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Problem 1 (Vasile Pop, Cluj-Napoca). Let $P, Q, R \in \mathbb{C}[X]$ be nonzero polynomials and a, b, c distinct nonzero complex numbers.

Proof that if the set $Z = \{z_n = P(n)a^n + Q(n)b^n + R(n)c^n : n \in \mathbb{N}\}$ is finite, then there exists $p \in \mathbb{N} \setminus \{0\}$ such that $z_{n+p} = z_n$, for every $n \in \mathbb{N}$.

Proof. Since Z is finite, then

$$Z_1 = \{z_{n+1} - az_n : n \in \mathbb{N}\}$$

is also finite. Moreover, $z_{n+1} - az_n = [a(P(n+1) - P(n))] \cdot a^n + [bQ(n+1) - aQ(n)] \cdot b^n + [cR(n+1) - aR(n)] \cdot c^n = P_1(n)a^n + Q_1(n)b^n + R_1(n)c^n$, where $\deg P_1 < \deg P$, $\deg Q_1 = \deg Q$, $\deg R_1 = \deg R$. If P_1 is nonzero, we may apply the same procedure as for the set Z to the set Z_1, obtaining a set $Z_2 = \{P_2(n)a^n + Q_2(n)b^n + R_2(n)c^n : n \in \mathbb{N}\}$, with $\deg P_2 < \deg P_1$, $\deg Q_2 = \deg Q$, $\deg R_2 = \deg R$. If P_2 is nonzero, we can obtain, in a finite number of steps, a finite set $Z' = Z_k$ with $P_k = 0$, that is

$$Z' = \{Q'(n)b^n + R'(n)c^n : n \in \mathbb{N}\}$$

where $\deg Q' = \deg Q$, $\deg R' = \deg R$.

By replacing the finite set $Z' = \{z'_n : n \in \mathbb{N}\}$ with the set

$$Z'_1 = \{z'_{n+1} - bz'_n : n \in \mathbb{N}\}$$

which will be also finite, the degree of Q' decreases and we apply this procedure until Q' becomes 0 and Z' becomes

$$Z'' = \{R''(n)c^n : n \in \mathbb{N}\}$$

where $\deg R'' = \deg R$.

Now, if $\deg R'' > 1$, by replacing the finite set $Z'' = \{z''_n : n \in \mathbb{N}\}$ with the set $Z''_1 = \{z''_{n+1} - bz''_n : n \in \mathbb{N}\}$, which will be also finite, the degree of R'' decreases with 1 and this procedure is applied until R'' is replaced by a constant. We arrive, in a finite number of steps, to a finite set

$$Z''' = \{\alpha \cdot c^n : n \in \mathbb{N}\}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$.

The set $Z'''$is finite if and only if it exists p_1 such that $c^{p_1} = 1$. By analogy (or from the symmetry of the problem), we obtain similar results for a and b, i.e. there exist p_2, p_3 such that $a^{p_2} = 1$ and $b^{p_3} = 1$. From here, for $p = p_1p_2p_3$, we obtain

$$a^p = b^p = c^p.$$

The set $U = \{z_{kp} : k \in \mathbb{N}\}$ is finite, therefore $\{P(kp) + Q(kp) + R(kp) : k \in \mathbb{N}\}$ is finite which means that the polynomial $P + Q + R$ is constant

$$P(X) + Q(X) + R(X) = a_0 \tag{1}$$

Also the sets $V = \{z_{kp+1} : k \in \mathbb{N}\}$ and $W = \{z_{kp+2} : k \in \mathbb{N}\}$ are finite, which leads to the existence of the constants a_1 and a_2 such that

$$aP(X) + bQ(X) + cR(X) = a_1 \tag{2}$$

and

$$a^2P(X) + b^2Q(X) + c^2R(X) = a_2 \tag{3}$$

Since a, b, c are distinct, from (1), (2) and (3) we obtain that P, Q, R are constant polynomials, therefore $z_{n+p} = z_n$ for every $n \in \mathbb{N}$.

Remark: Using mathematical induction, it can be proved that (z_n) verifies a linear recurrence of order $N = \deg P + \deg Q + \deg R$. Then use that every recurrent sequence with finite number of values is periodical.

Problem 2 (Vasile Pop, Cluj-Napoca). Consider the sets

$$S_n = \{(x_1, x_2, \ldots, x_n) : \forall i = \overline{1,n},\ x_i \in \{0,1,2\}\}$$

$$A_n = \{(x_1, x_2, \ldots, x_n) \in S_n : \forall i = \overline{1,n-2},\ |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\}$$

and[1]

$$B_n = \{(x_1, x_2, \ldots, x_n) \in S_n : (x_i = x_{i+1} \Rightarrow x_i \neq 0)\}.$$

Prove that $|A_{n+1}| = 3 \cdot |B_n|$.

Solution 1. Consider the following sets

$$A'_n = \{(x_1, x_2, \ldots, x_n) \in A_n : x_{n-1} = x_n\}$$

$$A''_n = A_n \setminus A'_n$$

$$B'_n = \{(x_1, x_2, \ldots, x_n) \in B_n : x_n = 0\}$$

$$B''_n = B_n \setminus B'_n$$

and denote $a_n = |A_n|$, $a'_n = |A'_n|$, $a''_n = |A''_n|$, $b_n = |B_n|$, $b'_n = |B'_n|$, $b''_n = |B''_n|$.

It is easy to observe the following relations between the a–sequences

$$\begin{cases} a_n &= a'_n + a''_n \\ a'_{n+1} &= a''_n \\ a''_{n+1} &= 2a'_n + 2a''_n \end{cases},$$

which lead to $a_{n+1} = 2a_n + 2a_{n-1}$.

For the b–sequences we have the same relations

$$\begin{cases} b_n &= b'_n + b''_n \\ b'_{n+1} &= b''_n \\ b''_{n+1} &= 2b'_n + 2b''_n \end{cases},$$

therefore $b_{n+1} = 2b_n + 2b_{n-1}$.

By computing the first values of (a_n) and (b_n) we obtain

$$\begin{cases} a_1 = 3, & a_2 = 9, & a_3 = 24 \\ & b_1 = 3, & b_2 = 8 \end{cases}$$

which leads to

$$\begin{cases} a_2 = 3b_1 \\ a_3 = 3b_2 \end{cases}$$

Now, reasoning by induction, it is easy to prove that $a_{n+1} = 3b_n$ for every $n \geq 1$.

Solution 2. Regarding x_i to be elements of $\mathbb{Z}_3$ and working "modulo 3", we have that

$$(x_1, x_2, \ldots, x_n) \in A_n \Rightarrow (x_1+1, x_2+1, \ldots, x_n+1) \in A_n,\ (x_1+2, x_2+2, \ldots, x_n+2) \in A_n$$

which means that 1/3 of the elements of A_n start with 0. We establish a bijection between the subset of all the vectors in A_{n+1} which start with 0 and the set B_n by

$$(0, x_1, x_2, \ldots, x_n) \in A_{n+1} \longmapsto (y_1, y_2, \ldots, y_n) \in B_n$$
$$y_1 = x_1, y_2 = x_2 - x_1, y_3 = x_3 - x_2, \ldots, y_n = x_n - x_{n-1}$$

[1] $|A|$ denotes the number of elements of the set A.

(if $y_k = y_{k+1} = 0$ then $x_k - x_{k-1} = x_{k+1} - x_k = 0$ (where $x_0 = 0$), which gives $x_{k-1} = x_k = x_{k+1}$, which is not possible because of the definition of the sets A_p; therefore, the definition of the above function is correct).

The inverse is defined by

$$(y_1, y_2, \dots, y_n) \in B_n \longmapsto (0, x_1, x_2, \dots, x_n) \in A_{n+1}$$
$$x_1 = y_1, x_2 = y_1 + y_2, \dots, x_n = y_1 + y_2 + \dots + y_n$$

Problem 3 (Vasile Pop, Cluj-Napoca). Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $0 < a_n \leq 1$, $\forall n \geq 1$ and $\lim_{n\to\infty}(a_1 + a_2 + \dots + a_n) = \infty$.

(1) Prove that for every $l \in [1, \infty) \cup \{\infty\}$, there exists an increasing function $L : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \dots + a_{L(n+1)}}{a_1 + a_2 + \dots + a_{L(n)}} = l.$$

(2) Find a function L for the sequence $a_n = \frac{1}{\sqrt{n}}$, $n \geq 1$.

Solution. Denote $S_k = a_1 + a_2 + \dots + a_k$, $k \geq 1$. The intervals $[S_k, S_{k+1})$, $k \geq 1$ form a partition of the set $[a_1, \infty)$.

(1)

Case I. If $l > 1$ then for every $n \geq 1$ there is a unique $k \in \mathbb{N} \setminus \{0\}$ such that $l^n \in [S_k, S_{k+1})$. Define $L(n) := k$, so $l^n \in [S_{L(n)}, S_{L(n)+1})$. Since $l^{n+1} - l^n > 1 \geq a_{L(n)+1}$ we obtain that $S_{L(n+1)} \geq S_{L(n)+1}$, therefore $L(n+1) > L(n)$, for every $n \geq 1$.

We have that

$$S_{L(n)} \leq l^n < S_{L(n)+1} = S_{L(n)} + a_{L(n)+1}$$
$$S_{L(n+1)} \leq l^{n+1} < S_{L(n+1)+1}$$

which lead to

$$\frac{l^{n+1} - 1}{l^n} < \frac{S_{L(n+1)}}{S_{L(n)}} < \frac{l^{n+1}}{l^n - 1}, \quad n \geq 1$$

Passing to limit with $n \to \infty$, we obtain $\lim_{n\to\infty} \frac{S_{L(n+1)}}{S_{L(n)}} = l$.

Case II. If $l = 1$, we take $L(n) = n$ and obtain

$$\lim_{n\to\infty} \frac{S_{L(n+1)}}{S_{L(n)}} = \lim_{n\to\infty} \frac{S_{L(n)} + a_{L(n+1)}}{S_{L(n)}} = 1 + \lim_{n\to\infty} \frac{a_{L(n+1)}}{S_{L(n)}} = 1 = l.$$

Case III. If $l = \infty$, we chose $L(n)$ such that $n^n \in [S_{L(n)}, S_{L(n)+1})$, $n \geq 1$. We obtain:

$$\frac{S_{L(n+1)}}{S_{L(n)}} \geq \frac{(n+1)^{n+1} - 1}{n^n} \longrightarrow \infty.$$

(2)

It is well known that the sequence $x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$ is convergent. Using this observation, we have that

$$\frac{S_{L(n+1)}}{S_{L(n)}} = \frac{x_{L(n+1)} + 2\sqrt{L(n+1)}}{x_{L(n)} + 2\sqrt{L(n)}}$$
$$l = \lim_{n\to\infty} \frac{S_{L(n+1)}}{S_{L(n)}} = \sqrt{\lim_{n\to\infty} \frac{L(n+1)}{L(n)}}$$

For $l = 1$ we chose $L(n) = n$.

For $l < 1$ we chose $L(n) = [l^{2n}]$ (the integer part of l^{2n})

For $l = \infty$ we chose $L(n) = n^n$.

Problem 4 (Dorian Popa, Cluj-Napoca). Let $I = [0, \infty)$ and $f \in C^2(I)$ such that

$$|f''(x) + 2xf'(x) + (x^2+1)f(x)| \leq 1$$

for every $x \in I$. Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

Proof. Put $z(x) = f'(x) + xf(x), x \in I$. One gets

$$|z'(x) + xz(x)| \leq 1, \; x \in I.$$

Let $g(x) = z'(x) + xz(x), \; x \in I$. It follows

$$\left(z(x)e^{\frac{x^2}{2}}\right)' = g(x)e^{\frac{x^2}{2}}$$

$$z(x) = \frac{\int\limits_0^x g(t)e^{\frac{t^2}{2}}dt + c}{e^{\frac{x^2}{2}}}, \; c \in \mathbb{R}.$$

One gets

$$|z(x)| \leq \frac{\int\limits_0^x |g(t)|e^{\frac{t^2}{2}}dt + |c|}{e^{\frac{t^2}{2}}} \leq \frac{\int\limits_0^x e^{\frac{t^2}{2}}dt + |c|}{e^{\frac{x^2}{2}}}, \; x \geq 0.$$

Since $\lim\limits_{x\to\infty} \frac{\int_0^x e^{\frac{t^2}{2}}dt+|c|}{e^{\frac{x^2}{2}}} = 0$ (l'Hospital) it follows $\lim\limits_{x\to\infty} z(x) = 0$

By the relation $f'(x) + xf(x) = z(x), \; x \in I$, it follows

$$f(x) = \frac{\int\limits_0^x z(t)e^{\frac{t^2}{2}}dt + c}{e^{\frac{x^2}{2}}}, \quad c \in \mathbb{R}$$

and using the boundedness of z one gets $\lim\limits_{x\to\infty} f(x) = 0$.

Problem 5 (Dorian Popa, Cluj-Napoca). Let $f : [0,1] \to [0,\infty)$ be a differentiable function with integrable derivative on $[0,1]$. Prove the inequality:

$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right| \leq M\left(\int_0^1 f(x)\,dx\right)^2$$

where $M = \sup\limits_{0\leq x\leq 1} |f'(x)|$.

Proof. For $M = \infty$ the inequality is obvious. Suppose that $M \neq \infty$. By the inequality $-M \leq f'(x) \leq M$, $x \in [0,1]$ it follows:

$$-Mf(x) \leq f(x)f'(x) \leq Mf(x), \; x \in [0,1].$$

One gets:

$$-M\int_0^x f(t)\,dt \leq \frac{1}{2}f^2(x) - \frac{1}{2}f^2(0) \leq M\int_0^x f(t)\,dt, \; x \in [0,1]$$

$$-Mf(x)\int_0^x f(t)\,dt \leq \frac{1}{2}f^3(x) - \frac{1}{2}f^2(0)f(x) \leq Mf(x)\int_0^x f(t)\,dt, \; x \in [0,1].$$

Integrating the last inequality on $[0,1]$ it follows:

$$-M\int_0^1 f^2(x)\,dx \leq \int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx \leq M\int_0^1 f^2(x)\,dx \Leftrightarrow$$
$$\left|\int_0^1 f^3(x)\,dx - f^2(0)\int_0^1 f(x)\,dx\right| \leq M\left(\int_0^1 f(x)\,dx\right)^2.$$

Problem 6 (Pop Vasile, Cluj-Napoca). For a matrix $A \in \mathcal{M}_n(\mathbb{C})$ we denote by $r(A)$ the rank of A and $C(A) = \{X \in \mathcal{M}_n(\mathbb{C}) \mid AX = XA\}$. Prove that:

1. $r(A^n) = r(A^{n+1})$.

2. If $A, B \in \mathcal{M}_n(\mathbb{C})$, $r(A^{n-1}) \neq r(A^n)$, $r(B^{n-1}) \neq r(B^n)$ then there is an invertible matrix $P \in GL_n(\mathbb{C})$ such $B = P^{-1} \cdot A \cdot P$ and the linear subspaces $C(A)$ and $C(B)$ are isomorphic.

Proof. 1. We have $r(A^{n+1}) = r(A^n \cdot A) \leq r(A^n)$. If $r(A^{n+1}) < r(A^n)$ it follows that there is a column matrix $X \neq 0$ such that $A^n \cdot X \neq 0$ and $A^{n+1} \cdot X = 0$. By Cayley- Hamilton theorem one gets:

$$a_0 I_n + a_1 A + \cdots + a_{n-1} A^{n-1} + A^n = 0 \tag{4}$$

$$a_0 X + a_1 A \cdot X + \cdots + a_{n-1} A^{n-1} \cdot X + A^n \cdot X = 0 \tag{5}$$

and multiplying by $A^n, A^{n-1}, \ldots, A$ it follows:

$$a_0 A^n X = 0,\ a_1 A^n X = 0, \ldots, a_{n-1} A^n X = 0,$$

hence $a_0 = a_1 = \ldots = a_{n-1} = 0$ and by (4) it follows $A^n = 0 \Rightarrow A^{n+1} = 0$, contradiction.

2. Since $r(A^{n-1}) \neq r(A^n)$ one can choose a column matrix X_0 such that $A^{n-1} \cdot X \neq 0$ and $A^n \cdot X = 0$. We prove that the set $\{X_0, A \cdot X_0, \ldots, A^{n-1} \cdot X_0\}$ is a basis in $\mathbb{C}^n$. Suppose that:

$$\alpha_0 X_0 + \alpha_1 A \cdot X_0 + \cdots + \alpha_{n-1} A^{n-1} \cdot X_0 = 0, \tag{6}$$

$\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$. Multiplying (6) by $A^{n-1}, A^{n-2}, \ldots, A$ one gets $\alpha_0 = \alpha_1 = \ldots, \alpha_{n-1} = 0$. Denote $E_1 = X_0, E_2 = A \cdot X_0, \ldots, E_n = A^{n-1} \cdot X_0$ it follows that:

$$A \cdot E_1 = E_2,\ A \cdot E_2 = E_3, \ldots, A \cdot E_{n-1} = E_n,\ A \cdot E_n = 0,$$

hence $A \cdot P = [E_2, E_3, \ldots, E_n, 0]$, $P = [E_1, E_2, \ldots, E_n]$ and

$$P^{-1} \cdot A \cdot P = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} = J_0.$$

Therefore **every matrix A with $r(A^{n-1}) \neq r(A^n)$ is similar with the Jordan cell** J_0. Now we prove that **if A, B are similar matrices then** $C(A)$ **and** $C(B)$ **are isomorphic subspaces.** If $B = P^{-1} \cdot A \cdot P$ define the map $f : C(A) \to C(B)$ by:

$$f(X) = P^{-1} \cdot X \cdot P,\ X \in C(A).$$

Then the inverse of f is $g : C(A) \to C(B)$:

$$g(Y) = P \cdot Y \cdot P^{-1},\ Y \in C(B).$$

We have $X \in C(A) \Leftrightarrow A \cdot X = X \cdot A \Leftrightarrow P \cdot B \cdot P^{-1} \cdot X = X \cdot P \cdot B \cdot P^{-1} \Leftrightarrow B \cdot P^{-1} \cdot X \cdot P = P^{-1} \cdot X \cdot P \cdot B$ $\Leftrightarrow B \cdot f(X) = f(X) \cdot B \Leftrightarrow f(X) \in C(B)$, hence f is an isomorphism.

