# Concursul TRAIAN LALESCU - faza naţională
# Constanţa, 14 mai 2011

**1.** Suppose $A \in \mathcal{M}_n(\mathbb{C})$. Prove that the sequence $(a_k)_{k\geq 0}$ is nondecreasing, where $a_k = \operatorname{rank}(A^{k+1}) - \operatorname{rank}(A^k)$.

**2.** Let $n \geq 2$ be a square-free positive integer, and denote by $D_n$ the set of its divisors. Consider $D \subseteq D_n$, a set with the following properties:

a) $1 \in D$;

b) $x \in D \Rightarrow n/x \in D$;

c) $x, y \in D \Rightarrow \gcd(x, y) \in D$.

Show that there exists a positive integer $k$ such that $|D| = 2^k$.

**3.** Let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that $\int_0^\infty f(x)dx < \infty$.

a) Prove that if $f$ is uniformly continuous, then $f$ is bounded.

b) Prove that the converse of the previous statement is not true.

**4.** Denote $D = (0, \infty) \times (0, \infty)$. Let $u \in C^1(D)$ and $\varepsilon > 0$.

a) Show that $x\frac{\partial u}{\partial x}(x, y) + y\frac{\partial u}{\partial y}(x, y) = u(x, y)$ $\forall (x, y) \in D$ iff there exists $\varphi \in C^1(0, \infty)$ such that $u(x, y) = x\varphi(y/x)$, $\forall (x, y)D$.

b) Show that if $\left|x\frac{\partial u}{\partial x}(x, y) + y\frac{\partial u}{\partial y}(x, y) - u(x, y)\right| \leq \varepsilon$, $\forall (x, y) \in D$, then there exists a unique function $\varphi \in C(0, \infty)$ such that $|u(x, y) - x\varphi(y/x)| \leq \varepsilon$, $\forall (x, y) \in D$.

*(The original contest problem requested $\varphi \in C^1(0, \infty)$, but this is not true).*

*To be solved in 3 hours and 30 minutes. All the problems are mandatory.*

**Solutions**
*(these are "unofficial"; it seems that the official ones have not been published).*

1. We shall use Frobenius' inequality:

for $A, B, C \in \mathcal{M}_n(\mathbb{C}) \implies \operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(ABC) + \operatorname{rank}(B)$.

For completeness, we give a short proof of this inequality, based on the elementary operations for block matrices.

$\operatorname{rank}(ABC) + \operatorname{rank}(B) = \operatorname{rank}\begin{pmatrix} B & 0 \\ 0 & ABC \end{pmatrix} = \operatorname{rank}\begin{pmatrix} B & 0 \\ AB & ABC \end{pmatrix} =$

$\operatorname{rank}\begin{pmatrix} B & -BC \\ AB & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} BC & B \\ 0 & AB \end{pmatrix} \geq \operatorname{rank}(BC) + \operatorname{rank}(AB)$.

Taking in Frobenius' inequality $B = A^{k-1}$, $C = A$ one obtains $\operatorname{rank}(A^k) + \operatorname{rank}(A^k) \leq \operatorname{rank}(A^{k+1}) + \operatorname{rank}(A^{k-1})$ i.e. $a_{k-1} \leq a_k$.

2. If $n = p_1 \cdots p_m$ where $p_i$ are distinct primes then $|D_n| = 2^m$ and is naturaly identified with $\mathbb{Z}_2^m$, via the bijection $p_1^{t_1} \cdots p_m^{t_m} \xrightarrow{\varphi} (t_1, \cdots t_m)$. If $A = \varphi(D)$, we have: $0 \in A$, $a \in A \implies 1 - a \in A$, $a, b \in A \implies ab \in A$.

We assert that $a, b \in A \implies a + b \in A$. [this would imply the conclusion because $(A, +)$ is then a subgroup of $(\mathbb{Z}_2^n, +)$ and Lagrange's theorem applies].

In fact, $x = a(1-b) = a + ab \in A$ and $y = b(1-a) = b + ab \in A$. $xy = ab + ab + ab + ab = 0$ and $x + y = (1+x)(1+y) + 1 + xy = (1+x)(1+y) + 1 \in A$. But $x + y = a + b + ab + ab = a + b$, so $a + b \in A$. Q.E.D.

3. a) Suppose by contradiction that $f$ is not bounded. Then there exists a sequence $(x_n)$ such that $f(x_n) > n$.

$(x_n)$ is not bounded (by Weierstrass' theorem) so we may take $x_n \nearrow \infty$ and $x_{n+1} - x_n > 1$.

$f$ being uniformly continuous there exists $\delta \in (0,1)$ such that $|f(x) - f(y)| < 1$ if $|x - y| \leq \delta$.

For $x \in [x_n, x_n + \delta]$ we have $f(x) \geq f(x_n) - |f(x) - f(x_n)| > n - 1$, so $\int_{x_n}^{x_n+\delta} f(x)\mathrm{d}x > (n-1)\delta$.

