi*) *The 1-Sylvester property is preserved by (anti-)isomorphisms of rings. In particular, the 1-Sylvester property is invariant under conjugation.*

(*vii*) *If a is 1-Sylvester then ua is also 1-Sylvester, for every central unit u. In particular, $-a$ is also 1-Sylvester.*

(*viii*) *The 1-Sylvester property is not invariant under equivalences.*

Proof.

(i) Indeed, $1 = uv + 0 \cdot u$ takes care of right invertible elements and $1 = u \cdot 0 + wu$, of the left invertible ones. An example of 2×2 invertible matrix that has no ACI is given in Remark 1, after Proposition 3.4.

(ii) If 2 is invertible, then $(2u)^{-1}$ is an ACI for the unit u. In $M_2(R)$, the noncentral unit $E_{12} + E_{21}$ has the ACI E_{12}.

(iv) Suppose $e^2 = e$ and $eb + ce = 1$ for some b, c. We multiply that equation on the left by $1 - e$ and then by $1 - e$ on the right. It follows that $1 - e = 0$, so $e = 1$.

(v) If $tb + ct = 1$ for $t^2 = 0$, multiplying the equation by t gives $tbt = tct = t$.

(vii) If $ab + ca = 1$, then for any central unit u, $(au)(u^{-1}c) + (cu^{-1})(ua) = 1$.

(viii) In $\mathbb{M}_2(R)$ over any ring R, the nilpotent E_{12} is 1-Sylvester (see Section 1), but the idempotent $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} E_{12} = E_{22}$ is not 1-Sylvester (see Proposition 2.3). □

Remarks

(1) In many rings (e.g., integral quaternions), the only 1-Sylvester elements are the units. For example, this is the case whenever the two-sided ideal generated by every nonunit is proper. This holds in rings that are not simple or in domains. Otherwise, in simple rings (like matrix rings, the focus of our article), even non-units can generate the whole ring as a two-sided ideal. Actually, if a is 1-Sylvester in a ring R, then $RaR = R$ (as $r = r \cdot 1 = r(ab + ca) = (ra)b + (rc)a$), but the converse fails (e.g., E_{11} in $\mathbb{M}_2(\mathbb{Z})$).

(2) By a result of P. Ara [1], it follows that the zero-square 1-Sylvester elements of exchange rings are even unit-regular.

According to our definition, an $n \times n$ matrix A over a ring R is 1-Sylvester if there are $B, C \in \mathbb{M}_n(R)$ such that $I_n = AB + CA$.

As witnessed by Proposition 2.1 (i), all one-sided invertible matrices are 1-Sylvester. Also from the previous proposition (vi) and (vii), recall that the 1-Sylvester property for matrices is *invariant under similarity* (and *under negatives*). Moreover, 1-Sylvester (square) matrices are *invariant under transpose*. In particular, if B is an ACI of A, then B^T is an ACI of A^T.

Next, we describe the diagonal 1-Sylvester matrices.

Proposition 2.2. *Over any commutative ring, the diagonal 1-Sylvester matrices are precisely the invertible diagonal matrices.*

Proof. Let $D = \mathrm{diag}(d_1, \dots, d_n)$ be a diagonal 1-Sylvester matrix. Suppose $DB + CD = I_n$ for some $n \times n$ matrices B, C. We just emphasize the diagonal of $DB + CD$: it is $\mathrm{diag}(d_1(b_{11} + c_{11}), \dots, d_n(b_{nn} + c_{nn}))$. Hence, all d_i $(1 \le i \le n)$ are units and so is D. The converse follows from the previous proposition. □

To provide an example of a diagonal 1-Sylvester matrix over a noncommutative ring that is not invertible, it suffices to choose all the diagonal entries to be one-sided (but not two-sided) invertible elements in any ring that is not Dedekind finite (i.e., there exist elements such that $ab = 1$ but $ba \neq 1$). Such examples exist also over Dedekind finite rings. As mentioned in Section 1, if $R = \mathbb{M}_2(k)$ for a field k, then E_{12} is 1-Sylvester but not unit. Hence, any diagonal matrix (over R) having E_{12} entries on the diagonal, is 1-Sylvester but not invertible.

We proceed with a general result that yields several important, albeit mostly negative, consequences.