Problem 7 (Masha Vlasenko, Kyiv). For every matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ (i.e. with integral entries and determinant 1) consider a rational linear transformation of real line $F_g(x) = \frac{ax+b}{cx+d}$. Prove that for every real number $x = p + q\sqrt{7}$ with $p, q \in \mathbb{Q}$ there exist a matrix $g \in SL(2, \mathbb{Z})$ such that $F_g(x) = x$.

Proof. Consider the set

$$R = \{y \in \mathbb{R} | y, xy \in \mathbb{Z} + x\mathbb{Z}\}.$$

One has $\mathbb{Z} \subset R$, and for $y_1, y_2 \in R$ also $y_1 + y_2, y_1 y_2 \in R$. For $y \in R$ there exist $a, b, c, d \in \mathbb{Z}$ such that $yx = ax + b$, $y = cx + d$. The matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has integral entries and $F_g(x) = x$. But it may have determinant other than 1. Suppose we have $z \in R$ analogously represented by matrix h with integral entries, and $zy \in R$ is represented by matrix f. Then $f = hg$. So, if y is an invertible element of R (i.e. $y, \frac{1}{y} \in R$), then matrix g corresponding to y is invertible. Thus g has determinant either 1 of -1. If $det(g) = -1$, then $det(g^2) = 1$ and g^2 is a matrix we are looking for. So, we reduced the problem to finding an invertible element in R.

For some large $N \in \mathbb{N}$ we have $N\sqrt{7} \in R$. Indeed, consider $L \in \mathbb{N}$ such that $Lp, Lq, Lq^2, Lp^2 \in \mathbb{Z}$, take $N = |Lq|$. Then $N\sqrt{7} = \pm(Lx - Lp)$ and $N\sqrt{7}x = \pm(7Lq^2 + Lpq\sqrt{7}) = \pm(Lpx - Lp^2 + 7Lq^2)$.

So $\mathbb{Z} + \mathbb{Z}N\sqrt{7} \subset R$. For $z = p_1 + q_1\sqrt{7}$ with $p_1, q_1 \in \mathbb{Q}$ we denote $\bar{z} = p_1 - q_1\sqrt{7}$. Then obviously $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1 + \bar{z}_2$. Consider $\varepsilon = 8 + 3\sqrt{7}$. Then $\bar{\varepsilon} = 8 - 3\sqrt{7}$, and $\varepsilon\bar{\varepsilon} = 1$. We are going to prove than there exist n such that $\varepsilon^n \in 1 + N(\mathbb{Z} + \sqrt{7}\mathbb{Z})$. Then $\varepsilon^n \in R$, obviously $\varepsilon^n \neq 1$, and $\frac{1}{\varepsilon^n} = \bar{\varepsilon^n} \in R$. So we will get an invertible element of R.

We have $\varepsilon^k = a_k + b_k\varepsilon$ with $a_0 = 1, b_0 = 0$ and $a_{k+1} = -b_k$, $b_{k+1} = 16b_k + a_k$. Thus the sequence of pairs $(a_k \; mod \; N, b_k \; mod \; N)$ is periodic. In fact, it has no period because we can invert our formulas: $b_k = -a_{k+1}$, $a_k = b_{k+1} + 16a_{k+1}$. So, for some $n > 0$ we will get $(a_n, b_n) \equiv (1, 0) mod N$, thus $\varepsilon^n \in 1 + N(\mathbb{Z} + \mathbb{Z}\sqrt{7})$. This finishes the proof.

Problem 8 (Gergely Röst, Szeged). Suppose that a sequence x_n and $p \in R$ satisfies $x_{n+1} \leq px_n + (1-p)x_{n-1}$ for all $n \geq 2$. For which p does the $\lim x_n$ (finite or infinite) exist?

Solution. The sequence $x_n = (p-1)^n$ satisfies the condition, but for $p \leq 0$ the limit does not exist. If $p \geq 1$, then $x_{n+1} \geq x_n$ for all $n \in N$, or $x_{n+1} \leq x_n$ for all $n \geq k$; since the inequality $x_n \geq x_{n+1}$ implies $x_{n+1} \geq x_{n+2}$. So in this case the sequence is eventually monotone, tending to a finite or infinite limit. Finally, let $0 < p < 1$. Define $l = \inf x_n$, $L = \sup x_n$. Since $x_n \leq \max\{x_1, x_2\}$, $L < +\infty$. Suppose that $l < L$, that is the limit does not exist. Then there are numbers l' and L' such that $l < l' < L < L'$. For a large n, $x_{n-1} < L'$. We can choose an n such that $x_n < l'$, hence $x_{n+1} \leq pl' + (1-p)L' =: \lambda$. Clearly $x_n < l' < \lambda$, and $x_{n+2} < \lambda, x_{n+3} < \lambda, ...$ follows, therefore $L \leq \lambda$, that is $L \leq pl' + (1-p)L'$. But this is not true if we choose l' and L' sufficiently close to l and L, a contradiction. So $l = L$ and the limit exists.

Problem 9 (José Luis Díaz-Barrero, Barcelona). Let $A(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0)$ be a non-constant complex polynomial. Show that all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{3^k \binom{n}{k}}{4^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \quad \text{and} \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{4^n - 1}{3^k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

Solution by the proposer.

If we assume $|z| < r_1$ then from $A(z) = \sum_{k=0}^{n} a_k z^k$, we have

$$|A(z)| = \left| \sum_{k=0}^{n} a_k z^k \right| \geq |a_0| - \sum_{k=1}^{n} |a_k||z|^k > |a_0| - \sum_{k=1}^{n} |a_k| r_1^k$$

$$= |a_0| \left(1 - \sum_{k=1}^{n} \left| \frac{a_k}{a_0} \right| r_1^k \right) \tag{7}$$

¿From the expression of r_1 and taking into account the identity

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n,$$

immediately follows

$$\left| \frac{a_k}{a_0} \right| r_1^k \leq \frac{3^k \binom{n}{k}}{4^n - 1}, \quad 1 \leq k \leq n \tag{8}$$

Substituting (8) into (7), we have

$$|A(z)| > |a_0| \left(1 - \sum_{k=1}^{n} \left| \frac{a_k}{a_0} \right| r_1^k \right) \geq |a_0| \left(1 - \sum_{k=1}^{n} \frac{3^k \binom{n}{k}}{4^n - 1} \right) = 0.$$

Consequently, $A(z)$ does not have zeros in $\{z \in C : |z| < r_1\}$.

To prove the second inequality we will use the well known fact that all the zeros of $A(z)$ have modulus less than or equal to the unique positive root of the equation

$$B(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0.$$

Therefore, the second part of our statement will be proved if we show that $B(r_2) \geq 0$. In fact, from the expression of r_2 immediately follows

$$\left|\frac{a_{n-k}}{a_n}\right| \leq \frac{3^k\binom{n}{k}}{4^n - 1} r_2^k, \quad 1 \leq k \leq n$$

and

$$B(r_2) = |a_n|\left[r_2^n - \sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right| r_2^{n-k}\right] \geq |a_n|\left[r_2^n - \sum_{k=1}^{n}\left(\frac{3^k\binom{n}{k}}{4^n - 1} r_2^k\right) r_2^{n-k}\right]$$

$$= |a_n| r_2^n \left(1 - \sum_{k=1}^{n} \frac{3^k\binom{n}{k}}{4^n - 1}\right) = 0,$$

as desired. This completes the proof.

Problem 10 (David Preiss, UCL). Find the largest $R \in (0, \infty)$ with the following property: Whenever $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function such that $\|f'(0)\| = 1$ and $\|f'(u) - f'(v)\| \leq \|u - v\|$ for all $u, v \in \mathbb{R}^2$, then for each $0 < r \leq R$ the maximum of f on the disk $\{u \in \mathbb{R}^2 : \|u\| \leq r\}$ is attained at exactly one point.

Solution Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(s) = 1$ for $s \leq 0$, $h(s) = 1 - s$ for $0 \leq s \leq 1$ and $h(s) = 0$ for $s \geq 1$. Let $\psi(t) = \int_0^t h(s)ds$. Then $\psi'(0) = 1$ and $|\psi'(t_1) - \psi'(t_2)| = |h(t_1) - h(t_2)| \leq |t_1 - t_2|$ for all t_1, t_2. Let $g(x, y) = \psi(x) + y^2/2$. Then $g'(x, y) = (\psi'(x), y)$, so $\|g'(x_1, y_1) - g'(x_2, y_2)\|^2 = |\psi'(x_1) - \psi'(x_2)|^2 + |y_1 - y_2|^2 \leq |x_1 - x_2|^2 + |y_1 - y_2|^2 = \|(x_1, y_1) - (x_2, y_2)\|^2$. Suppose that g has a unique maximum on $D_r = \{x \in \mathbb{R}^2 : \|x\| \leq r\}$. Since $g(x, y) = g(x, -y)$ and $x \to g(x, 0)$ is non-decreasing, the unique maximum has to be attained at $(r, 0)$. Since $g(1, 0) = g(r, 0)$ for $r > 1$, we have $r \leq 1$. If $1/2 < r \leq 1$, we have $g(x, \sqrt{r^2 - x^2}) = x - x^2/2 + (r^2 - x^2)/2 = x - x^2 + r^2/2$. Since this function has negative derivative at $x = r$, g does not attain its maximum at $(0, r)$. So $R \leq 1/2$.

Suppose now that $r \leq 1/2$ and that f attains its maximum on D_r at u, v, $u \neq v$. Since $\|f'(z) - f'(0)\| \leq r$, $\|f'(z)\| \geq 1 - r > 0$ for all $z \in D_r$. Hence f may attain its maximum only at the boundary of D_r, and so we must have $\|u\| = \|v\| = r$ and $f'(u) = au$ and $f'(v) = bv$, where $a, b \geq 0$. Since $au = f'(u)$ and $bv = f'(v)$ belong to the disk D with centre $f'(0)$ and radius r, they do not belong to the interior of D_r. Hence $\|f'(u) - f'(v)\| = \|au - bv\| \geq \|u - v\|$ and this inequality is strict since $D \cap D_r$ contains no more than one point. But this contradicts the assumption that $\|f'(u) - f'(v)\| \leq \|u - v\|$. So $R = 1/2$.

Problem 11 (Vjekoslav Kovač, Zagreb). Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two decreasing sequences of positive real numbers such that $\prod_{j=1}^n x_j \geq \prod_{j=1}^n y_j$ for every $n \geq 1$. Prove that $\sum_{j=1}^n x_j \geq \sum_{j=1}^n y_j$ for every $n \geq 1$.

Solution.

We first note that for $n \geq 1$ and integers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$ we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \geq y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n}.$$

This is immediate from the problem hypothesis after rearranging factors:

$$(x_1 \ldots x_n)^{\alpha_n} (x_1 \ldots x_{n-1})^{\alpha_{n-1} - \alpha_n} \ldots (x_1 x_2)^{\alpha_2 - \alpha_3} x_1^{\alpha_1 - \alpha_2}$$
$$\geq (y_1 \ldots y_n)^{\alpha_n} (y_1 \ldots y_{n-1})^{\alpha_{n-1} - \alpha_n} \ldots (y_1 y_2)^{\alpha_2 - \alpha_3} y_1^{\alpha_1 - \alpha_2}.$$

Furthermore, note that

$$y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n} \ge y_1^{\alpha_{\sigma(1)}} y_2^{\alpha_{\sigma(2)}} \dots y_n^{\alpha_{\sigma(n)}}$$

for any permutation σ of $\{1, 2, \dots, n\}$.
For every integer $N \ge 1$ by using the multinomial theorem we obtain

$$\begin{aligned}(x_1 + x_2 + \dots + x_n)^N &= \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_n \ge 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \dots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \dots \alpha_n!} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \\ &\ge \sum_{\substack{\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n \ge 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \dots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \dots \alpha_n!} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \\ &\ge \sum_{\substack{\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n \ge 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \dots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \dots \alpha_n!} y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n} \\ &\ge \frac{1}{n!}(y_1 + y_2 + \dots + y_n)^N\end{aligned}$$

since each n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ has at most $n!$ distinct rearrangements.
Thus $\frac{x_1 + \dots + x_n}{y_1 + \dots + y_n} \ge \sqrt[N]{\frac{1}{n!}}$ and by taking $N \to \infty$ we obtain $x_1 + \dots + x_n \ge y_1 + \dots + y_n$.

Problem 12 (David Harutyunyan, Yerevan). Let $n \ge 2$ and

$$f(x_1, \dots, x_n) = \frac{x_1}{1 + x_2 \cdot \dots \cdot x_n} + \frac{x_2}{1 + x_1 x_3 \cdot \dots \cdot x_n} + \dots + \frac{x_n}{1 + x_1 \cdot \dots \cdot x_{n-1}} + x_1 \cdot \dots \cdot x_n.$$

Find the maximum value of f on the set $\{(x_1, \dots, x_n) : 0 \le x_i \le 1, i = 1, \dots, n\}$.
Solution Denote

$$g(n) = \max_{\substack{0 \le x_i \le 1 \\ i = 1, \dots, n}} f(x_1, \dots, x_n).$$

We are going to show that $g(2) = 2, g(3) = \frac{5}{2}$ and $g(n) = n - 1, n \ge 4$.
First, let's prove that for $n \in \mathbb{N}$ and $0 \le x_i \le 1$, $i = 1, \dots, n$ the following inequality holds:

$$x_1 + \dots + x_n \le n - 1 + x_1 \cdot \dots \cdot x_n \tag{9}$$

We prove this by induction: for $n = 1$ it is obvious. Assuming that the inequality holds for $n = k$, we'll get

$$x_1 + x_2 + \dots + x_{k+1} - x_1 \cdot \dots \cdot x_{k+1} = (x_1 + x_2 + \dots + x_k) + x_{k+1}(1 - x_1 \cdot \dots \cdot x_k) \le$$
$$\le x_1 + x_2 + \dots + x_k + 1 - x_1 \cdot \dots \cdot x_k \le 1 + k - 1 = k.$$

1) Case $n = 2$

$$f(x_1, x_2) = \frac{x_1}{1 + x_2} + \frac{x_2}{1 + x_1} + x_1 x_2 \le \frac{x_1}{1 + x_1 x_2} + \frac{x_2}{1 + x_1 x_2} + x_1 x_2 \le$$
$$\frac{x_1 + x_2}{1 + x_1 x_2} + x_1 x_2 \le 1 + 1 = 2$$

(by (9)) and $f(1, 1) = 2$, so $g(2) = 2$.
2) Case $n = 3$

$$f(x_1, x_2, x_3) = \frac{x_1}{1 + x_2 x_3} + \frac{x_2}{1 + x_1 x_3} + \frac{x_3}{1 + x_1 x_2} + x_1 x_2 x_3 \le$$

$$\leq \frac{x_1+x_2+x_3}{1+x_1x_2x_3} + x_1x_2x_3 = \frac{x_1+x_2+x_3+x_1x_2x_3+(x_1x_2x_3)^2}{1+x_1x_2x_3} \leq$$

$$\leq \frac{x_1+x_2+x_3+2x_1x_2x_3}{1+x_1x_2x_3}.$$

Let's show that the last fraction is not greater that $\frac{5}{2}$.

$$\frac{x_1+x_2+x_3+2x_1x_2x_3}{1+x_1x_2x_3} \leq \frac{5}{2} \Leftrightarrow x_1+x_2+x_3-\frac{1}{2}x_1x_2x_3 \leq \frac{5}{2}.$$

$$x_1+x_2+x_3-\frac{1}{2}x_1x_2x_3 = x_1+x_2+x_3(1-\frac{1}{2}x_1x_2) \leq$$

$$\leq x_1+x_2+1-\frac{1}{2}x_1x_2 = x_1+x_2(1-\frac{1}{2}x_1)+1 \leq \frac{x_1}{2}+2 \leq \frac{5}{2}.$$

Note also that $f(1,1,1)=\frac{5}{2}$, so $g(3)=\frac{5}{2}$.

3) Case $n \geq 4$

$$f(x_1,x_2,...,x_n) \leq \frac{x_1+...x_n}{1+x_1...x_n} + x_1...x_n \leq \frac{n-1+x_1...x_n}{1+x_1...x_n} + x_1...x_n$$

(here we have used (9)).

$$\frac{n-1+x_1...x_n}{1+x_1...x_n} + x_1...x_n = \frac{n-1+2x_1...x_n+(x_1...x_n)^2}{1+x_1...x_n} \leq \frac{n-1+3x_1...x_n}{1+x_1...x_n} =$$

$$= \frac{(n-1)(1+\frac{3}{n-1}x_1...x_n)}{1+x_1...x_n} \leq \frac{(n-1)(1+x_1...x_n)}{1+x_1...x_n} = n-1.$$

As $f(1,1,...,1,0)=n-1$, we get $g(n)=n-1$ for $n \geq 4$.

Problem 13 (David Harutyunyan, Yerevan). Find all $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(x)+f(y)) = f(y+f(x))+x+f(y)-f(f(y)) \tag{10}$$

for all $x,y\in\mathbb{R}$.

Solution We'll prove $f(x)\equiv x$ in 3 steps.

Step 1. f is injective.

Let $a,b\in\mathbb{R}$ and $f(a)=f(b)=c$. Putting $y=a$ and $y=b$ in (10) we'll get

$$f(x+f(x)+c) = f(a+f(x))+x+c-f(c)$$

and

$$f(x+f(x)+c) = f(b+f(x))+x+c-f(c),$$

so

$$f(a+f(x)) = f(b+f(x)) \quad \text{for all } x\in\mathbb{R} \tag{11}$$

Putting $x=a$ and $x=b$ in (10), we'll obtain

$$f(a+c+f(y)) = f(y+c)+a+f(y)-f(f(y)),$$

$$f(b+c+f(y)) = f(y+c)+b+f(y)-f(f(y))$$

so

$$a-b = f(a+c+f(y))-f(b+c+f(y)). \tag{12}$$

Now if we show that there exist y_0 and x_0 such that $c+f(y_0)=f(x_0)$, then from (11) and (12) we'll deduce that $a=b$. If we take $x=f(f(y))-f(y)+c$ in (10), then we'll obtain such x_0 and y_0.