So, $\int_0^\infty f(x)dx \geq \sum_n (n-1)\delta = \infty$, contradiction.

b) Define $f(x) = 2^n(x - n + 2^{-n})$ for $x \in [n - 2^{-n}, n]$ , $f(x) = 2^n(n + 2^{-n} - x)$ for $x \in [n, n + 2^{-n}]$, $(n \in \mathbb{N})$ and $f(x) = 0$ elsewhere. Then $f$ is continuous, bounded $(0 \leq f \leq 1)$ integrable $(\int_0^\infty f = \sum_n 2^{-n} = 1)$ but it is not uniformly continuous because $f(n + 2^{-n}) - f(n) = 1 \nrightarrow 0$.

4. a) "$\Longleftarrow$" is standard.

"$\implies$" For $(x, y) \in D$ (fixed), define $f(t) = u(tx, ty)$, $t \in (0, \infty)$.

$f'(t) = x\frac{\partial u}{\partial x}(tx, ty) + y\frac{\partial u}{\partial y}(tx, ty) = \frac{1}{t}u(tx, ty) = f(t)/t$.

One obtains $f(t) = ct$, where c is a constant; it will depend of course on $x, y$. So, there exists a function $F : D \to \mathbb{R}$ such that $f(t) = tF(x, y)$. For $t = 1$ we have $F(x, y) = u(x, y)$, so $u(tx, ty) = tu(x, y)$.

For $t = 1/x$ we obtain $u(x, y) = xu(1, y/x)$, so we may choose $\varphi(t) = u(1, t)$.

Note that this is Euler's equation for homogeneous (of order 1) functions.

b) The uniqueness is easy. Let $\psi$ another function verifying the inequality. We obtain $|x\varphi(y/x) - x\psi(y/x)| \leq 2\varepsilon$, $\forall (x, y) \in D$.

Replacing $y$ by $tx$ (for $t > 0$) we obtain $|\varphi(t) - \psi(t)| \leq 2\varepsilon/x$. For $x \to \infty \implies \varphi(t) = \psi(t)$.

It remains to prove the existence of such a $\varphi$. Using again $f(t) = u(tx, ty)$, ($f$ depends on $x, y$ considered fixed) we have $|tf'(t) - f(t)| \leq \varepsilon$.

Denoting $tf'(t) - f(t) = g(t)$ and solving as an ODE for $f$ we get $f(t) = t\left(\int g(s)/s^2 \mathrm{d}s + C\right)$, or

$$f(t) = t\left(-\int_t^\infty g(s)/s^2\mathrm{d}s + K\right), \tag{*}$$

where $K = K(x, y)$ is constant with respect to $t$.

So, $|f(t) - Kt| = t\left|-\int_t^\infty g(s)/s^2\mathrm{d}s\right| \le t\int_t^\infty \varepsilon/s^2\mathrm{d}s = t\varepsilon/t = \varepsilon$, because $|g| \le \varepsilon$.

Hence, $|f(t) - Kt| \le \varepsilon$, $|u(tx, ty) - Kt| \le \varepsilon$ and for $t = 1$ one gets $|u(x, y) - K(x, y)| \le \varepsilon$. It remains to verify that $K(x, y)$ is homogeneous and continuous.

Using (*) for $t = 1$ we obtain $K \in C(D)$ (because $f$ and $g$ are continuous with respect to $x$ and $y$) and for $t \to \infty$, $K = K(x, y) = \lim_{t\to\infty} f(t)/t = \lim_{t\to\infty} \frac{u(tx,ty)}{t}$.

(the limit exists because $\lim_{t\to\infty} \int_t^\infty g(s)/s^2\mathrm{d}s = 0$).

Now it's easy to check that $K(x, y)$ is 1-homogeneous: for $a > 0$,

$K(ax, ay) = \lim_{t\to\infty} \frac{u(tax,tay)}{t} = \lim_{t\to\infty} \frac{u(tax,tay)}{ta} \cdot a = \lim_{s\to\infty} \frac{u(sx,sy)}{s} \cdot a = K(x, y)a.$

So, $K(x, y) = xK(1, y/x)$, i.e. $\varphi(t) = K(1, t)$. Q.E.D.

Let us show that $\varphi$ cannot be always chosen in $C^1$. Consider the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^2/2$ for $|x| \le 1$ and $p(x) = |x|$ for $|x| > 1$. The function $p$ is in $C^1(\mathbb{R})$.

For the function $u \in C^1(D)$, $u(x, y) = p(x - y)$, denoting $h(x, y) = x\frac{\partial u}{\partial x}(x, y) + y\frac{\partial u}{\partial y}(x, y) - u(x, y)$ we have

$|h(x, y)| = |(x - y)p'(x - y) - p(x - y))|$, hence

$|h(x, y)| = 0$ for $|x - y| \ge 1$ and

$|h(x, y)| \le 1/2$ for for $|x - y| < 1$.

So, $u$ satisfies the hypothesis in b) with $\varepsilon = 1/2$. Taking $\varphi(t) = |1 - t|$ we have $x\varphi(y/x) = |x - y|$ and

$|u(x, y) - x\varphi(y/x)| = |p(x - y) - |x - y|| \le 1/2$. We know that $\varphi$ is unique, but it is not differentiable at $t = 1$.

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