Proposition 2.3. *Let $1 \le i \le n$ and let R be an arbitrary (not necessarily commutative) ring. Any matrix $A \in \mathbb{M}_n(R)$ with only zeros on its ith row and ith column is not 1-Sylvester.*

Proof. If A has the i-th row zero, so is AB for every $n \times n$ matrix B. Moreover, if A has the ith column zero, so is CA for every $n \times n$ matrix C. Hence, for every B, C, the sum $AB + CA$ has the (diagonal) (i, i) entry equal to zero, so the sum is $\neq I_n$, whence A is not 1-Sylvester. □

Corollary 2.4. *The diagonal $n \times n$ matrices with at least one zero diagonal entry are not 1-Sylvester.*

Furthermore, we establish several results concerning nonzero nilpotent matrices that are 1-Sylvester.

Lemma 2.5. *In $\mathbb{M}_2(R)$ over any ring R, nilpotents E_{12} and E_{21} are mutually ACIs. As such, these are 1-Sylvester.*

Proof. Just note that $E_{12}E_{21} + E_{21}E_{12} = I_2$. □

Next, we provide an example of a matrix that is 1-Sylvester but not an ACI.

Lemma 2.6. *Over any ring, the nilpotent $T_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has no ACI. However, it is 1-Sylvester.*

Proof. Suppose T_3 is ACI and let $A = [a_{ij}]$ for $1 \le i, j \le 3$. Then

$$AT_3 + T_3A = \begin{bmatrix} a_{21} + a_{31} & a_{11} + a_{22} + a_{32} & a_{11} + a_{12} + a_{23} + a_{33} \\ a_{31} & a_{21} + a_{32} & a_{21} + a_{22} + a_{33} \\ 0 & a_{31} & a_{31} + a_{32} \end{bmatrix}.$$

This sum is I_3 only if $a_{31} = 0$. This successively requires $a_{21} = 1$, and then $a_{32} = 0$. Hence, the entry (3, 3) is zero, a contradiction.

However, we can find A, B such that $AT_3 + T_3B = I_3$. For example, for $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = E_{21}$, the sum $AT_3 + T_3B = (E_{22} + E_{33}) + E_{11} = I_3$. □

Remark. The pair (A, B) given as example in the previous proof is far from being unique. One can replace the third column of A and the first row of B by arbitrary entries, and the result of this computation remains unchanged.

In the $n \times n$ case, we can generalize the nonzero nilpotent 1-Sylvester matrix described in the previous lemma.

Theorem 2.7. *Let n be a positive integer and let T_n be the strictly upper triangular $n \times n$ matrix, which has all entries above the diagonal equal to 1. Then T_n is 1-Sylvester.*

Proof. Take $A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ (excepting two zeros, the diagonal entries are equal to −1, and the subdiagonal entries are equal to 1) and $B = E_{21}$. Then $AT_n + T_nB = (E_{22} + E_{33} + \ldots + E_{nn}) + E_{11} = I_n$.

The details of the computation follow. We actually have

$$\begin{aligned} A &= \sum_{i=2}^{n-1} (E_{i,i-1} - E_{ii}) + E_{n,n-1} \\ &= E_{21} - E_{22} + E_{32} - E_{33} + \ldots + E_{n-1,n-2} - E_{n-1,n-1} + E_{n,n-1} \end{aligned}$$

and

$$T_n = \sum_{i,j=1,i<j}^{n} E_{ij} = (E_{12} + E_{13} + \ldots + E_{1n}) + (E_{23} + E_{24} + \ldots + E_{2n}) + \ldots + E_{n-1,n}.$$

For AT_n, the product starts with

$$E_{21}(E_{12} + E_{13} + \ldots + E_{1n}) - E_{22}(E_{23} + E_{24} + \ldots + E_{2n}) = E_{22} + E_{23} + \ldots + E_{2n} - E_{23} - E_{24} - \ldots - E_{2n} = E_{22},$$

and so on. □

Further, recall that every nilpotent matrix over a field is similar to a *block diagonal* matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes), a special case of the *Jordan canonical form* for matrices. A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e., $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as an $n \times n$ matrix. When $n = 1$, $S = 0$.

In [2] Theorem 3.3, the following result was proved.

Theorem 2.8. *The following are equivalent for a ring R.*
(1) *Every nilpotent matrix over R is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).*
(2) *R is a division ring.*

Thus, according to Proposition 2.1, (vi), it follows

Theorem 2.9. *Over any division ring, each nonzero nilpotent matrix is 1-Sylvester.*

Proof. Any block diagonal matrix with each block a shift matrix is strictly upper triangular with only superdiagonal nonzero entries, which are equal to 1, that is, $S = E_{12} + E_{23} + \ldots + E_{n-1,n}$. Then if $T = E_{21} + E_{32} + \ldots + E_{n,n-1}$ is the subdiagonal, we have $TS + SE_{21} = (E_{22} + E_{33} + \ldots + E_{nn}) + E_{11} = I_n$, as desired. □

It follows from the previous theorem that, over any ring, all shift matrices are 1-Sylvester. However, only shift matrices of even size admit an ACI.