Step 2. $f(f(x))=x$ for all $x\in\mathbb{R}$.

We take $x=f(f(y))-f(y)$ in (10), then $f(x+f(x)+f(y))=f(y+f(x))$, and since f is injective, we'll get $x+f(x)+f(y)=y+f(x)$, so $x=y-f(y)=f(f(y))-f(y)$. Hence $f(f(y))=y$.

Step 3. $f(x) \equiv x$.

Take $y = 0$ in (10). Then

$$f(x + f(x) + f(0)) = f(f(x)) + x + f(0) - f(f(0)) = 2x + f(0).$$

If we take $f(x)$ instead of x in the last equality, we'll obtain

$$f(f(x) + f(f(x)) + f(0)) = 2f(x) + f(0),$$

but using **Step 2**, we'll get also

$$f(f(x) + f(f(x)) + f(0)) = f(f(x) + x + f(0)) = 2x + f(0),$$

so $2x + f(0) = 2f(x) + f(0)$, hence $f(x) = x$ for any x.

Problem 14 (Artur Barkhudaryan, Yerevan). Let $x_1, x_2, \ldots, x_{2005}$ be real numbers, $|x_n| \leq 1$. Define the sequence y_n as follows:

$$y_{-1} = y_0 = 0, \quad y_n = x_n + 2\cos\left(\tfrac{1}{2005}\right) y_{n-1} - y_{n-2} \quad (n = 1, 2, \ldots, 2005).$$

Prove that

$$|y_{2005}| < \frac{1}{2\sin^2(\frac{1}{2005})}.$$

Solution For $x = (x_1, \ldots, x_{2005}) \in \mathbb{R}^{2005}$ let Ax denote the sequence $(y_1, \ldots, y_{2005})$ as defined above. It can be easily seen that A is linear: $A(x+y) = Ax + Ay$ and $A(\lambda x) = \lambda Ax$ for $\lambda \in \mathbb{R}$. For any sequence $x = (x_1, \ldots, x_{2005})$ and for $n = 0, 1, \ldots, 2004$ define

$$S_n x = (\underbrace{0, 0, \ldots, 0}_{n}, x_1, x_2, \ldots, x_{2005-n}).$$

Notice that A commutes with S_n: $AS_n x = S_n Ax$.

Denote by u the sequence $(1, 0, 0, \ldots, 0)$ and denote $v = Au$. It is easily seen that

$$v_n = \frac{\sin\left(\frac{n}{2005}\right)}{\sin(\frac{1}{2005})}.$$

Now, for any $x = (x_1, \ldots, x_{2005})$,

$$y = Ax = A\left(\sum_{n=1}^{2005} x_n S_{n-1} u\right) = \sum_{n=1}^{2005} x_n A S_{n-1} u = \sum_{n=1}^{2005} x_n S_{n-1} A u = \sum_{n=1}^{2005} x_n S_{n-1} v$$

and hence

$$y_{2005} = \sum_{n=1}^{2005} x_n v_{2006-n}.$$

Thus, if $|x_n| \leq 1$ for all n, we have

$$|y_{2005}| = \left|\sum_{n=1}^{2005} x_n v_{2006-n}\right| \leq \sum_{n=1}^{2005} |x_n v_{2006-n}| \leq \sum_{n=1}^{2005} |v_n| = \sum_{n=1}^{2005} \frac{\sin\left(\frac{n}{2005}\right)}{\sin(\frac{1}{2005})} = \frac{1 + \cos(\frac{1}{2005}) - \cos(1) - \cos(\frac{2006}{2005})}{2\sin^2(\frac{1}{2005})} < \frac{1}{2\sin^2(\frac{1}{2005})}.$$

Problem 15 (Artur Barkhudaryan, Yerevan). For any maps $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ define

$$f \cdot g(x) = f(g(x)).$$

Together with this operation the set M_n of maps from $\{1, 2, \ldots, n\}$ to itself is a monoid (i.e a semigroup with identity). A monoid homomorphism is a map between monoids that preserves the monoid operation and the identity.

Is there a monoid homomorphism $F : M_{2005} \to M_{2006}$ such that

1. $F(f)(n) = f(n)$ for any $f \in M_{2005}$ and $n \in \{1, 2, \ldots, 2005\}$,
2. $F(f)(2006) \leq 2005$ whenever $f \in M_{2005}$ is not bijective?

Solution Suppose $F : M_{2005} \to M_{2006}$ is a monoid homomorphism satisfying both 1 and 2. Let $g \in M_{2005}$ be a bijection. Then

$$gg^{-1} = g^{-1}g = 1_{2005}$$

(where 1_n denotes the identity on $\{1, 2, \ldots, n\}$), and hence

$$F(g)F(g^{-1}) = F(g^{-1})F(g) = F(1_{2005}) = 1_{2006},$$

thus $F(g)$ is also a bijection. According to 1,

$$F(g)(\{1, \ldots, 2005\}) = \{1, \ldots, 2005\},$$

so $F(g)(2006) = 2006$.

Now take $f, g \in M_{2005}$,

$$f(n) = 5\left\lceil \frac{n}{5} \right\rceil,$$

$$g(n) = \begin{cases} n + 5, & n \leq 2000, \\ n - 2000, & n > 2000. \end{cases}$$

Clearly g is bijective, f is not and

$$g \cdot f = f \cdot g.$$

Then $F(g)F(f) = F(f)F(g)$ and hence

$$F(g)F(f)(2006) = F(f)F(g)(2006) = F(f)(2006),$$

but on the other hand, according to 2, $n = F(f)(2006) \leq 2005$ and

$$F(g)(n) = g(n) \neq n.$$

This contradiction proves that no such homomorphism exists.

Problem 16 (Artur Barkhudaryan, Yerevan). For any maps $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ define

$$f \cdot g(x) = f(g(x)).$$

Together with this operation the set M_n of maps from $\{1, 2, \ldots, n\}$ to itself is a monoid (i.e a semigroup with identity). A monoid homomorphism is a map between monoids that preserves the monoid operation and the identity.

Suppose $N \geq 2005$ and suppose $F : M_{2005} \to M_N$ is a monoid homomorphism such that

1. $F(f)(n) = f(n)$ for any $f \in M_{2005}$ and $n \in \{1, 2, \ldots, 2005\}$,
2. $F(f)(n) \leq 2005$ for every $n \in \{1, 2, \ldots, N\}$ whenever $f \in M_{2005}$ is not bijective.

Prove that N is a multiple of 2005.

Solution Denote $R = \{2006, 2007, \ldots, N\}$. Let $g \in M_{2005}$ be a bijection. Then

$$gg^{-1} = g^{-1}g = 1_{2005}$$

(where 1_n denotes the identity on $\{1, 2, \ldots, n\}$), and hence

$$F(g)F(g^{-1}) = F(g^{-1})F(g) = F(1_{2005}) = 1_N,$$

thus $F(g)$ is also a bijection. According to 1,

$$F(g)(\{1, \ldots, 2005\}) = \{1, \ldots, 2005\},$$

so $F(g)(R) = R$.

Let p be either 5 or 401. Take $f, g \in M_{2005}$,

$$f(n) = p\left\lceil \frac{n}{p} \right\rceil,$$

$$g(n) = \begin{cases} n + p, & n \leq 2005 - p, \\ n + p - 2005, & n > 2005 - p. \end{cases}$$

Clearly g is bijective, f is not and

$$g \cdot f = f \cdot g.$$

Then $F(g)F(f) = F(f)F(g)$ and, more generally,

$$(F(g))^n F(f) = F(f)(F(g))^n \quad \text{for any } n \in \mathbb{Z}.$$

For $a, b \in R$ write $a \sim b$ if $(F(g))^n(a) = b$ for some $n \in \mathbb{Z}$. The relation $\sim$ is easily seen to be an equivalence on the set R. Let

$$A = \{(F(g))^n(a); n \in \mathbb{Z}\} \subseteq R$$

be a class of equivalence of $\sim$. Note that, according to 2, $c = F(f)(a) \leq 2005$ and thus

$$F(f)(F(g))^n(a) = (F(g))^n F(f)(a) = (F(g))^n(c) = \\ g^n(c) \neq c = F(f)(a)$$

for $n = 1, 2, \ldots, \frac{2005}{p} = q$, which implies

$$(F(g))^n(a) \neq a.$$

On the other hand,

$$(F(g))^q(a) = F(g^q)(a) = F(1_{2005})(a) = 1_N(a) = a.$$

So the number of elements of A is q. As A was an arbitrary class of equivalence of R, this proves that the number of elements of R is a multiple of q (which is either 5 or 401).

So, we have N is a multiple of 5 and a multiple of 401, which follows N is a multiple of 2005.

Problem 17 (Géza Kós, Budapest). Let A be the $n \times n$ matrix where the (i, j)th entry is $i + j$, What is the rank of A?

Solution. For $n = 1$ the rank is 1. Now assume $n \geq 2$. Since $A = (i)_{i,j=1}^n + (j)_{i,j=1}^n$, matrix A is the sum of two matrixes of rank 1. Therefore, the rank of A is at most 2. The determinant of the top-left 2×2 minor is -1, so the rank is exactly 2.

The rank of A is 1 for $n = 1$ and 2 for $n \geq 2$.

Problem 18 (Géza Kós, Budapest). Prove that for an arbitrary, three times differentiable $\mathbb{R} \to \mathbb{R}$ function f, there exists a real number $\xi \in (-1, 1)$ such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 1 Let

$$g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that $g(\pm 1) = f(\pm 1)$, $g(0) = f(0)$ and $g'(0) = f'(0)$.

Apply Rolle's theorem for the function $h(x) = f(x) - g(x)$ and its derivatives. Since $h(-1) = h(0) = h(1) = 0$, there exist $\eta \in (-1,0)$ and $\vartheta \in (0,1)$ such that $h'(\eta) = h'(\vartheta) = 0$. We also have $h'(0) = 0$, so there exist $\varrho \in (\eta, 0)$ and $\sigma \in (0, \vartheta)$ such that $h''(\varrho) = h''(\sigma) = 0$. Finally, there exists a $\xi \in (\varrho, \sigma) \subset (-1,1)$ where $h'''(\xi) = 0$. Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2} \cdot 6 - f(0) \cdot 0 + \frac{f(1)}{2} \cdot 6 - f'(0) \cdot 6.$$

Solution 2. The expression $\frac{f(1) - f(-1)}{2} - f'(0)$ is the divided difference $f[-1,0,0,1]$ and there exists a number $\xi \in (-1,1)$, such that $f[-1,0,0,1] = \frac{f'''(\xi)}{3!}$.

Solution 3. f(x)

Problem 19 (Géza Kós, Budapest). Let f be a polynomial of degree n and suppose there exist real numbers $1 = x_0 > x_1 > \cdots > x_n \geq -1$ such that $f(x_k) = (-1)^k$ for all $k = 0, 1, \ldots, n$. Prove that

$$f\left(1 + \frac{2}{n^2}\right) > \frac{3}{2}.$$

Solution. Let T_n be the nth Chebishev polynomial. For an arbitrary $\varepsilon > 0$, consider the polynomial $g_\varepsilon(x) = (1+\varepsilon)f(x) - T_n(x)$. The value of this polynomial is positive at x_{2k} and negative at x_{2k+1}. Therefore, g_ε has a root in each interval (x_k, x_{k+1}). This is already n disjoint roots; so g_ε cannot have any root in $[1, \infty)$. Since $g_\varepsilon(1) = \varepsilon > 0$, we have $g_\varepsilon(x) > 0$ for all $x \geq 1$ and thus $f(x) > \frac{1}{1+\varepsilon}T_n(x)$. By $\varepsilon \to 0$ we obtain $f(x) \geq T_n(x)$ for all $x > 1$.

Then

$$f\left(1 + \frac{2}{n^2}\right) \geq T_n\left(1 + \frac{2}{n^2}\right) = \cosh\left(n \cdot \operatorname{ar\,cosh}\left(1 + \frac{2}{n^2}\right)\right) =$$

$$= \cosh\left(n \cdot \ln\left(1 + \frac{2}{n^2} + \sqrt{\frac{4}{n^2} + \frac{4}{n^4}}\right)\right) > \frac{1}{2}\exp\left(n \cdot \ln\left(1 + \frac{2}{n}\right)\right) =$$

$$= \frac{1}{2}\left(1 + \frac{2}{n}\right)^n > \frac{1}{2}\left(1 + n \cdot \frac{2}{n}\right) = \frac{3}{2}.$$

Problem 20 (Yuri Syroid). Find the maximal possible dimension of a subspace V of the space of real $n \times n$ matrices satisfying the following condition

$$trace(XY) = 0, \forall X, Y \in V$$

Solution If A is a nonzero symmetric matrix, then $trace(A^2) > 0$. Indeed, eigenvalues of A are real. If $\lambda_1, \ldots, \lambda_n$ are these eigenvalues (counting with multiplicity), then

$$trace(A^2) = \sum_{i=1}^{n} \lambda_i^2 > 0.$$

Therefore V does not contain a nonzero symmetric matrix. Dimension of the space of symmetric $n \times n$ matrices is equal to $\frac{n(n+1)}{2}$. Thus $\dim V \leqslant n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the condition of the problem.

Problem 21 (Yuri Syroid). Let A, B be complex square matrices of size 6 similar to diagonal matrices $diag(2,2,2,2,1,1)$, $diag(3,3,3,1,1,1)$, respectively. Prove that there exists a linear subspace $U \subset \mathbb{C}^6$ not equal to $\{0\}$ and $\mathbb{C}^6$ such that $AU \subset U, BU \subset U$.

Solution Let $M_6(\mathbb{C})$ denote the space of all complex square matrices of size 6. Put $Z_A = \{X \in M_6(\mathbb{C}) | AX = XA\}, Z_B = \{X \in M_6(\mathbb{C}) | BX = XB\}$. One immediately checks that $\dim Z_A = 4^2 + 2^2 = 20, \dim Z_B = 3^2 + 3^2 = 18$. One obviously has the inequality

$$\dim Z_A \cap Z_B = \dim Z_A + \dim Z_B - \dim(Z_A + Z_B) \geqslant 18 + 20 - 36 = 2.$$

Thus there exists a matrix $C \in Z_A \cap Z_B, C \neq xE$ for any $x \in \mathbb{C}$ (here E denotes the identity matrix). Let λ be an eigenvalue of C. Put $U = \{v \in \mathbb{C}^6 | Cv = \lambda v\}$. $C \neq xE$ implies $U \neq \{0\}, \mathbb{C}^6$. Since A and B commute with C, it follows that $AU \subset U, BU \subset U$. This completes the proof.

Problem 22 (Yuri Syroid). Prove that any square complex matrix is similar to a symmetric matrix.

Solution Let A be a square complex matrix. Changing A by its Jordan normal form we reduce to the case $A = aE + J$, where J is a Jordan cell. Clearly, we may assume that $a = 0$. Let n be the size of J.

For $x, y \in \mathbb{C}^n$ put $(x,y) = \sum_{i=1}^n x_i y_i$. Let B be a non-degenerate matrix. Put $(x,y)_B = (Bx, By)$.

Let X be a matrix of size n. Now we show that the matrix $Y = BXB^{-1}$ is symmetric iff $(Xx,y)_B = (x, Xy)_B$ for any $x, y \in \mathbb{C}^n$.

Indeed, $(Xx,y)_B = (BXx, By) = (YBx, By)$, $(x, Xy)_B = (Bx, YBy)$. Since B is non-degenerate, the following statements are equivalent:

1. $(Xx,y)_B = (x,Xy)_B$ for all x, y.

2. $(Yx,y) = (x,Yy)$ for all x, y.

The last condition is an equivalent definition of a symmetric matrix, so we are done.

Thus it is enough to find a non-degenerate matrix B such that $(Jx,y)_B = (x,Jy)_B$ for all $x, y \in \mathbb{C}^n$.

Let $g(x,y)$ be a non-degenerate symmetric bilinear form on $\mathbb{C}^n$. It is known that there exists an orthonormal basis for $g(x,y)$. In other words, there exists a non-degenerate matrix B such that $g(x,y) = (x,y)_B$. One easily checks that $g(Jx,y) = g(x,Jy)$ for $g(x,y) = \sum_{i=1}^n x_i y_{n+1-i}$.

Problem 23 (Tamás Keleti, Budapest). Characterise those differentiable $\mathbb{R} \longrightarrow \mathbb{R}$ functions that have periodic derivative.

Answer. These are the functions of the form $f(x) = g(x) + c \cdot x$, where g is periodic differentiable function and $c \in \mathbb{R}$.

Proof. If $f(x) = g(x) + c \cdot x$, where g is periodic and $c \in \mathbb{R}$ then $f'(x) = g'(x) + c$ is clearly periodic.

Now suppose that $f'(x)$ is periodic; that is, there exists a $p > 0$ such that $f'(x+p) = f'(x)$ for every $x \in \mathbb{R}$. Then

$$(f(x+p) - f(x))' = f'(x+p) - f'(x) = 0$$

for every $x \in \mathbb{R}$, which implies that $f(x+p) - f(x) = d$ for a constant $d \in \mathbb{R}$. Letting $g(x) = f(x) - \frac{d}{p}x$ we get

$$g(x+p) - g(x) = f(x+p) - \frac{d}{p}(x+p) - (f(x) - \frac{d}{p}x) = f(x+p) - f(x) - d = 0$$

for any $x \in \mathbb{R}$. Therefore choosing $c = \frac{d}{p}$, f is indeed of the form $f(x) = g(x) + cx$, where g is periodic and $c \in \mathbb{R}$.

Remark. It would be natural to use integral in the proof. However, arguments using the integral of f' are complete (most likely) only for reasonably nice functions (for example for C^1 functions) since in general a derivative is not always Riemann integrable and Newton-Leibniz is not true in general even if we take Lebesgue integral.