Proposition 2.10. *Over any ring, the shift matrices are ACI iff they are of even size.*

Proof. Let $T = [t_{ij}]$ be an arbitrary $n \times n$ matrix, and let S be the shift matrix of size n. We focus on the diagonal entries of the sum $ST + TS$.

These are $t_{21}, t_{32} + t_{21}, \ldots, t_{n,n-1} + t_{n-1,n-2}, t_{n,n-1}$. If $ST + TS = I_n$, then all these entries equal 1. Hence, $t_{21} = 1$, $t_{32} = 0$, $t_{43} = 1$ and so on. If n is odd, then $t_{n,n-1} = 0$, a contradiction. If n is even, the alternation ends with $t_{n-1,n-2} = 0$ and $t_{n,n-1} = 1$. All the other entries of T can be chosen equal to zero and so T is an ACI for S.

More precisely, in the even case, for $S = E_{12} + E_{23} + \ldots + E_{2n-1,2n}$, the matrix $T = E_{21} + E_{43} + \ldots + E_{2n,2n-1}$ (i.e., on the subdiagonal, we alternate 1, 0, 1, 0, …) is an ACI for S. Indeed,

$$ST + TS = (E_{11} + E_{33} + \ldots + E_{2n-1,2n-1}) + (E_{22} + E_{44} + \ldots + E_{2n,2n}) = I_{2n}.$$ □

3 The 1-Sylvester 2×2 matrices

In order to describe the 1-Sylvester 2×2 matrices over commutative rings, we start with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, $C = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$.

The Sylvester equation $AB + CA = I_2$ reduces to a nonhomogeneous linear system of 4 equations and 8 unknowns:

$$\begin{cases} ax_1 + bx_3 + ay_1 + cy_2 = 1 \\ ax_2 + bx_4 + by_1 + dy_2 = 0 \\ cx_1 + dx_3 + ay_3 + cy_4 = 0 \\ cx_2 + dx_4 + by_3 + dy_4 = 1 \end{cases}$$

with the system matrix $\begin{bmatrix} a & 0 & b & 0 & a & c & 0 & 0 \\ 0 & a & 0 & b & b & d & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & a & c \\ 0 & c & 0 & d & 0 & 0 & b & d \end{bmatrix}$, augmented by the column $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

By applying the Kronecker product and vectorization, we will show at the end of this section that the Sylvester equation can be reduced to a matrix equation of the form $PX = Q$.

Next, we describe the *upper triangular* 1-Sylvester 2×2 matrices over commutative domains.

Proposition 3.1. *Let R be a commutative domain. The matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(R)$ is 1-Sylvester iff both of the following conditions hold*:
(1) *a and b are coprime.*
(2) *There exists $u \in U(R)$ such that $d = au$ and $a|1 + u^{-1}$.*

Proof. (⇒) For $c = 0$ the above system becomes $\begin{cases} a(x_1 + y_1) + bx_3 = 1 \\ ax_2 + bx_4 + by_1 + dy_2 = 0 \\ dx_3 + ay_3 = 0 \\ d(x_4 + y_4) + by_3 = 1 \end{cases}$.

From the first and fourth equations follows that a, b are coprime (and b, d are coprime).

Multiplying first equation by d and replacing the third equation shows that $a|d$. Analogously, multiplying the fourth equation by a and replacing the third equation shows $d|a$.

Therefore, in general, a and d are associates (i.e., $d = au$ for some unit u). If $a = 0$, then $d = 0$, and we can take $u = -1$. If $a \neq 0$, then $d \neq 0$. Then from the third equation, $y_3 = -ux_3$, and the forth equation becomes $a(x_4 + y_4) - bx_3 = u^{-1}$. Adding the first equation gives $a(x_1 + y_1 + x_4 + y_4) = 1 + u^{-1}$.

(⇐) From conditions 1 and 2, there exist $x_1, x_3 \in R$ and $u \in U(R)$ such that $ax_1 + bx_3 = 1$, $d = au$, and $a|1 + u^{-1}$. Take $x_2 = x_4 = y_1 = y_2 = 0$, $y_3 = -ux_3$, and $y_4 = v - x_1$, where $av = 1 + u^{-1}$. Then, one may check that these choices produces a solution to the required system of equations. Equivalently, one may compute that $\begin{bmatrix} a & b \\ 0 & au \end{bmatrix}\begin{bmatrix} x_1 & 0 \\ x_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -ux_3 & v - x_1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & au \end{bmatrix} = I_2$. □

In particular, as $U(\mathbb{Z}) = \{\pm 1\}$, we characterize the upper triangular integral 2×2 matrices that are 1-Sylvester.