Problem 24 (Tamás Keleti, Budapest). Let A be a real $n \times n$ matrix, I be the $n \times n$ identity matrix and let $f(t) = \det(I + tA)$ for every $t \in \mathbb{R}$. Determine $f'(0)$.

Answer: $f'(0) = \text{tr}(A)$.

Proof. Let $A = (a_{i,j})$, $B(t) = I + tA = (b_{i,j}(t))$; that is, $b_{i,j}(t) = ta_{i,j}$ if $i \neq j$ and $b_{i,j}(t) = 1 + ta_{i,j}$ if $i = j$. By definition,

$$f(t) = det(B(t)) = \sum_{\pi \in \text{Perm}(1,\dots,n)} \varepsilon(\pi) \cdot b_{1,\pi(1)}(t) \cdot \ldots \cdot b_{n,\pi(n)}(t),$$

where Perm$(1, \dots, n)$ denotes the set of permutations of $\{1, \dots, n\}$, and $\varepsilon(\pi)$ is 1 if π is an even permutation and -1 if it is odd. This shows that $f(t)$ is a polynomial, and so $f'(0)$ is the coefficient of the linear term. If π is not the identity then for at least two i-s $i \neq \pi(i)$, so in $\varepsilon(\pi) \cdot b_{1,\pi(1)}(t) \cdot \ldots b_{n,\pi(n)}(t)$ at least two $ta_{i,j}$ type factors appear, so we get no linear term. Therefore it is enough to find the linear term in

$$b_{1,1}(t) \cdot \ldots \cdot b_{n,n}(t) = (1 + ta_{1,1}) \cdot \ldots \cdot (1 + ta_{n,n}),$$

which is clearly $(a_{1,1} + \ldots + a_{n,n})t$, thus indeed $f'(0) = a_{1,1} + \ldots + a_{n,n} = \text{tr}(A)$.

Problem 25 (Tamás Keleti, Budapest). Let f be an $\mathbb{R} \to \mathbb{R}$ function such that f^n is a polynomial for every $n = 2, 3, \ldots$. Does this imply that f itself is a polynomial?

Answer: Yes, it is even enough to assume that f^2 and f^3 are polynomials.

First Proof. Let $p = f^2$ and $q = f^3$. Write these polynomials in the form of

$$p = a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}, \qquad q = b \cdot q_1^{b_1} \cdot \ldots \cdot q_l^{b_l},$$

where $a, b \in \mathbb{R}$, $a_1, \dots, a_k, b_1, \dots b_l$ are positive integers and $p_1, \dots, p_k, q_1, \dots, q_l$ are irreducible polynomials with leading coefficients 1. Then using that $p^3 = q^2$ and that the factorisation of $p^3 = q^2$ is unique we get that $a^3 = b^2$, $k = l$ and for some $(i_1, \dots, i_k)$ permutation of $(1, \dots, k)$ we have $p_1 = q_{i_1}, \dots, p_k = q_{i_k}$ and $3a_1 = 2b_{i_1}, \dots, 3a_k = 2b_{i_k}$. Hence $b_1, \dots, b_l$ are divisible by 3 and so $r = b^{1/3} \cdot q_1^{b_1/3} \cdot \ldots \cdot q_l^{b_l/3}$ is a polynomial. Since $r^3 = q = f^3$ we have $f = r$ and the proof is complete.

Second Proof (with less explanations) Let $\frac{p}{q}$ be the simplest form of the rational function $\frac{f^3}{f^2}$. Then the simplest form of its square is $\frac{p^2}{q^2}$. On the other hand $\frac{p^2}{q^2} = \left(\frac{f^3}{f^2}\right)^2 = f^2$ is a polynomial therefore q must be a constant and so $f = \frac{f^3}{f^2} = \frac{p}{q}$ is a polynomial.

Problem 26 (Alexander Fomin).

1. Do there exist continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: $f(g(x)) = x^3$, $g(f(x)) = x^5$ for every $x \in \mathbb{R}$?
2. The same question for differentiable functions.

Solution.

1. Yes, there are. An arbitrary continuous increasing function $f : [e, e^5] \to [e, e^3]$, where $f(e) = e$, $f(e^5) = e^3$ and e is the natural logarithm base (also known as Euler's number), can be extended to a continuous increasing function $f : R \to R$, satisfying the condition $(f(x))^3 = f(x^5)$ for all $x \in \mathbb{R}$. The functions $f(x)$ and $g(x) = f^{-1}(x^3)$ satisfy the condition of the problem.
2. No, there are not. The differentiability fails at the points $x = 1, -1$.

Problem 27 (Colombia). Find all functions $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f(u \times v) = f(u) \times f(v)$$

for all vectors u, v in $\mathbb{R}^3$.

Solution. Clearly $f \equiv 0$ satisfies the condition. Let us assume that $f \neq 0$

1. If $v \neq 0$, then $f(0) = f(0 \times v) = f(0) \times f(v)$ which implies that $f(0) \perp f(0)$, concluding that $\mathbf{f(0) = 0}$.

2. Given $u \in \mathbb{R}^3$, there exist $v, w \in \mathbb{R}^3$ such that $u = v \times w$ (choose appropriate v, w in $< u >^{\perp}$). Then

$$\begin{aligned} f(-u) &= f(w \times v) = f(w) \times f(v) \\ &= -f(v) \times f(w) = -f(v \times w) \\ &= -f(u) \end{aligned}$$

concluding that $\mathbf{f(-u) = -f(u)}$ **for all** $\mathbf{u \neq 0}$.

3. Let's suppose that $f(u) = 0$ for some $u \neq 0$.

Let v be an arbitrary vector such that $v \cdot u = 0$. We can find a vector w such that $v = u \times w$. This implies $f(v) = f(u \times w) = f(u) \times f(w) = 0$. Now, if $x \in \mathbb{R}^3$, taking $w = u \times x$, we get $f(w) = 0$ (from the previous reasoning), but also $x \cdot w = 0$ so $f(x) = 0$. As x was chosen arbitrarily, we conclude that $f \equiv 0$, contradicting our initial assumption.

Therefore $\mathbf{f(u) \neq 0}$ **for all** $\mathbf{u \neq 0}$.

4. First let us note the following observation: If $u, v, w \in \mathbb{R}^3$ and $u \times w = v \times w$ then $u = v + \gamma w$ for some $\gamma \in \mathbb{R}$.

Let u, v be two linearly independent vectors or equivalently two vectors such that $u \times v \neq 0$. From the previous observation and the equalities

$$\begin{aligned} f(u+v) \times f(u) &= f((u+v) \times u) = f(v \times u) = f(v) \times f(u) \\ f(u+v) \times f(v) &= f((u+v) \times v) = f(u \times v) = f(u) \times f(v) \end{aligned}$$

we conclude that $f(u+v) = f(v) + \gamma_1 f(u)$ and $f(u+v) = f(u) + \gamma_2 f(v)$ for some $\gamma_1, \gamma_2 \in \mathbb{R}$.

Subtracting both equations we have $(1 - \gamma_2) f(v) + (\gamma_1 - 1) f(u) = 0$ and

$$\begin{aligned} 0 &= ((1 - \gamma_2) f(v) + (\gamma_1 - 1) f(u)) \times f(u) \\ &= (1 - \gamma_2) f(v \times u) + (\gamma_1 - 1) f(u \times u) \\ &= (1 - \gamma_2) f(v \times u) \end{aligned}$$

In view of the above conclusion, last equation implies $\gamma_2 = 1$, so $f(u+v) = f(u) + f(v)$.

If u and v are linearly dependent ($v = \gamma u$), let r, s be vectors such that r, s, u are linearly independent. If $\gamma \neq -1$, we guarantee the vectors $r + v - s$ and $s + u - r$ are linearly independent. Because of the previous analysis, we have

$$\begin{aligned} f(u+v) &= f((s+u-r) + (r+v-s)) \\ &= f(s+u-r) + f(r+v-s) \\ &= f(s) + f(u) + f(-r) + f(r) + f(v) + f(-s) \\ &= f(u) + f(v) \end{aligned}$$

The case $\gamma = -1$ is trivial, because $f(u-u) = f(0) = 0 = f(u) - f(u) = f(u) + f(-u)$. Therefore for all $u, v \in \mathbb{R}^3$ we have that $\mathbf{f(u+v) = f(u) + f(v)}$.

5. From the previous item, is easily deduced that $f(tu) = tf(u)$for all $t \in \mathbb{Q}$ but in general we have this for all $t \in \mathbb{R}$. To see this note that $f(tv) \times f(v) = f(tv \times v) = f(0) = 0$ for all $v \in \mathbb{R}^3$. This implies $f(tv) = s_{t,v} f(v)$ for some $s_{t,v} \in \mathbb{R}$ which may depend on t and v.

If $f(tv_1) = s_{t,v_1} f(v_1)$ and $f(tv_2) = s_{t,v_2} f(v_2)$

$$\begin{aligned} s_{t,v_1} f(v_1 \times v_2) &= s_{t,v_1} f(v_1) \times f(v_2) = f(s_{t,v_1} v_1) \times f(v_2) \\ &= f(tv_1) \times f(v_2) = f(tv_1 \times v_2) \\ &= f(v_1 \times tv_2) = f(v_1) \times f(tv_2) \\ &= f(v_1) \times s_{t,v_2} f(v_2) = s_{t,v_2} f(v_1 \times v_2) \end{aligned}$$

so $s_{t,v_1}=s_{t,v_2}$ when $v_1\times v_2\neq 0$. If we would have $v_1\times v_2=0$ ($v_2=\gamma v_1$), there would exist u_1, u_2, such that $u_1\times u_2=v_1$ and $u_1\times v_1\neq 0$ (taking appropriately u_1, u_2 in $<v_1>^{\perp}$) and then

$$\begin{aligned} s_{t,v_2}f(v_2) &= f(tv_2)=f(t\gamma v_1)=f(t\gamma u_1\times u_2)=f(t\gamma u_1)\times f(u_2)\\ &= s_{t,v_1}f(\gamma u_1)\times f(u_2)=s_{t,v_1}f(\gamma u_1\times u_2)=s_{t,v_1}f(v_2)\end{aligned}$$

also having $s_{t,v_1}=s_{t,v_2}$. Consequently, $s_{t,v}$ only depends on t.

Moreover, if $t>0$, we have $s_tf(v)=f(tv)=f(\sqrt{t}\sqrt{t}v)=s_{\sqrt{t}}f(\sqrt{t}v)=\left(s_{\sqrt{t}}\right)^2f(v)$, having that $s_t=\left(s_{\sqrt{t}}\right)^2>0$. In particular for $t_1>t_2$ it is satisfied that

$$\begin{aligned} s_{t_1}f(v) &= f(t_1v)=f(t_2v+(t_1-t_2)\,v)=f(t_2v)+f((t_1-t_2)\,v)\\ &= s_{t_2}f(v)+s_{t_1-t_2}f(v)=(s_{t_2}+s_{t_1-t_2})\,f(v)\end{aligned}$$

and $s_{t_1}=s_{t_2}+s_{t_1-t_2}>s_{t_2}$, thus s_t is increasing respect to t. As $s_t=t$ for all $t\in\mathbb{Q}$ we conclude that $s_t=t$ for all $t\in\mathbb{R}$; equivalently, $\mathbf{f(tu)=tf(u)}$ **for all** $\mathbf{t\in\mathbb{R}}$

6. From the last two steps, we infer that f is a linear map with trivial kernel. If $\{e_1,e_2,e_3\}$ constitutes the ordered canonical basis of $\mathbb{R}^3$, the following equalities hold:

$$f(e_1)=f(e_2)\times f(e_3)\quad;\quad f(e_2)=f(e_3)\times f(e_1)\quad;\quad f(e_3)=f(e_1)\times f(e_2)$$

Therefore $\{f(e_1),f(e_2),f(e_3)\}$ is a orthogonal basis of $\mathbb{R}^3$. Furthermore, from the equalities

$$\|f(e_1)\|=\|f(e_2)\|\|f(e_3)\|\quad;\quad \|f(e_2)\|=\|f(e_3)\|\|f(e_1)\|\quad;\quad \|f(e_3)\|=\|f(e_1)\|\|f(e_2)\|$$

it is obtained that $f(e_1),f(e_2),f(e_3)$ are unitary vectors, hence the associate matrix Q of f is orthogonal. As $\det Q=f(e_1)\cdot(f(e_2)\times f(e_3))=f(e_1)\cdot f(e_1)=1$, we also conclude that $\mathbf{f(v)=Qv}$ **where** Q **is an orthogonalmatrix of determinant 1.**

7. Conversely, let Q an arbitrary 3×3 orthogonal matrix. Define $f:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ as $f(u)=Qu$.

If u,v are arbitrary vectors in $\mathbb{R}^3$, then $Q(u\times v)\cdot Qu=(u\times v)\cdot u=0$ and $Q(u\times v)\cdot Qv=(u\times v)\cdot v=0$ (recall that an orthogonal transformation preserves scalar product, norms and angles). Then, $f(u\times v)=Q(u\times v)$ is parallel to $Qu\times Qv=f(u)\times f(v)$.

Let $\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3$ as before. As $f(e_1)=f(e_2\times e_3)$ is a unitary vector parallel to $f(e_2)\times f(e_3)$ and $f(e_1)\cdot(f(e_2)\times f(e_3))=\det Q=1$, we get $f(e_2)\times f(e_3)=f(e_1)=f(e_2\times e_3)$. Similarly, $f(e_3)\times f(e_1)=f(e_3\times e_1)$ and $f(e_1)\times f(e_2)=f(e_1\times e_2)$. The distributivity of cross product with respect to vector addition and the linearity of f, allows us to conclude from the previous equalities that $f(u\times v)=f(u)\times f(v)$ for all $u,v\in\mathbb{R}^3$.

8. From the last two steps, we finally conclude that a function $f:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ satisfies the equality $f(u\times v)=f(u)\times f(v)$ for all vectors u,v in $\mathbb{R}^3$ if and only $f(u)=0$ for all $u\in\mathbb{R}^3$ or there exists an orthogonal matrix Q of determinant 1 such that $f(u)=Qu$ for all $u\in\mathbb{R}$.

Problem 28 (Colombia). Let $p>2$ be an even number and let $g:\mathbb{R}\longrightarrow\mathbb{R}$ defined as $g(x)=x(1-x)^p+x^p(1-x)$.

Show that g attains its absolute maxima in exactly two values on the interval $[0,1]$

Solution. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined as $f(x)=g'\left(x+\frac{1}{2}\right)$. From the binomial theorem applied to the function g, it follows that

$$f(x)=\sum_{k=1}^{p-1}C(k)\left(\frac{1}{2}\right)^{p-k}x^k \tag{13}$$

where

$$C(k)=(-1)^k\binom{p}{k}+(-1)^{k+1}p\binom{p-1}{k}+(-1)^kp\binom{p-1}{k-1}+p\binom{p-1}{k}-p\binom{p-1}{k-1}-\binom{p}{k}$$

Note that if k is an even number, $C(k)=0$. If $k=2t-1$ is an odd number, then

$$C(k)=2\left[p\binom{p-1}{2t-1}-p\binom{p-1}{2t-2}-\binom{p}{2t-1}\right]=2t\left[\binom{p}{2t}-\binom{p}{2t-1}\right]$$

Therefore equation (13) can be simplified to get the following identity:

$$f(x)=\sum_{t=1}^{p/2}\left[\binom{p}{2t}-\binom{p}{2t-1}\right]\frac{t}{2^{p-2t-1}}\,x^{2t-1} \tag{14}$$

Given a positive integer $n>1$, the sequence of binomial coefficients $\left(\binom{n}{0},\binom{n}{1},\dots\binom{n}{n-1},\binom{n}{n}\right)$ is unimodal. Thus, there exists an integer $T>0$ such that $\binom{p}{2t}-\binom{p}{2t-1}>0$ for all $t<T$ and $\binom{p}{2t}-\binom{p}{2t-1}<0$ for all $t>T$. Consequently, $f(x)$ is a polynomial such that its sequence of coefficients has exactly one variation of sign; moreover, as f does not have monomials of even degree, the sequence of coefficients of $f(-x)$ has exactly one variation of sign. By Descartes's rule of signs, this implies that f has at most one positive root and at most one negative root. As $f(0)=0$, it follows that f has exactly three real roots.

Separating the linear monomial of f, we get $f(x)=x\left[\frac{1}{2^{p-3}}\left(\frac{p(p-1)}{2}-p\right)+x^2h(x)\right]$ where $h(x)$ is a polynomial. If $S(x)=\left[\frac{1}{2^{p-3}}\left(\frac{p(p-1)}{2}-p\right)+x^2h(x)\right]$, note that $S(0)=\frac{1}{2^{p-3}}\left(\frac{p(p-1)}{2}-p\right)>0$ (because $p\geq 4$). Therefore there exists an open neighbourhood V of 0 in which S is positive. On the other hand, $f(x)=xf(x)$ so there exist $a\in\left(-\frac{1}{2},0\right)$ and $b\in\left(0,\frac{1}{2}\right)$ such that $f(a)<0$ and $f(b)>0$. As $f\left(-\frac{1}{2}\right)=1$ and $f\left(\frac{1}{2}\right)=-1$, by the intermediate value theorem we guarantee the existence of one root in the interval $\left(-\frac{1}{2},0\right)$ and one root in the interval $\left(0,\frac{1}{2}\right)$. Consequently, f has exactly three real roots or equivalently, g has exactly three critical points: one in the open interval $\left(0,\frac{1}{2}\right)$, $x=\frac{1}{2}$ and one in the open interval $\left(\frac{1}{2},1\right)$.