Proposition 3.2. *The upper triangular 1-Sylvester integral* 2×2 *matrices are*:

(*i*) $\pm E_{12}$;

(*ii*) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ *with coprime a, b, and* $a \in \{\pm 1, \pm 2\}$;

(*iii*) $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ *with coprime* a *and* b.

Proof. Just note that in case (ii) $u = 1$ and so $a|2$, and in case (iii), $u = -1$. □

Building on Proposition 3.1, we can readily characterize all upper triangular 2×2 ACI matrices over a commutative domain.

Proposition 3.3. *Let R be a commutative domain. The matrix* $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(R)$ *is ACI iff all three of the following conditions hold*:

(1) $2a$ *and* b *are coprime.*

(2) *There exists* $u \in U(R)$ *such that* $d = au$ *and* $a|1 + u^{-1}$.

(3) *Either* $u = -1$, *or* $2a$ *is a unit.*

Proof. (⇒) Constructing the system of equations as in Proposition 3.1 proves condition 1. The second condition holds because any ACI matrix is 1-Sylvester. For condition 3, use the equation $(a + d)x_3 = 0$. Clearly, $a + d = 0$ implies that $u = -1$. When $a + d \neq 0$, we have $x_3 = 0$. Then the equation $2ax_2 + bx_3 = 1$ shows that $2a$ is a unit.

(⇐) If all three conditions hold, let $x_1, x_3 \in R$ be such that $2ax_1 + bx_3 = 1$. Consider two cases.

When $u = -1$, then $\begin{bmatrix} x_1 & 0 \\ x_3 & -x_1 \end{bmatrix}$ is an ACI for $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$.

When $2a \in U(R)$, take $x_1 = 2a^{-1}$, $x_4 = 2d^{-1}$, $x_2 = -2bx_1x_4$ and $x_3 = 0$. These choices satisfy the system of equations for x_1, x_2, x_3, and x_4.

Alternatively, one can verify that

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 2a^{-1} & -2bx_1x_4 \\ 0 & 2d^{-1} \end{bmatrix} + \begin{bmatrix} 2a^{-1} & -2bx_1x_4 \\ 0 & 2d^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = I_2.$$ □

Finally, for integral ACIs, we have the following characterization.

Proposition 3.4. *An upper triangular* 2×2 *matrix* $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ *has an ACI over* $\mathbb{Z}$ *iff* $A = \pm E_{12}$ *or else* $d = -a$ *and* $2a, b$ *are coprime.*

Remarks

(1) Since $b \neq 0$, it follows from the first equation of the linear system above that, over $\mathbb{Z}$, A has an ACI only if b is odd. If so, $\gcd(2a, b) = 1$ iff $\gcd(a, b) = 1$.

As an example, $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is 1-Sylvester by Proposition 3.2 (or directly, since it is invertible) but not ACI (actually, over any ring where 2 is not a unit). Indeed, for any matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, the sum $AB + BA = 2\begin{bmatrix} x+z & x+w \\ 0 & z-w \end{bmatrix} \neq I_2$. Here, $\gcd(1,2) = 1 \neq 2 = \gcd(2,2)$. This is also an example of *unit that has no ACI.*

(2) Since in the ACI case of the proof of the previous proposition, the entry x_2 is arbitrary, the matrices $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ (with coprime $2a$ and b) have infinitely many ACI over $\mathbb{Z}$.

Not only invertible matrices may have an ACI. We also have the following result.

Proposition 3.5. *Let a be an element of an arbitrary ring R. All upper triangular matrices $A_a = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in \mathbb{M}_2(R)$ are ACI. As such, these are 1-Sylvester.*

Proof. This follows as $\begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix}E_{21} + E_{21}\begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} = I_2$. It is easy to see that, for a commutative ring R and arbitrary $x, y \in R$, $\begin{bmatrix} x & y \\ 1-2ax & -x \end{bmatrix}$ are all the ACIs of A_a. □

Remarks

(1) Observe that $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ is a unit iff a is a unit. So all the A_a, where a is not a unit, are not invertible upper triangular matrices that have many ACIs.
(2) An ACI of the transpose $(A_a)^T$ is $E_{12} = (E_{21})^T$.

To conclude this section, we establish a result concerning the uniqueness of the ACI.