As $g'\left(\frac{1}{2}\right)=\frac{p(p-3)}{2^{p-2}}>0$, the function g attains a local minimum in $x=\frac{1}{2}$. Besides $\left[\frac{1}{2},1\right]$ and $\left[0,\frac{1}{2}\right]$ are closed and bounded intervals, so g attains an absolute maximum in each of them. The graph of g in the coordinate plane is symmetric with respect to the vertical line $x=\frac{1}{2}$, so the maximum value of f in the closed intervals $\left[\frac{1}{2},1\right]$ and $\left[0,\frac{1}{2}\right]$ must be the same. This absolute maxima must be attained in a interior point of each interval, hence in a critical point (because $g(0)=g(1)=0<g\left(\frac{1}{2}\right)$. Consequently, g restricted to the interval $[0,1]$ attains its maximum value in exactly two points p,q (the two critical points of g different from $x=\frac{1}{2}$). As g only takes negative values outside the closed interval $[0,1]$, we conclude that $f(p)=f(q)$ is the global maximum of g. This completes the solution.

Problem 29 (Colombia). Let $f(x)=x^4+a_1x^3+a_2x^2+a_3x+a_4$ be a polynomial with real coefficients which have local maximum M and absolute minimum m.

Prove that

$$\frac{3}{10}\left(\frac{1}{4}a_1^2-\frac{2}{3}a_2\right)^2<M-m<3\left(\frac{1}{4}a_1^2-\frac{2}{3}a_2\right)^2$$

Solution. Let $f(a)=m_1,f(b)=m_2,a<b$, be points of the local minima and $f(c)=M,a<c<b$, the point of the local maximum. First we want to find the maximum and the minimum of the difference $M-m$ by a fixed $d=b-a$. Since a parallel translation does not change the difference $M-m$, we may assume that $a=m_1=0$ and $b=d$ without loss of generality. By our assumption the derivative and the polynomial are of the form

$$\begin{aligned}f'(x)&=4x(x-c)(x-d)\\ f(x)&=x^4+\frac{4}{3}(c+d)x^3+2cdx^2.\end{aligned}$$

Then $f(c)=-\frac{1}{3}c^4+\frac{2}{3}c^3d$ is the local maximum and $f(d)=-\frac{1}{3}d^4+\frac{2}{3}d^3c$ and $f(0)=0$ are two local minima.

The standard investigation shows that $\Delta_1=f(c)-f(0)$ increases and $\Delta_2=f(c)-f(d)$ decreases by increasing c from 0 to d. Thus $\Delta_1=f(c)-f(0)$ reaches its maximum when $c=d$ and $\Delta_2=f(c)-f(d)$ reaches its maximum when $c=0$. In both cases it is equal to $\frac{1}{3}d^4$. If $c=\frac{1}{2}d$ then $f(c)=\frac{1}{16}d^4$ and $f(d)=f(0)=0,\Delta_1=\Delta_2=\frac{1}{16}d^4$. This is the minimum for the difference $M-m$, because otherwise either Δ_1 or Δ_2 is greater than $\frac{1}{16}d^4$. Thus it is proved that

$$\frac{1}{16}d^4 \le M - m < \frac{1}{3}d^4. \tag{15}$$

Now we need to estimate d. It is the distance between the extreme roots of the polynomial $f'(x) = 4x^3 + 3a_1x^2 + 2a_2x + a_3$ if this polynomial has three distinct roots. Let D be the distance between the points of extremum of this polynomial. It is equal to the distance between roots of the polynomial $f''(x) = 12x^2 + 6a_1x + 2a_2$. Hence

$$D = \sqrt{\frac{1}{4}a_1^2 - \frac{2}{3}a_2}. \tag{16}$$

Note that $\frac{1}{4}a_1^2 - \frac{2}{3}a_2 > 0$, otherwise the polynomial $f(x)$ cannot have a local maximum. The graph of the polynomial $g(x) = 4x^3 - 3D^2x$ can be obtained by a parallel translation from the graph of the polynomial $f'(x) = 4x^3 + 3a_1x^2 + 2a_2x + a_3$ because $g(x)$ has the same distance D between extremes and the same leading coefficient 4. The distance between the extreme roots of the equation $g(x) = c$ decreases by increasing the absolute value of c until the equation has three distinct roots. Its maximum is $\sqrt{3}D$, where $c = 0$, its minimum is $\frac{3}{2}D$, where $c = \pm D^3$. This means that if $f'(x) = 4x^3 + 3a_1x^2 + 2a_2x + a_3$ has three distinct real roots then the distance between the extreme roots satisfies the following inequalities

$$\frac{3}{2}D < d \le \sqrt{3}D.$$

Taking in account (1) and (2) we obtain finally

$$\frac{3}{10}\left(\frac{1}{4}a_1^2 - \frac{2}{3}a_2\right)^2 < \frac{81}{256}\left(\frac{1}{4}a_1^2 - \frac{2}{3}a_2\right)^2 < M - m < 3\left(\frac{1}{4}a_1^2 - \frac{2}{3}a_2\right)^2$$

Problem 30. Let A be an $n \times n$ matrix with real coefficients. Define $\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$, I stands here for the unit $n \times n$ matrix. It is known that the series is convergent for every matrix A. We identify in a natural way $n \times n$ matrix with a point of $\mathbb{R}^{n^2}$, so exp maps $\mathbb{R}^{n^2}$ into itself. Prove or disprove the following statements.

(a) The set $\exp(\mathbb{R}^{n^2})$ is an open subset of $\mathbb{R}^{n^2}$.

(b) The set $\exp(\mathbb{R}^{n^2})$ is a closed subset of $\mathbb{R}^{n^2}$.

(c) The map $\exp\colon \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$ is differentiable at each point $A \in \mathbb{R}^{n^2}$, i.e. there exists a linear map $D\exp(A)\colon \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$ such that

$$\lim_{\|H\|\to 0} \frac{1}{\|H\|}\|\exp(A+H) - \exp(A) - D\exp(A)(H)\| = 0$$

and for each matrix A the map $D\exp(A)$ is a linear isomorphism of $\mathbb{R}^{n^2}$ onto itself.

Solution. It is easy to see that if $A\vec{v} = \lambda\vec{v}$ then $\exp(A)\vec{v} = e^{\lambda}\vec{v}$, more generally if $(A - \lambda I)^k\vec{v} = \vec{0}$, then $\big(\exp(A) - e^{\lambda}I\big)^k\vec{v} = \vec{0}$. Thus all eigenvalues of $\exp(A)$ are of this form (there are precisely n of them if counted with multiplicities). Therefore no matrix with an eigenvalue 0 is of the form $\exp(A)$. Obviously $e^{-n}I = \exp(-nI)$ and $\lim_{n\to\infty} e^{-n}I = 0 \in \mathbb{R}^{n^2}$. This way we proved that the image of exp is **not** a closed set. We shall prove that for $n > 1$ it is not open either. By straightforward calculation one proves that

$$\exp\left(\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

so $\exp\left(\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so for $n = 2$ the matrix $-I$ is of the form $\exp(A)$. Let $\varepsilon > 0$ and $B = \begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$. Suppose $B = \exp(A)$ for some real matrix A. If λ is an eigenvalue of A, then e^{λ} is an eigenvalue of $\exp(A)$, so $-1 = e^\lambda$. Therefore $\lambda = \pi i + 2m\pi$ for some integer m. Since $\lambda \notin \mathbb{R}$ the number $\bar{\lambda}$ is also an eigenvalue of the **real** matrix A. So A is diagonalisable. This implies that B is diagonalisable, too. This is a contradiction, so B cannot be written in the form $\exp(A)$, so $\exp(\mathbb{R}^4)$ is not open. Obviously this argument works for each $n \geq 2$, it is enough to notice that $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$.

Clearly the map exp is differentiable and

$$D(\exp)(A)(H) = H + AH + HA + A^2H + AHA + HA^2 + A^3H + A^2HA + AHA^2 + HA^3 + \cdots,$$

note that sometimes $AH \neq HA$. If the differential of the map $F\colon \mathbb{R}^m \to \mathbb{R}^m$ is an isomorphism at every point and depends continuously on the point then the image of F is an open subset of $\mathbb{R}^m$ (Implicit Function Theorem or Inverse Map Theorem). We know that in the case of exp it is not the case. Therefore there are matrices A such that $D(\exp)(A)$ is NOT an isomorphism. We may give an example although we do not have to.

Let $A_\alpha = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $A = \alpha C$. Clearly $C^2 = -I$. Therefore $A_\alpha^2 = -\alpha^2 I$, $A_\alpha^3 = -\alpha^3 C$, $A_\alpha^4 = \alpha^4 I, \ldots$ Using these formulas together with standard Taylor expansions of sine and cosine we get

$$D(\exp)(A_\alpha)(H) = \frac{1}{2}\Big[(\sin\alpha + \cos\alpha)H + \sin\alpha CH + \sin\alpha HC + (\sin\alpha - \cos\alpha)CHC\Big].$$

Thus $D\exp(A_\pi)(H) = -H + CHC = C^2H + CHC = C(CH + HC)$. It is easy to see that $CH + HC = 0$ iff H is a symmetric matrix whose trace is 0. This means that the dimension of the kernel of $D(\exp)(A_\pi)$ is 2.

Remark. The problem is relatively easy. One of my colleagues told me that some time ago A.T.Fomenko wrote something about the exponential defined on real matrices. I did not have a chance to look into the paper. I looked into few books on ODE's in which Floquet theory is described. In no such book the authors say anything about exp defined on real matrices although it is an obvious place to say something about it; Coddington, Levinson for example avoid it by looking at B^2 instead of B. The result is rather unexpected: the differential of exp does not have to be an isomorphism.

One can restate the problem e.g.

For what real matrices B there exists a real matrix A such that $B = \exp(A)$? — it is somewhat harder that the problem above. The answer is: B is invertible and the Jordan blocks corresponding to real negative eigenvalues have even size and each such block is repeated an even number of times (this means in fact that negative reals are considered as non-real numbers so they come in pairs: a number together with its conjugate, of course this is true also for the Jordan blocks). The interior of the image of exp consists of all matrices whose all real eigenvalues are positive. Each matrix B of the form $\exp(A)$ may be approximated by a matrices without negative real eigenvalues, it is enough to change B so that negative real eigenvalues will be replaced with non-real ones.

Instead of 1.3 one could ask: *Find the rank of the linear map $D(\exp)(A)$ for any real matrix A* (maybe only for $n = 2$).

Problem 31. Let $S^1 = \{z \in \mathbb{C} : \quad |z| = 1\}$ be the unit circle. A map $F\colon S^1 \to S^1$ is called a C^r map, $r = 0, 1, 2, \ldots$, of degree $k \in \mathbb{Z}$ iff there exists a C^r function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + k$ for all $x \in \mathbb{R}$ and $F(e^{2\pi i t}) = e^{2\pi i f(t)}$ for all $t \in \mathbb{R}$. Prove that there exits a sequence (F_n) of C^∞ maps of degree 1 such that for each nonnegative integer r the sequence $\big(f_n^{(r)}\big)$ is uniformly convergent to the identity and for no n there exists a continuous map $G : S^1 \to S^1$ such that $F_n = G \circ G$.

Solution. We shall show that it is possible to define F_n so that F_n will have precisely one periodic orbit of prime period $2(2n+1)$. Suppose that $F_n = G \circ G$. If a point $p \in S^1$ is m–periodic for G, i.e. $\underbrace{G \circ G \circ \cdots \circ G}_{m}(p) = p$, then it m–periodic for F_n if m is odd and it is $\frac{m}{2}$–periodic for F_n if m is even. In both cases the number of periodic points for G and for F_n is the same while the number of periodic orbits of G may be smaller then those of F_n. Since F_n has one periodic orbit, G must also have one periodic orbit consisting of $2(2n+1)$ points, so the prime period is $2(2n+1)$. But in such a situation F_n has 2 periodic orbits, the prime period of each of them is $2n+1$, a contradiction. To complete the proof it suffices to define f_n:

$$f_n(x) = x + \frac{1}{2(2n+1)} + \left(\frac{1}{2(2n+1)}\right)^{2n} \sin^2 2(2n+1)x\ .$$

Clearly $f_n(x+1) = f(x)+1$, so F_n is well defined C^∞ map of degree 1. One can see that $1 = e^{2\pi i 0}$ is a periodic point of prime period $2(2n+1)$. The function f_n maps the segment $\left[0, \frac{1}{2(2n+1)}\right]$ onto the segment $\left[\frac{1}{2(2n+1)}, \frac{2}{2(2n+1)}\right]$, this one onto $\left[\frac{2}{2(2n+1)}, \frac{3}{2(2n+1)}\right]$, etc.

Remark. The problem is rather easy, known to people interested in generalisations of the Implicit Function Theorem to infinitely dimensional spaces.

One can show that all arbitrarily close to the identity C^∞ diffeomorphisms of the real line are „squares" if multiplication is replaced with composition.

Problem 32 (Roman Karasev). Let $f(x)$ be a polynomial with real coefficients of degree n having n distinct real roots $x_1, x_2, \ldots, x_n$. Prove that for any non-negative integer $k \le n-2$

$$\sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} = 0.$$

Solution. Consider the rational function $r(z) = \dfrac{z^k}{f(z)}$ of complex variable z. Note that by the degree reasons $r(z) \le \dfrac{C}{|z|^2}$ for some constant C and large enough $|z|$. So for the integral over the circumference S_R of radius R and centre at the origin we have

$$\left| \oint_{S_R} r(z)dz \right| \le \frac{2\pi C}{R}$$

for large enough R. On the over hand, for large enough R we have the equality

$$\oint_{S_R} r(z)dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{x_i} r(z) = 2\pi i \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)}.$$

Thus for large enough R we have

$$\left| \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} \right| \le \frac{C}{R},$$

and going to the limit $R \to +\infty$ we obtain that the sum considered equals 0.

Problem 33 (J. Rodrigo). Let A_{3x3} be a symmetrical matrix with distinct eigenvalues, let V be the set of matrices B_{3x3} such that they commute with A, $BA = AB$. Show that the matrices of V are also symmetrical. Is the result true if A has multiple eigenvalues?

Solution. The matrices I, A, A^2 commute with A. We see that V has dimension 3: The dimension of V is the dimension of the matrices that commute with $J = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, where J is the canonical Jordan matrix of A (it's just a change of basis). The matrices commuting with J are the matrices $\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}$ such that $\begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This is equivalent to $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a-c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b-c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c-a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j \\ k \\ l \\ m \\ n \\ o \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Since the three eigenvalues are different, the rank of the coefficient matrix is 6, so the dimension of V is 3. Then, $V = L(I, A, A^2)$. Since I, A, A^2 are symmetrical, all the matrices of V are symmetrical. (This can be generalised to any dimension n).

The result is not true if A has multiple eigenvalues. As an example, $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ commutes with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and B is not symmetrical.

Problem 34 (Dierk Schleicher, Bonn; modified by G. Kós). Let $\gamma\colon (0,1) \to \mathbb{R}^2$ be an injective continuous curve. We say that γ satisfies the *triangle condition* if for every $t \in [0,1]$ and every three disjoint sequences $t_n, t'_n, t''_n \in [0,1]$ of parameters converging to t, the largest angle of triangle Δ_n with vertices $\gamma(t_n), \gamma(t'_n), \gamma(t''_n)$ converges to 180°.

a) Does all continously differentiable curves satisfy the triangle condition?

b) Are all curves satisfying the triangle condition differentiable?

c) Assume that γ is differentiable and $|\dot\gamma| = 1$ everywhere. Prove that γ satisfies the triangle condition if and only if $\dot\gamma$ is continuous.

Sketch of solution. a) No. A V-shaped polyline can be parametrised such that it is continuously differentiable. For example, $\gamma(t) = \left((2t-1)^3; |2t-1|^3\right)$ is continuously differentiable, but it does not satisfy the triangle property.

b) No. Take an increasing function $g : [0,1] \to \mathbb{R}$ which is not everywhere differentiable and set $\gamma(t) = (g(t), 0)$. The shape of the curve is a single line segment, so the triangle condition is obvious, but the curve is not differentiable.

c) First, assume that $\dot\gamma$ is continuous. If (t_n), (t'_n) are disjoint sequences converging to a certain t, then Cauchy's theorem shows that the direction of the line between $\gamma(t_n)$ and $\gamma(t'_n)$ converges to $\dot\gamma(t)$. This implies the triangle condition.

Next, assume that $\dot\gamma$ is not continuous at a point t. The set of accumulation points of $\dot\gamma$ at t is connected, so it is an arc of the unit circle or it is the complete circle. So there exist (disjoint) sequences (t_n) and (t'_n) such that $\lim \dot\gamma(t_n)$ and $\lim \dot\gamma(t'_n)$ exist and their angle is in $(0, \pi)$. Now take two sequences, (u_n) and (u'_n) such than the directions $(\gamma(t_n), \gamma(u_n)$ and $(\gamma(t'_n), \gamma(u'_n)$ converge to $\lim \dot\gamma(t_n)$ and $\lim \dot\gamma(t'_n)$, respectively. Then the sequences t_n and t'_n, together with either u_n or u'_n, fail the triangle condition.

Problem 35 (Dierk Schleicher, Bonn). Let $\gamma\colon [0,\infty) \to \mathbb{R}^2$ be a continuously differentiable curve with $\gamma(t) = (t, 0)$ for $t \geq 1$ and $t = 0$. Suppose that γ is injective and disjoint from all of its $(0,n)$-translates for $n \in \mathbb{Z}$. Is there an upper bound on the algebraic winding number of γ around the point $(0,1)$?

Note: the algebraic winding number of γ is defined as the integral $\frac{1}{2\pi}\int_0^\infty d\arg(\gamma(t) - (0,1)$ (this is the (fractional) number of times the curve γ turns around $(0,1)$ before landing at $(0,0)$).

Sketch of solution. Start with the curve $\gamma(t) = (t+2, 0)$ and apply a sequence of homotopies to the entire plane. These homotopies will keep disjoint curves disjoint.

The first homotopy is the identity for all (x,y) with $x \leq 1 or x \geq 3$, and maps $(x,y) \to (x, y + 1.5s|x-1|)$ for $s \in [0,1]$ (it raises a neighbourhood of the points with $x = 1$).

The second homotopy is similar, but shifts all points with y-coordinate near 0.5 by 2 units to the left, and does the same for points with y-coordinate near $1.5, 2.5, \cdots - 0.5, -1.5, \ldots$.

The third homotopy is similar to the first: $(x, y) \to (x, y - 1.5s|x+1|)$ (it lowers points with x-coordinate near -1).

The fourth homotopy is the inverse of the second.

Composing these four homotopies in order, and repeating this process any finite number of times, applied to the initial curve γ yields a curve with the desired properties.

Problem 36 (Comenius University, Bratislava). Let $x, y, z \in \mathbb{R}$ such that $xyz = 1$. Find the maximal and minimal value of the term

$$\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx}.$$

Solution.

$$\begin{aligned}
&\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx} = \\
&= \frac{z}{z(1+x+xy)} + \frac{xz}{xz(1+y+yz)} + \frac{1}{1+z+zx} = \\
&= \frac{z}{z+zx+1} + \frac{xz}{xz+1+z} + \frac{1}{1+z+zx} = 1,
\end{aligned}$$

so the minimum and maximum is equal to 1.

Problem 37 (Comenius University, Bratislava). Let A, B, C be angles of triangle. Find the infimum and supremum (maximal and minimal value if they exist) of the term

$$\cos A + \cos B + \cos C.$$

Solution. We know that $A + B + C = \pi$. Then we obtain

$$\begin{aligned}
\cos A + \cos B + \cos C &= \cos A + \cos B + \cos(\pi - (A+B)) = \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \left(2\cos^2\frac{A+B}{2} - 1\right) = \\
&= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1 = \\
&= 4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} + 1.
\end{aligned}$$

Because A, B, C are angles of triangle we know that $0 < A, B, C < \pi$. Then we obtain that

$$\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} > 0$$

and it is easy to prove that the term $\cos A + \cos B + \cos C$ has infimum 1 and no minimum. On the other hand we have

$$\cos A + \cos B + \cos C = 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right) + 1 \le$$

$$\le 2\sin\frac{C}{2}\left(1 - \sin\frac{C}{2}\right) + 1 = 2\left[-\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2 + \frac{1}{4}\right] + 1 \le \frac{3}{2}.$$

The equality holds true for A, B, C such that $\cos\frac{A-B}{2} = 1$ and $\sin\frac{C}{2} = \frac{1}{2}$. It means that $A = B = C = \frac{\pi}{3}$. Then supremum and maximum of the term $\cos A + \cos B + \cos C$ is $\frac{3}{2}$.

Problem 38 (Baidilda Tureshbayev). Let $f(x) = a_1\cos x + \cdots + a_5\cos 5x$, where $a_1, \ldots, a_5$ are real numbers, such that $\forall x \in \mathbb{R}\ |f(x)| \leqslant 1$. Prove inequality

$$|f^{(10)}(x)| \leqslant 3 \cdot 5^{10} \quad \forall x \in \mathbb{R}.$$

Solution. We consider Fourier series of the function $f(x)$:

$$f(x) = \frac{1}{\pi}\int_{\pi}^{\pi} D_5(t-x)f(t)dt,$$

where

$$D_5(t) = \frac{1}{2} + \sum_{k=1}^{5}\cos kt$$

is Dirichet's kernel.

Then we have

$$|f^{(10)}(x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}\left|D_5^{(10)}(t-x)\right| |f(t)|dt \leqslant$$

$$\leqslant 2\max_{t\in[\pi;\pi]}\sum_{k=1}^{5} k^{10}|\cos k(t-x)| \leqslant$$

$$\leqslant 2\sum_{k=1}^{5} k^{10} \leqslant 3\cdot 5^{10}.$$

Problem 39 (Baidilda Tureshbayev).

Let $f(x) = ax^2 + bx + c$, $a \neq 0, b, c$ — real numbers. Denote

$$M = \{x \in \mathbb{R} : \ |f(x)| \leqslant 1\}.$$

Prove following inequality

$$|M| \leqslant \frac{2\sqrt{2}}{\sqrt{|a|}}$$

Solution.

We may assume that $a > 0$. We denote quantity $|M|$ by μ.

Case 1: Equation $f(x) = -1$ has a root. Then there are x_1, x_2, x_3, x_4 such that

$$f(x_1) = f(x_4) = 1, \quad f(x_2) = f(x_3) = -1$$

$$x_1 < x_2 \leqslant x_3 < x_4.$$

Therefor $M = [x_1; x_2] \cup [x_3; x_4]$ and $|x_2 - x_1| = |x_4 - x_3| = \frac{\mu}{2}$. We consider only x_1, x_2, x_4, then

$$\begin{cases} ax_1^2 + bx_1 + c = 1, \\ ax_2^2 + bx_2 + c = -1, \\ ax_3^2 + bx_3 + c = 1 \end{cases} \Rightarrow a = \frac{\begin{vmatrix} 1 & x_1 & 1 \\ 1 & x_2 & -1 \\ 1 & x_3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}} = \frac{2}{(x_2 - x_1)(x_3 - x_1)} \Rightarrow$$

$$\Rightarrow |a| \leqslant \frac{2}{\frac{\mu}{2}\cdot\frac{\mu}{2}} = \frac{8}{\mu^2} \Rightarrow \mu \leqslant \frac{2\sqrt{2}}{\sqrt{|a|}}.$$

Case 2: Equation $f(x) = -1$ has no roots but equation $f(x) = 1$ has roots x_1, x_2.

Let $x_3 = \frac{x_1+x_2}{2}$, then $(|f(x_3)| < 1)$

$$\begin{cases} ax_1^2 + bx_1 + c = 1 \\ ax_2^2 + bx_2 + c = 1 \\ ax_3^2 + bx_3 + c = f(x_3) \end{cases} \Rightarrow a = \frac{\begin{vmatrix} 1 & x_1 & 1 \\ 1 & x_2 & 1 \\ 1 & x_3 & f(x_3) \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}} = \frac{f(x_3) - 1}{(x_3 - x_2)(x_3 - x_1)} \Rightarrow$$

$$\Rightarrow |a| \leqslant \frac{2}{|x_3 - x_2| \cdot |x_3 - x_1|} \leqslant \frac{8}{\mu^2} \Rightarrow \mu \leqslant \frac{2\sqrt{2}}{\sqrt{|a|}}.$$

Problem 40 (Baidilda Tureshbayev).
Let $x \geqslant 1$ be a real number. Prove that there are exist natural numbers

$$m, n, k \in [x; x + 8\sqrt{x}]$$

such that

$$\ln m, \ln n, \ln k$$

are $\mathbb{Q}$-linear dependent.
Solution: $n^2, n(n+1), (n+1)^2$.

Problem 41 (Baidilda Tureshbayev). Let

$$\sum_{n=1}^{\infty} a_n \cos nx$$

be a convergence series on the interval $[\alpha; \beta]$. Prove that $a_n \to 0$ as $n \to \infty$.

Problem 42 (Javier ...). Let f be a function with derivatives of any order in $(0, \infty)$ and such that there exists $\lim_{x\to\infty} f^{(n)}(x)$ for every n. Show that, if f has an horizontal asymptotic line in ∞, then $\lim_{x\to\infty} f^{(n)}(x) = 0$ for every natural number n. Is it the counterpart of this statement true?

Solution. Since f has an horizontal asymptotic line in ∞, $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$. Then, $\lim_{x\to\infty} \big(f(x+1) - f(x)\big) = \lim_{x\to\infty} f(x+1) - \lim_{x\to\infty} f(x) = l - l = 0$. On the other hand, applying the mean value theorem of Lagrange, $f(x+1) - f(x) = f'(\alpha)$ with $\alpha \in (x, x+1)$, so $\lim_{x\to\infty} \big(f(x+1) - f(x)\big) = \lim_{x\to\infty} f'(x)$, and then $\lim_{x\to\infty} f'(x) = 0$.

Assume that $\lim_{x\to\infty} f^{(n-1)}(x) = 0$. Hence $f^{(n-1)}$ has an horizontal asymptotic line in ∞, and then applying the previous paragraph we have that $\lim_{x\to\infty} f^{(n)}(x) = \lim_{x\to\infty} \big(f^{(n-1)}(x)\big)' = 0$.

The counterpart of this statement is not true: the function $f(x) = \sqrt{x}\sin x^{1/3}$ has derivatives of any order in $(0, \infty)$ and satisfies that $f'(x) = \frac{1}{2\sqrt{x}} \sin x^{1/3} + \frac{1}{3}\frac{\sqrt{x}}{x^{2/3}} \cos x^{1/3} \to 0$ as $x \to \infty$, and then $\lim_{x\to\infty} f^{(n)}(x) = 0$ for every $n \in \mathbb{N}$, but has not an horizontal asymptotic line in ∞, since $\lim_{x\to} f(x)$ doesn't exist.

Problem 43 (Javier ...). With the same conditions than in the problem 42, show that if f has a no horizontal asymptotic line in ∞, then $\lim_{x\to\infty} f'(x) = m \neq 0$, $\lim_{x\to\infty} f^{(n)}(x) = 0$, for every $n \in \mathbb{N}$, $n \geq 2$.

Is the reciprocal true?

Solution. As f has a no horizontal asymptotic line in ∞, $\lim_{x\to\infty} \big(f(x) - (mx+n)\big) = 0$ for some $m \neq 0$, $n \in \mathbb{R}$. Consequently, $g(x) = f(x) - (mx+n)$ has an horizontal asymptotic line in ∞, and then by problem 42 we have that $\lim_{x\to\infty} g'(x) = \lim_{x\to\infty} \big(f'(x) - m\big) = 0$, so $\lim_{x\to\infty} f'(x) = m \neq 0$. Moreover $f'(x)$ has an horizontal asymptotic line in ∞, so we can apply again the problem 42 and we obtain that $\lim_{x\to\infty} \big(f'(x)\big)^{(n)} = \lim_{x\to\infty} f^{(n+1)}(x) = 0$ for every $n \geq 1$, so $\lim_{x\to\infty} f^{(n)}(x) = 0$ for every $n \geq 2$.

The reciprocal also is false in this case: the function $f(x) = \sqrt{x}\sin x^{1/3} + x$ verifies that $f'(x) = \big(\sqrt{x}\sin x^{1/3}\big)' + 1 \to 1 \neq 0$ as as $x \to \infty$, $f^{(n)}(x) = \big(\sqrt{x}\sin x^{1/3}\big)^{(n)} \to 0$ as $x \to \infty$ for $n \geq 2$, but f hasn't a no horizontal asymptotic line in ∞, since $m = \lim_{x\to\infty} \frac{\sqrt{x}\sin x^{1/3} + x}{x} = 1$, but $n = \lim_{x\to\infty} \big(\sqrt{x}\sin x^{1/3} + x - x\big) = \lim_{x\to\infty} \sqrt{x}\sin x^{1/3}$, and this last limit doesn't exist.

Problem 44 (Alexander Fomin). Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients. Prove that its graph has an axis of symmetry if and only if $f'(-a/4) = 0$.
Proof. It is easy to see that an axis of symmetry can be only vertical ($x = p$). It takes place if and only if the function $g(x) = f(x+p)$ is even. Using the formula $f(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2}(x-p)^2 + \frac{f'''(p)}{6}(x-p)^3 + (x-p)^4$ we obtain $g(x) = f(p) + f'(p)x + \frac{f''(p)}{2}x^2 + \frac{f'''(p)}{6}x^3 + x^4$. The function g is even if and only if $f'''(0) = f'(0) = 0$, i.e. $p = -a/4$ and $f'(-a/4) = 0$.

Problem 45 (Alexander Fomin). Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ be a polynomial with real coefficients. Prove that its graph has a centre of symmetry if and only if $15ab = 4a^3 + 25c$.
Proof. A point $(p;q)$ is a centre of symmetry if and only if the function $g(x) = f(x+p) - q$ is odd. Using the formula $f(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2}(x-p)^2 + \frac{f'''(p)}{3!}(x-p)^3 + \frac{f^{IV}}{4!}(x-p)^4 + (x-p)^5$ we obtain $f(x) = (f(p) - q) + f'(p)x + \frac{f''(p)}{2}x^2 + \frac{f'''(p)}{3!}x^3 + \frac{f^{IV}}{4!}x^4 + x^5$. The function $g(x)$ is odd if and only if $f(p) - q = f''(p) = f^{IV}(p) = 0$. It is possible if and only if $p = -a/5$, $q = f(-a/5)$ and $15ab = 4a^3 + 25c$.

Problem 46 (Valeriu Anisiu, Cluj-Napoca). Define the function $f : [1,\infty) \to [1,\infty)$ by $f(x) = 1 + \log x$ and denote by f_n the n-th iterate of the function f for $n \geq 1$ and $f_0(x) = x$.

Show that the series $\sum_{n=1}^{\infty} a_n$ is convergent, where $a_n = \dfrac{1}{f_0(n)f_1(n)\cdots f_{n-1}(n)}$.

Solution

Observe first that $a_n = f_n'(n)$.
It is easy to see that:
(1) $f(x) \leq -\ln(b) + bx$ for $0 < b < 1, x \geq 1$
(2) $f_n(x) \leq -\log(b)(1-b^n)/(1-b) + b^n x$ [by iterating (1)]
(3) $f_{n+1}(x) \leq f_n(x)$ for $x \geq 1$, $n \in \mathbb{N}$
(4) $f_n(x)$ is an increasing function of x
(5) $f_n'(x)$ is a decreasing function of x

The mean value theorem implies

$$f_n(n) - f_n(n-1) = f_n'(n-\theta) > f_n'(n)$$

where $\theta = \theta_n \in (0,1)$. One obtains:
(6) $a_n < f_n(n) - f_n(n-1) = (f_n(n) - f_{n-1}(n-1)) + (f_{n-1}(n-1) - f_n(n-1))$.
The sequence $f_n(n)$ is bounded (cf. (2) for $b = 1/2$), so the partial sums

$$\sum_{n=2}^{N} (f_n(n) - f_{n-1}(n-1))$$

are also bounded ($N \in \mathbb{N}$).
Using (6), in order to show that the series $\sum_n a_n$ is convergent it is sufficient to show that

$$\sum_{n>1} (f_{n-1}(n-1) - f_n(n-1)) < \infty$$

We shall estimate $e_n = f_n(n) - f_{n+1}(n)$. Because $0 \leq x - f(x) \leq \frac{1}{2}(x-1)^2$, we obtain:

$$0 \leq e_n \leq \frac{1}{2}(f_n(n) - 1)^2$$

Using now (2) with $b = 1 - n^{-2/3}$ we get:

$$e_n \leq u_n = \frac{1}{2}\left(-\log(b)(1-b^n)/(1-b) + nb^n - 1\right)^2$$

It is not difficult to see that the series $\sum_n u_n$ is convergent because u_n is equivalent to $\frac{1}{8}n^{-4/3}$.

Problem 47 (Valeriu Anisiu, Cluj-Napoca). Let $\mathcal{U}_n$ be the set of all $n \times n$ matrices with entries in the set $\{0,1\}$ and $\mathcal{V}_n$ the set of the matrices in $\mathcal{U}_n$ having an odd determinant.

Show that for $n \geq 2$, $\frac{2}{7} < \text{card}(\mathcal{V}_n)/\text{card}(\mathcal{U}_n) \leq \frac{3}{8}$.

Solution

We shall identify $\mathcal{U}_n$ with $\mathcal{M}_n(\mathbb{Z}_2)$, the set of $n \times n$ matrices over the field $\mathbb{Z}_2$.

Obviously $\text{card}(\mathcal{U}_n) = 2^{n^2}$.

A matrix $A \in \mathcal{M}_n(\mathbb{Z}_2)$ belongs to $\mathcal{U}_n$ if and only if it is invertible in $\mathcal{M}_n(\mathbb{Z}_2)$ and this is equivalent with the fact that its columns are linearly independent (over $\mathbb{Z}_2$).

In order to compute the number of invertible elements of the ring $\mathcal{M}_n(\mathbb{Z}_2)$, observe that for first column of such a matrix there are $2^n - 1$ possibilities (only a null column has to be excluded); the second column must not be a multiple of the first, so there are $2^n - 2$ possibilities. The third column must not be a linear combination of the first two columns, so there are $2^n - 2^2$ possibilities, and so on.

One obtains that the number of the invertible elements of the ring $\mathcal{M}_n(\mathbb{Z}_2)$ is $\text{card}(\mathcal{V}_n) = (2^n - 1)(2^n - 2^1)(2^n - 2^2)\cdots(2^n - 2^{n-1})$.

Hence, $\text{card}(\mathcal{V}_n)/\text{card}(\mathcal{U}_n) = (1 - 1/2)(1 - 1/4)(1 - 1/8)\cdots(1 - 1/2^n)$.

For $n \geq 2$, one has $\text{card}(\mathcal{V}_n)/\text{card}(\mathcal{U}_n) \leq (1 - 1/2)(1 - 1/4) = 3/8$.

On the other hand, $\text{card}(\mathcal{V}_n)/\text{card}(\mathcal{U}_n) > \prod_{n=1}^{\infty}(1 - 2^{-n})$.

Using the inequality $\prod_n(1 + x_n) > 1 + \sum x_n$ which is valid for $-1 < x_n < 0$, we obtain:

$\prod_{n=1}^{\infty}(1 - 2^{-n}) = (1 - 1/2)(1 - 1/4)(1 - 1/8)\prod_{n=4}^{\infty}(1 - 2^{-n}) >$
$\frac{21}{64}(1 - \sum_{n=4}^{\infty} 2^{-n}) = \frac{21}{64} \cdot \frac{7}{8} > \frac{2}{7}$.

Problem 48 (Artur Michalak, Poznan). Let $\mathcal{K}$ be the set of all nonempty subsets of $\{1, 2, \ldots, n\}$ equipped with the Hausdorff metric d_H given by

$$d_H(A, B) = \max\{\max_{a \in A} \min_{b \in B} |a - b|, \max_{b \in B} \min_{a \in A} |a - b|\}$$

for every $A, B \in \mathcal{K}$. Find the smallest number of elements of a $(1 + \varepsilon)$-net in $\mathcal{K}$, for $0 < \varepsilon < 1$. (A δ-net in $\mathcal{K}$ is a family $\mathcal{F}$ of elements of $\mathcal{K}$ such that for every $A \in \mathcal{K}$ there exists $B \in \mathcal{F}$ with $d_H(A, B) < \delta$.)