Lemma 3.6. *If $2a$ is a unit then $\frac{a^{-1}}{2}$ is an ACI for a.*

Proof. As 2 and a commute, the hypothesis is equivalent to $2, a \in U(R)$. □

In the remainder of this section, we apply two successive simplifications to establish a uniqueness result for ACI matrices. These simplifications also allow to investigate properties of 1-Sylvester 3×3 matrices as well.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the *Kronecker product* $A \otimes B$ is the $pm \times qn$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Especially in linear algebra and matrix theory, the *vectorization* of a matrix is a linear transformation, which converts the matrix into a vector. Specifically, the vectorization of a $m \times n$ matrix A, denoted vec(A), is the $mn \times 1$ column vector obtained by stacking the columns of the matrix A on top of one another:

$$\text{vec}(A) = [a_{11}, \ldots a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{mn}]^T.$$

By using the Kronecker product notation and the vectorization operator *vec*, we can rewrite Sylvester's equation (i.e., $AX + XB = C$) in the form

$$(I_m \otimes A + B^T \otimes I_n)\text{vec}X = \text{vec}C,$$

where (in general) A is of dimension $n \times n$, B is of dimension $m \times m$, X is of dimension $n \times m$, and I_k is the $k \times k$ identity matrix. In this form, the equation can be seen as a linear system of dimension $mn \times mn$.

If we take matrices of the same size and $C = I_n$, we obtain the 1-Sylvester $n \times n$ matrices studied in this paper. In particular, if we also take $A = B$, we obtain the ACI matrices. That is, the ACI equation $AB + BA = I_n$ is represented as $(I_n \otimes A + A^T \otimes I_n)\mathrm{vec}(B) = \mathrm{vec}(I_n)$.

From this representation, it follows that A has a unique ACI iff $I_n \otimes A + A^T \otimes I_n$ is invertible.

Theorem 3.7. *Let R be a commutative ring and let $A \in \mathbb{M}_2(R)$. Then, A has a unique ACI if and only if 2, $\mathrm{Tr}(A)$, and $\det(A)$ are units of R. In this case, the unique ACI is $\frac{1}{2}A^{-1}$.*

Proof. Written as 2×2 blocks, we have

$$I_2 \otimes A + A^T \otimes I_2 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} aI_2 & cI_2 \\ bI_2 & dI_2 \end{bmatrix} = \begin{bmatrix} A + aI_2 & cI_2 \\ bI_2 & A + dI_2 \end{bmatrix}.$$

To check when this matrix is invertible, we have to compute the determinant of the 4×4 matrix

$$M = \begin{bmatrix} 2a & b & c & 0 \\ c & a+d & 0 & c \\ b & 0 & a+d & b \\ 0 & b & c & 2d \end{bmatrix}.$$

Subtracting $\mathrm{col}_4(M)$ from $\mathrm{col}_1(M)$ and $\mathrm{row}_4(M)$ from $\mathrm{row}_1(M)$, simplify a lot the computation of $\det(M) = 4\mathrm{Tr}^2(A)\det(A)$.

By computation, we also obtain $\Delta_x = 2d\mathrm{Tr}^2(A)$, $\Delta_y = -2b\mathrm{Tr}^2(A)$, $\Delta_z = -2c\mathrm{Tr}^2(A)$, $\Delta_w = 2a\mathrm{Tr}^2(A)$.

Hence, if $2\det(A)$ is a unit (and $\mathrm{Tr}^2(A) \neq 0$), we obtain $x = \frac{d}{2\det(A)}$, $y = -\frac{b}{2\det(A)}$, $z = -\frac{c}{2\det(A)}$, $w = \frac{a}{2\det(A)}$. This gives $B = \frac{1}{2\det(A)}\mathrm{adj}(A)$, where the adjugate matrix is $\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Finally, $B = \frac{1}{2}A^{-1}$.

Conversely, if $4\mathrm{Tr}^2(A)\det(A)$ is a unit, then 2, $\mathrm{Tr}(A)$ and $\det(A)$ must be units. □

Remark. The existence of $\frac{1}{2}A^{-1}$ in $\mathbb{M}_2(R)$ does not imply that A has a unique ACI. For an explicit example, let R be any ring for which $2 \in U(R)$, and take $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$. Then, both $\frac{1}{2}A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$ and $B = -E_{21}$ are ACIs of A.

In closing, to determine suitable conditions characterizing ACI 3×3 matrices, one may apply Jameson's approach [3] for solving the Sylvester equation. However, the resulting conditions are rather unwieldy. A sample is given below.

Theorem 3.8. *Let R be any commutative ring and A a 3×3 matrix over R. The matrix A is ACI iff 2, $\det(A)$ and $\det(\mathrm{Tr}(A)A^2 + \det(A)I_3)$ are units in R.*

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