Solution. For $n = 3k$ and $n = 3k - 1$ we put $T = \{2 + 3s : s = 0, 1, \ldots, k - 1\}$. For $n = 3k - 2$ we put $T = \{1 + 3s : s = 0, 1, \ldots, k - 1\}$. It is clear that for every $a \in \{1, \ldots, n\}$ there exists a unique element $\varphi(a)$ in T such that $|a - \varphi(a)| \leqslant 1$. First, we show that $d_H(A, \varphi(A)) \leqslant 1$ for every $A \in \mathcal{K}$. For $a \in A$ we have $\min_{b \in \varphi(A)} |a - b| \leqslant |a - \varphi(a)| \leqslant 1$. For $b \in \varphi(A)$ there exists $a \in A$ such that $b = \varphi(a)$. Then $\min_{c \in A} |c - b| \leqslant |a - \varphi(a)| \leqslant 1$. Consequently, $d_H(A, \varphi(A)) \leqslant 1$. Let $\mathcal{F}$ be the family of all nonempty subset of T. It is clear that $\mathcal{F}$ has $2^k - 1$ elements. Moreover, it is a $(1 + \varepsilon)$-net in $\mathcal{K}$.

Let $\mathcal{H}$ be any $(1 + \varepsilon)$-net in $\mathcal{K}$. Let $\mathcal{P} = \{\varphi(A) : A \in \mathcal{H}\}$. It is clear that $\mathcal{P} \subset \mathcal{F}$. Let $A \in \mathcal{F}$. Then there exists $B \in \mathcal{H}$ such that $d_H(A, B) \leqslant 1$. Hence, $\{a - 1, a, a + 1\} \cap B \neq \emptyset$ for every $a \in A$. But $\varphi(\{a - 1, a, a + 1\}) = a$ for every $a \in T$. Consequently, $\varphi(B) = A$. This shows that $\mathcal{P} = \mathcal{F}$. Consequently, $\mathcal{H}$ has at least $2^k - 1$ elements.

Problem 49 (Artur Michalak, Poznan). Find minimum and maximum of the function

$$f(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} 2x_i x_j$$

on the unit sphere $x_1^2 + \cdots + x_n^2 = 1$ in $\mathbb{R}^n$.

Solution. f is a quadratic form. From standard facts from the linear algebra it follows that there exists an othonormal base $g_1, \ldots, g_n$ in $\mathbb{R}^n$ such that

$$f(\sum_{i=1}^{n} x_i g_i) = \sum_{i=1}^{n} \lambda_i x_i^2,$$

where λ_i are eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

and $S = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 1\} = \{\sum_{i=1}^n y_i g_i : y_1^2 + \cdots + y_n^2 = 1\}$. Consequently, $\min f(S) = \min_{1\leqslant i\leqslant n} \lambda_i$ and $\max f(S) = \max_{1\leqslant i \leqslant n} \lambda_i$. Then it is enough to find λ_i. We have

$$\det \begin{bmatrix} -x & 1 & \cdots & 1 \\ 1 & -x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -x \end{bmatrix} = \det \begin{bmatrix} -x+n-1 & 1 & \cdots & 1 \\ -x+n-1 & -x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -x+n-1 & 1 & \cdots & -x \end{bmatrix}$$

$$= (-x+n-1)\det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & -x-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -x-1 \end{bmatrix} = (-x+n-1)(-x-1)^{n-1}.$$

Consequently, $\min f(S) = -1$ and $\max f(S) = n - 1$.

Problem 50 (Djordje Milicevic, Princeton). Let $U \subset \mathbb{C}$ be the set of complex roots of unity ($U = \{z \in \mathbb{C} : z^n = 1 \text{ for some } n \in \mathbb{N}$). Prove or disprove: For every positive integer $N > 0$, there is an $M = M(N)$, with the property that whenever f and g are two polynomials with rational coefficients of degree at most N such that

$$\#|f^{-1}(U) \cap g^{-1}(U)| > M,$$

we must have $f^r = g^s$ for some $r, s \in \mathbb{N}$.

Alternative formulation: Let f and g be two polynomials with rational coefficients of degree at most N such that

$$\#|f^{-1}(U) \cap g^{-1}(U)| \gg N^{5+\epsilon}.$$

Prove that $f^r = g^s$ for some $r, s \in \mathbb{N}$.

Solution. The answer is yes. Pick a $z \in f^{-1}(U) \cap g^{-1}(U)$; then

$$f(z) = \zeta_n^\alpha \quad \text{and} \quad g(z) = \zeta_n^\beta \tag{1}$$

for a certain n^{th} primitive root of unity $\zeta_n = e^{2\pi i/n}$, $0 \leqslant \alpha, \beta < n$ and $(\alpha, \beta, n) = 1$. Note that, as $\#|f^{-1}(U) \cap g^{-1}(U)| > M$, n may eventually be assumed as large as needed by taking a large M (as there are only $\varphi(k)$ primitive k^{th} roots of unity, and $\sum_{k\leqslant n} \varphi(k) \ll n^2$, it suffices to take $M \gg Nn^2_{\text{req'd}}$). In fact, we really need *only one* z, but with large n.

This means that the rational polynomial

$$\Delta = f^\beta - g^\alpha$$

has a complex (algebraic!) root z. Now, for any $0 < j < n$, $(j, n) = 1$, we can act on (1) by the Galois automorphism $\sigma_j \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ extending the automorphism $\zeta_n \mapsto \zeta_n^j$ of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. As f and g are rational polynomials, this will yield

$$f(\sigma_j(z)) = \zeta_n^{j\alpha} \quad \text{and} \quad g(\sigma_j(z)) = \zeta_n^{j\beta};$$

ie. $\varphi(n)$ $\sigma_j(z)$'s (which are clearly all different) are all roots of Δ. If we had

$$\varphi(n) > N \max(\alpha, \beta),$$

the problem would thus be solved.

However, given that we have no control over n, it is conceivable that $(\alpha, \beta) \in [0, n)^2$ might both be very large compared to $\varphi(n)$. Note, however, that on the other hand from (1) we also have

$$f^m(z) = \zeta_n^{m\alpha} \quad \text{and} \quad g^m(z) = \zeta_n^{m\beta},$$

for an arbitrary $0 < m < n$. By dividing $[0, n)^2$ into $\ll n$ squares of size $\ll \sqrt{n}$ and an application of the Pigeonhole Principle, we can ensure that $f^m(z) = \zeta_n^{\alpha'}$ and $g^m(z) = \zeta_n^{\beta'}$ with $(0,0) \neq (\alpha', \beta') = (m\alpha \bmod n, m\beta \bmod n) \ll \sqrt{n}$, so that, instead of Δ, we can consider the polynomial

$$\tilde{\Delta} = f^{\beta'} - g^{\alpha'}$$

of degree $\ll N\sqrt{n}$.

Now, from $\varphi(n)/n = \prod_{p|n}(1 - 1/p) \gg_\epsilon \prod_{p|n} 1/p^\epsilon \geqslant 1/n^\epsilon$, we have the standard lower bound $\varphi(n) \gg_\epsilon n^{1-\epsilon}$. So, for large enough n (explicitly, for $n \gg N^{2+\epsilon}$), the polynomial $\tilde{\Delta}$ will have more zeroes than its degree. This means that $\tilde{\Delta} = 0$, which completes the proof.

Problem 51 (Djordje Milicevic, Princeton). Can the additive group $\mathbb{R}$ be written as a union of four proper sub-semigroups?

Solution. Yes. Indeed it is clear how one would devise such a partition of the additive group $\mathbb{R}^2$: we can simply divide it into four semi-open quadrants (which are certainly sub-semigroups), one open half-axis in each, and origin added to one of them at will.

Yet, $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as $\mathbb{Q}$-vector spaces (both have countable Hamel bases), and hence as additive groups. This completes the proof.

Problem 52 (Djordje Milicevic, Princeton). For a noncommutative group G, and an automorphism $\phi : G \to G$, write

$$G[\phi] = \{g \in G : \phi(g) = g^{-1}\}.$$

How large can $|G[\phi]|/|G|$ be?

Solution. $|G[\phi]|/|G|$ can be as large as 3/4, as shown by the automorphism of $D_8 = \{\tau^i\sigma^j : 0 \leqslant i < 2, 0 \leqslant j < 4\}$ ($\tau^2 = \sigma^4 = e$, $\tau\sigma = \sigma^3\tau$) given by $\phi(\tau) = \tau$, $\phi(\sigma) = \sigma^3$.

Now suppose that $|G[\phi]| > (3/4)|G|$, and take $x_0 \in G[\phi]$. Pick an arbitrary $y \in E = G[\phi] \cap x_0 G[\phi]$; here note that $|E| > |G|/2$. From $\phi(y) = y^{-1}$, $\phi(x_0) = x_0^{-1}$ and $\phi(x_0^{-1}y) = y^{-1}x_0$, we obtain that $x_0 y = y x_0$. This means that E is contained in the normalizer $N(x_0)$, but as $|E| > |G|/2$, we must have $N(x_0) = G$, i.e. x_0 must be in the center $Z(G)$. We have just proved that $G[\phi] \subset Z(G)$, so that $|Z(G)| > (3/4)|G|$; this is possible only if G is commutative.

Problem 53 (Djordje Milicevic, Princeton).

Given a group G, let $G(m)$ denote the subgroup generated by m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that then so is $G(\gcd(m, n))$.

Solution. Write $d = \gcd(m, n)$. It is easy to see that $\langle G(m), G(n) \rangle = G(d)$; hence, it will suffice to check commutativity for any two elements in $G(m) \cup G(n)$, and so for any two generators a^m and b^n. Consider their commutator $z = a^{-m}b^{-n}a^m b^n$; then the relations

$$z = (a^{-m}ba^m)^{-n}b^n = a^{-m}(b^{-n}ab^n)^m$$

show that $z \in G(m) \cap G(n)$. But then z is in the center of $G(d)$. Now, from the relation $a^m b^n = b^n a^m z$, it easily follows by induction that

$$a^{ml}b^{nl} = b^{nl}a^{ml}z^{l^2}.$$

Plugging in $l = m/d$ and $l = n/d$ yields $z^{(m/d)^2} = z^{(n/d)^2} = e$, but this implies that $z = e$ as well.

Problem 54 (Moubinool Omarjee, Paris). Let $A, B \in M_3(\mathbb{Z})$ such that

$$AB = \begin{pmatrix} 1 & 2k & k(2k+1) \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}, quad k \in \mathbb{N}.$$

Prove that there exists $C \in m_3(\mathbb{Z})$ such that $BA = C^k$.

Solution.

$$\begin{pmatrix} 1 & 2k & k(2k+1) \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^2 = M^k,$$

$$BA = A^{-1}(AB)A = A^{-1}M^kA = (A^{-1}MA)^k.$$

Problem 55 (Moubinool Omarjee, Paris). For a $P(x)$ real polynomial of degree $n-1$, consider the matrix $A = \left(P(i+j-1)\right)_{1\le i,j,\le n}$. Compute $\det(A)$.

Solution. $f \in L(R_{n-1}[X])$linear map of $E = R_{n-1}[X]$ given by $f(Q) = Q(x+1)$

Consider the basis Lagrange interpolation $(L_i)_{1\le i\le n}$ of E

$$L_i(x) = \prod_{j\neq i} \left(\frac{x-j}{i-j}\right)$$

$$P(x) = \sum_{i=1}^{n} P(i)L_i(x)$$

By induction $f^{j-1}(P) = \sum_{i=1}^{n} P(i+j-1)L_i$ where f^{j-1} iterate of f $j-1$ times, we have $\det(A) = \det_{(L_1,\dots,L_n)}(P, f(P), \dots, f^{n-1}(P))$

Let $g = f - Id$. By induction $\det(A) = \det_{(L_1,\dots,L_n)}(P, g(P), \dots, g^{n-1}(P))$

Let $(S_1, \dots, S_n)$ basis of $E = R_{n-1}[X]$ given by $S_1 = 1$, $S_k = \frac{(x-1)(x-2)\dots(x-k+1)}{(k-1)!}$, we have $g(S_1) = 0$, $g(S_{k+1}) = S_k$

Decompose P in the basis $(S_1, \dots, S_n)$, $P = \sum_{i=1}^{n} w_i S_i$ we get

$$\det(A) = \begin{vmatrix} w_1 & w_2 & & & w_n \\ w_2 & & & \dots & 0 \\ \dots & & \dots & & 0 \\ \dots & w_n & & & \dots \\ w_n & \dots & \dots & 0 & 0 \end{vmatrix} \det_{(L_1,\dots,L_n)}(S_1, \dots, S_n)$$

The matrix $(S_j(i))$ is triangular inferior with $S_i(i) = 1$

$$\det(A) = (-1)^{\frac{n(n-1)}{2}} w_n^n = (-1)^{\frac{n(n-1)}{2}} ((n-1)! a_{n-1})^n$$

where $a_n - 1$ is coefficient of x^{n-1} in P

Problem 56 (Moubinool Omarjee, Paris). Let $A \in M_n(R) \setminus \{0\}$, $u \in \mathbb{R}^n$, $||u|| = 1$. Prove that

$$(\det A)^2 \left(\frac{n-1}{tr(A^tA)}\right)^{n-1} \le ||Au||^2.$$

Solution. f morphism associate to A eith an orthonormal basis f^* adjoint of f

$g = f^* \circ f$ is self-adjoint there exist an orthonormal basis of eigenvector $c = (c_1, \dots, c_n)$ of g

$$||Au||^2 = (f(u), f(u)) = (u, g(u)) = \sum_{i=1}^{n} a_i (c_i, u)^2 \ge a_1 ||u||^2 = a_1$$

$a_1 \le \dots \le a_n$ eigenvalues corresponding $(c_1, \dots, c_n)$

If $a_1 = 0$ it is finish

If $a_1 = ||f(c_1)||^2 > 0$ use AM-GM

$$\frac{tr(g)}{n-1} = \frac{a_1 + \dots + a_n}{n-1} \ge (a_1 \dots a_n)^{\frac{1}{n-1}}$$

$$\det(g)\left(\frac{n-1}{tr(g)}\right)^{n-1} < \frac{a_1\dots a_n}{a_2\dots a_n} = a_1 \le ||Au||^2$$

Problem 57 (Moubinool Omarjee, Paris). $A, B \in M_n(\mathbb{R})$ with $rank(N) = 1$. Prove that

$$\det((A-B)(A+B)) \le (\det A)^2.$$

Solution. If $B = J = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & 0 \end{pmatrix}$ then $\det(A+B) = \det\begin{pmatrix} a_{11}+1 & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{1j} & & \vdots & a_{nn} \end{pmatrix}$ expand according to the first row $\det(A+B) = (a_{11}+1)\Delta_{11} + \sum_{j=2}^{n}(-1)^{j+1}a_{1j}\Delta_{1j} = \det(A) + \Delta_{11}$

Similarly $\det(A-B) = \det(A) - \Delta_{11}$

$$\det((A-B)(A+B)) \le (\det A)^2 - \Delta_{11}^2 \le (\det A)^2$$

If $rank(B) = 1$ then B is equivalent to J, $B = RJS$ where R, S invertiblle matrix let $\tilde{A} = R^{-1}AS^{-1}$ then $\det((A-B)(A+B)) = (\det R)^2(\det S)^2 \det((\tilde{A}+J)(\tilde{A}-J)) \le (\det R)^2(\det S)^2(\det \tilde{A})^2 = (\det A)^2$.

Problem 58 (Moubinool Omarjee, Paris). $A \in M_2(\mathbb{R})$, $\det A = 1$ but $A \ne \pm I_2$. For $X \in \mathbb{R}^2$ we denote $W(X) = \{A^n X,\ n \in \mathbb{N}\}$.

Prove that there exists $X \in \mathbb{R}^2 suchthat$W(X) is not bounded if and only if $|tr(A)| \ge 2$.

Solution. Characteristic polynomial of A is $p(x) = x^2 - (trA)x + 1$, discriminant $\Delta = (trA)^2 - 4$

If $|trA| > 2$, $p(x)$ has two eigenvalues one $b > 1$. X is eigenvector $W(X) = \{b^n X,\ m \in N\}$ bounded

If $|trA| = 2$, $p(x)$ has a root $b = \pm 1$ multiplicity 2. $A = bI_2 + N$ where N nilpotent since $A \ne I_2$. Choose X such that $NX \ne 0$. $W(X) = \{b^n X + nb^{n-1}NX\}$ not bounded

If $|trA| < 2$, $Sp(A) = \{e^{iq}, e^{-iq}\}$, A is similar to $\begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix}$ and $W(X)$ is bounded

Problem 59 (Miloš Milosavljević, Niš). Let $n \in N$. Find all the polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, of degree n, satisfying the following two conditions:

i) $\{a_0, a_1, \dots, a_n\} = \{0, 1, \dots, n\}$

ii) all the zeros of $P(x)$ are rational numbers.

Solution: Note that $P(x)$ does not have any positive zeros because $P(x) > 0$ for every $x > 0$. Thus, we can represent them in the form $-\alpha_i$, $i = \overline{1,n}$, where $\alpha_i \ge 0$. If $a_0 \ne 0$ then there is a $k \in N, 1 \le k \le n-1$, with $a_k = 0$, so using Viete's formulae we get

$$\alpha_1\alpha_2\dots\alpha_{n-k-1}\alpha_{n-k} + \alpha_1\alpha_2\dots\alpha_{n-k-1}\alpha_{n-k+1} + \dots + \alpha_{k+1}\alpha_{k+2}\dots\alpha_{n-1}\alpha_n = \frac{a_k}{a_n} = 0,$$

which is impossible because the left side of the equality is positive. Therefore $a_0 = 0$ and one of the zeros of the polynomial, say α_n, must be equal to zero. Consider the polynomial $Q(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \dots + a_1$. It has zeros $-\alpha_i$, $i = \overline{1, n-1}$. Again, Viete's formulae, for $n \ge 3$, yield:

$$\alpha_1\alpha_2\dots\alpha_{n-1} = \tfrac{a_1}{a_n} \qquad (1)$$

$$\alpha_1\alpha_2\dots\alpha_{n-2} + \alpha_1\alpha_2\dots\alpha_{n-3}\alpha_{n-1} + \dots + \alpha_2\alpha_3\dots\alpha_{n-1} = \tfrac{a_2}{a_n} \qquad (2)$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_{n-1} = \tfrac{a_{n-1}}{a_n}. \qquad (3)$$

Dividing (2) by (1) we get

$$\tfrac{1}{\alpha_1} + \tfrac{1}{\alpha_2} + \dots + \tfrac{1}{\alpha_{n-1}} = \tfrac{a_2}{a_1}. \qquad (4)$$

Using the inequality between the arithmetic and the harmonic mean, as well as (3) and (4) we have that:

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}}{n-1} \ge \frac{n-1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \dots + \frac{1}{\alpha_{n-1}}} = \frac{(n-1)a_1}{a_2},$$

i.e. $\frac{a_2a_{n-1}}{a_1a_n} \geq (n-1)^2$. Hence $\frac{n^2}{2} \geq \frac{a_2a_{n-1}}{a_1a_n} \geq (n-1)^2$, and finally $n = 3$. So, the only polynomials possibly satisfying (i) and (ii) are those of degree at most three. These polynomials can easily be found and they are $P(x) = x$, $P(x) = x^2 + 2x$, $P(x) = 2x^2 + x$, $P(x) = x^3 + 3x^2 + 2x$ and $P(x) = 2x^3 + 3x^2 + x$. □

Remark:

Let us observe that this solution does not really need the fact that all the zeros of the polynomial are rational but only that they are real. The reason why this "rational" version of the formulation of this problem is suggested is because mentioning rational numbers may direct one to use certain facts about polynomials valid generally only for rational parameters involved, facts which, in my opinion, can hardly be the way to the solution, thus facts completely misleading and making this problem even more difficult to solve.

Problem 60 (Miloš Milosavljević, Niš).

Let $A = [a_{ij}]_{n\times n}$ be a matrix with nonnegative elements the sum of which being equal to n.

(a) Prove $|detA| \leq 1$;

(b) If $|detA| = 1$ and $\lambda \in C$ arbitrary eigen-value of A, show that $|\lambda| = 1$.

Solution: (a) By H'adamar's theorem:

$$|detA| \leq \prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2. \tag{17}$$

As the numbers a_{ij} are nonnegative we have that, for $i = \overline{1,n}$:

$$\sum_{j=1}^{n} a_{ij}^2 \leq (\sum_{j=1}^{n} a_{ij})^2. \tag{18}$$

Using the inequality between the arithmetic and the geometric mean:

$$(\prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij})^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}}{n} = 1 \tag{19}$$

The required inequality follows directly from (1), (2) and (3).

(b) If $|detA| = 1$ the inequalities (1), (2) and (3) actually become equalities. In (3) equality holds iff the sum of elements of each row is equal 1, whereas in (2) equality holds iff in each row at most one of it's elements is distinct from zero. Hence, for $|detA| = 1$ to hold it is necessary that each row contains exactly one element equal to 1 and $n-1$ elements equal to zero. Since in (1) equality is reached iff all the vectors $a_i = [a_{i1}\ a_{i2}\ ...\ a_{in}]^T$, $i = \overline{1,n}$, are orthogonal to one another, in view of the above mentioned necessary condition, we have that $|detA| = 1$ iff each row and each column of A contains exactly one element equal to 1 and $n-1$ elements equal to zero. Therefore there are $\{k_1, k_2, ..., k_n\} = \{1, 2, ..., n\}$ such that $a_{1k_1} = a_{2k_2} = ... = a_{nk_n} = 1$. Let λ arbitrary eigen-value of A and $x = [x_1\ x_2\ ...\ x_n]^T$ the corresponding eigen-vector. We have

$$|\lambda|^2\langle x, x\rangle = \lambda\overline{\lambda}\langle x, x\rangle = \langle \lambda x, \lambda x\rangle = \langle Ax, Ax\rangle =$$

$$= \langle [x_{k_1}\ x_{k_2}\ ...\ x_{k_n}]^T, [x_{k_1}\ x_{k_2}\ ...\ x_{k_n}]^T\rangle = \sum_{i=1}^{n} x_{k_i}^2 = \sum_{i=1}^{n} x_i^2 = \langle x, x\rangle,$$

whence $|\lambda| = 1$. □

Problem 61 (Miloš Milosavljević, Niš). Let k be a fixed positive integer. In an infinite row distinct positions S_n, $n = 0, 1, 2, ...$ are marked. A ball jumps from one position to another following the next two rules:

1) it starts at S_0;

2) if it gets at S_n an $i \in \{1, 2, ..., k\}$ is randomly chosen (the probability of a fixed element of $\{1, 2, ..., k\}$ being chose being $\frac{1}{k}$), and the ball jumps to the position S_{n+i}.

Denote by p_n the probability of the ball getting at S_n any time during its movement. Find $\lim_{n\to\infty} p_n$.

Solution:

Lemma1. The polynomial $Q(x) = kx^k - (x^{k-1}+x^{k-2}+...+x+1)$ has k distinct zeros, all of the zeros, except $x = 1$, being with complex modulus smaller than 1.

Proof: Suppose there is a zero of $Q(x)$ with multiplicity greater than 1 and denote it by α. We have $Q(\alpha) = Q'(\alpha) = 0$. As

$$(x-1)Q(x) = kx^{k+1} - (k+1)x^k + 1$$

we get $(x-1)Q'(x) + Q(x) = k(k+1)x^{k-1}(x-1)$. Since $Q(\alpha) = Q'(\alpha) = 0$ the relation above yields

$$(\alpha - 1)Q'(\alpha) + Q(\alpha) = k(k+1)\alpha^{k-1}(\alpha - 1)$$

so $\alpha^{k-1}(\alpha-1) = 0$, i.e. $\alpha = 0$ or $\alpha = 1$. $x = 0$ is clearly not a solution to $Q(x) = 0$, and $Q'(1) = k^2 - \frac{k(k-1)}{2} \neq 0$, so $x = 1$ cannot be a multiple zero of $Q(x)$. Thus $Q(x)$ does not have any multiple zeros.

Let us now prove the second part of this lemma. Let z be any complex number with $|z| = R \geq 1$. Since

$$|z^{k-1} + z^{k-2} + ... + z + 1| \leq |z^{k-1}| + |z^{k-2}| + ... + |z| + 1 = R^{k-1} + R^{k-2} + ... + R + 1 \leq$$

$$\leq \underbrace{R^k + R^k + ... + R^k}_{k} = kR^k = |kz^k|,$$

we conclude

$$|Q(z)| = |kz^k - (z^{k-1} + z^{k-2} + ... + z + 1)| \geq |kz^k| - |z^{k-1} + z^{k-2} + ... + z + 1| \geq 0.$$

The last two inequalities imply that $Q(z) = 0$ iff the complex arguments of all the numbers $z^{k-1}, z^{k-2}, ..., z, 1$ are equal and $R = 1$. The last condition can be satisfied only if $z = 1$. Therefore the only zero of $Q(x)$ with modulus greater or equal than 1 is $z = 1$ which proves the second part of the lemma. •

Lemma2. Let i, l be positive integers with $l \leq i$. The set

$$A_{i,l} = \{(x_1, x_2, ..., x_l) \in N^l \mid x_1 + x_2 + \ldots x_l = i\}$$

has $a_{i,l} = \binom{i-1}{l-1}$.

Proof:

We will prove the assertion by mathematical induction on l. For $l = 1$ we have $a_{i,1} = 1 = \binom{i-1}{l-1}$. Suppose the assertion holds for a $l \in N$ and prove $a_{i,l+1} = \binom{i-1}{l}$. If the sum of $l+1$ positive integers is equal i, then sum of the first l of them equals one of the numbers $l, l+1, ..., i-1$. Hence

$$a_{i,l+1} = a_{l,l} + a_{l+1,l} + ... + a_{i-1,l}.$$

Using the inductive hypothesis:

$$\binom{i-1}{l} - a_{i,l+1} = \binom{i-1}{l} - \left(\binom{i-2}{l-1} + \binom{i-3}{l-1} + ... + \binom{l-1}{l-1}\right) =$$

$$= \binom{i-2}{l} - \left(\binom{i-3}{l-1} + \binom{i-4}{l-1} + ... + \binom{l-1}{l-1}\right) =$$

$$= \binom{i-3}{l} - \left(\binom{i-4}{l-1} + \binom{i-5}{l-1} + ... + \binom{l-1}{l-1}\right) =$$

$$= ... = \binom{l}{l} - \binom{l-1}{l-1} = 0$$

(where we have used several times the well known relation $\binom{a}{b}+\binom{a}{b+1}=\binom{a+1}{b+1}$, for $a>b$), which proves $a_{i,l+1}=\binom{i-1}{l}$. •

We now take up solving the problem itself. Consider the position S_{n+k}, $n\in N$. The positions from which the ball can directly to this position in one move are $S_{n+k-1}, S_{n+k-2}, \ldots, S_n$. Therefore

$$p_{n+k}=\tfrac{1}{k}p_{n+k-1}+\tfrac{1}{k}p_{n+k-2}+\ldots+\tfrac{1}{k}p_n,\ n\in N. \tag{1}$$

This represents a recurrence equation. As is well known, to solve it we need to consider the polynomial

$$Q(x)=kx^k-(x^{k-1}+x^{k-2}+\ldots+x+1)$$

the zeros of which are z_j, $j=\overline{0,k-1}$. By Lemma1. these zeros are distinct and all but $z_0=1$ are with complex modulus smaller than 1. Hence the general solution of the recurrence equation (1) is given by:

$$p_n=\alpha_0 z_0{}^n+\alpha_1 z_1{}^n+\ldots+\alpha_{k-1}z_{k-1}{}^n,\ \alpha_0,\alpha_1,\ldots,\alpha_{k-1}\in C. \tag{2}$$

As $|z_0|=1>|z_1|,|z_2|,\ldots,|z_{k-1}|$, (2) implies

$$\lim_{n\to\infty}p_n=\alpha_0.$$

Thus the problem reduces to finding α_0. Let us now find the probabilities $p_1,p_2,\ldots,p_k$ that we will need later to calculate α_0.

Let $1\le i\le k$ and $1\le l\le i$. By Lemma2., the number i can be represented as the sum of l positive integers in $a_{i,l}=\binom{i-1}{l-1}$ different ways. Each of them is with probability $(\frac{1}{k})^l$. Therefore we have that

$$p_i=\sum_{l=1}^{i}(\frac{1}{k})^l a_{i,l}=\sum_{l=1}^{i}(\frac{1}{k})^l\binom{i-1}{l-1}=\sum_{l'=0}^{i-1}\binom{i-1}{l'}(\frac{1}{k})^{l'+1}=$$

$$=\frac{1}{k}\sum_{l'=0}^{i-1}\binom{i-1}{l'}(\frac{1}{k})^{l'}=\frac{1}{k}(1+\frac{1}{k})^{i-1}.$$

The numbers $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}$ can now be find from the following system of linear equations (with the unknowns $\alpha_0,\alpha_1,\ldots,\alpha_{k-1}$):

$$\begin{aligned}
\frac{1}{k}(1+\frac{1}{k})^0=p_1 &= \alpha_0\cdot 1+\alpha_1\cdot z_1+\ldots+\alpha_{k-1}\cdot z_{k-1}\\
\frac{1}{k}(1+\frac{1}{k})^1=p_2 &= \alpha_0\cdot 1^2+\alpha_1\cdot z_1{}^2+\ldots+\alpha_{k-1}\cdot z_{k-1}{}^2\\
&\ \vdots\\
\frac{1}{k}(1+\frac{1}{k})^{k-1}=p_k &= \alpha_0\cdot 1^k+\alpha_1\cdot z_1{}^k+\ldots+\alpha_{k-1}\cdot z_{k-1}{}^k
\end{aligned}.$$

The determinant of the system's matrix is

$$D=\begin{vmatrix}1 & z_1 & \ldots & z_{k-1}\\ 1 & z_1{}^2 & \ldots & z_{k-1}{}^2\\ \vdots & \vdots & & \vdots\\ 1 & z_1{}^k & \ldots & z_{k-1}{}^k\end{vmatrix}=\prod_{i=1}^{k-1}z_i\cdot\begin{vmatrix}1 & 1 & \ldots & 1\\ 1 & z_1 & \ldots & z_{k-1}\\ \vdots & \vdots & & \vdots\\ 1 & z_1{}^k & \ldots & z_{k-1}{}^{k-1}\end{vmatrix}=$$

$$=\prod_{i=1}^{k-1}z_i\cdot W(1,z_1,\ldots,z_{k-1})=\prod_{i=1}^{k-1}z_i\cdot\prod_{i=1}^{k-1}(z_i-1)\cdot\prod_{0<j<i<k}(z_i-z_j)\neq 0.$$

Also we have

$$D_{\alpha_0}=\begin{vmatrix}\frac{1}{k}(1+\frac{1}{k})^0 & z_1 & \ldots & z_{k-1}\\ \frac{1}{k}(1+\frac{1}{k})^1 & z_1{}^2 & \ldots & z_{k-1}{}^2\\ \vdots & \vdots & & \vdots\\ \frac{1}{k}(1+\frac{1}{k})^{k-1} & z_1{}^k & \ldots & z_{k-1}{}^k\end{vmatrix}=\frac{1}{k}\cdot\prod_{i=1}^{k-1}z_i\cdot\begin{vmatrix}1 & 1 & \ldots & 1\\ 1+\frac{1}{k} & z_1 & \ldots & z_{k-1}\\ \vdots & \vdots & & \vdots\\ (1+\frac{1}{k})^{k-1} & z_1{}^{k-1} & \ldots & z_{k-1}{}^{k-1}\end{vmatrix}=$$

$$= \frac{1}{k}\cdot\prod_{i=1}^{k-1} z_i \cdot W(1+\frac{1}{k}, z_1, ..., z_{k-1}) = \frac{1}{k}\cdot\prod_{i=1}^{k-1} z_i \cdot \prod_{i=1}^{k-1}(z_i - (1+\frac{1}{k}))\cdot \prod_{0<j<i<k}(z_i - z_j).$$

The last two equalities yield:

$$\alpha_0 = \frac{D_{\alpha_0}}{D} = \frac{\frac{1}{k}\cdot\prod_{i=1}^{k-1}(z_i - (1+\frac{1}{k}))}{\prod_{i=1}^{k-1}(z_i - 1)} = \frac{\prod_{i=0}^{k-1}((1+\frac{1}{k}) - z_i)}{\prod_{i=1}^{k-1}(1-z_i)}.$$

Since $Q(x) = k(x - z_0)(x - z_1)...(x - z_{k-1})$ we have

$$\prod_{i=0}^{k-1}((1+\frac{1}{k}) - z_i) = \frac{Q(1+\frac{1}{k})}{k} = \frac{k(1+\frac{1}{k})^k - \frac{(1+\frac{1}{k})^k - 1}{1+\frac{1}{k}-1}}{k} = 1.$$

Let us note that $Q'(x) = k\sum_{i=0}^{k-1}\prod_{j\neq i}(x - z_j)$, having in mind $z_0 = 1$, yields $Q'(1) = k\prod_{i=1}^{k-1}(1-z_i)$. Therefore

$$\prod_{i=1}^{k-1}(1-z_i) = \frac{Q'(1)}{k} = \frac{k^2 - (k-1+...+1)}{k} = \frac{k+1}{2}.$$

Finally, we find that

$$\alpha_0 = \frac{\prod_{i=0}^{k-1}((1+\frac{1}{k}) - z_i)}{\prod_{i=1}^{k-1}(1-z_i)} = \frac{1}{\frac{k+1}{2}} = \frac{2}{k+1}.$$

Thus $\lim_{n\to\infty} p_n = \frac{2}{k+1}$. □

Problem 62 (Miloš Milosavljević, Niš). If $z_1, z_2, ..., z_n$ are complex numbers then there is a subset S of $\{1, 2, ..., n\}$ for which

$$|\sum_{k\in S} z_k| \geq \frac{1}{\pi}\sum_{k=1}^{n}|z_k|.$$

Solution: Write $z_k = |z_k|e^{i\alpha_k}$. For $-\pi \leq \theta \leq \pi$, let $S(\theta)$ be the set of all k for which $\cos(\alpha_k - \theta) > 0$. Then

$$|\sum_{k\in S(\theta)} z_k| = |\sum_{k\in S(\theta)} e^{-i\theta}z_k| \geq Re\sum_{k\in S(\theta)} e^{-i\theta}z_k = \sum_{k=1}^{n}|z_k|\cos^+(\alpha_k - \theta).$$

Choose θ_0 so as to maximize the last sum, and put $S = S(\theta_0)$. This maximum is at least as large as the average of the sum over $[-\pi, \pi]$, and this average is $\frac{1}{\pi}\sum_{k=1}^{n}|z_k|$, because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^+(\alpha - \theta)d\theta = \frac{1}{\pi}$$

for every α. □

Problem 63 (Robert Strich, Goettingen). For which positive integers n is the following statement true: If $P \in \mathbb{R}[X_1, \ldots, X_n]$ is a polynomial in n variables such that $P(x) > 0$ for all $x \in \mathbb{R}^n$ then there exists an $\varepsilon > 0$ such that $P(x) > \varepsilon$ for all $x \in \mathbb{R}^n$.

Solution: The statement is true for $n = 1$: Since P must have even degree in this case we have $\lim_{x\to\pm\infty} P(x) = \infty$ and thus we can find $M > 0$ such that $f(x) > 1$ for $|x| > M$. Furthermore p attains its minimum value $\delta > 0$ on the compact set $[-M, M]$ and thus we can take $\varepsilon = \min(1, \delta)$.

The statement is false for all $n \geq 2$. A counterexample is for instance the polynomial $P = X_1^2 + (X_1X_2 - 1)^2$. Clearly $P(x) > 0$ for all $x \in \mathbb{R}^n$ since not both summands in P can be zero. But for $x_1 \neq 0$, $x_2 = 1/x_1$ and arbitrary $x_3, \ldots, x_n$ we have $P(x_1, \ldots, x_n) = x_1^2$ which can obviously be arbitrarily small.