# Chapter 1

# Mathematical models in economy. Short descriptions

## 1.1 Arrow-Debreu model of an economy via Walras equilibrium problem.

Let us consider first the so-called Arrow-Debreu model. The presentation will be brief. A more detailed description and several justifications can be found in Debreu [6], Border [5] or Isac [8]. Let's start by presenting the main elements of an abstract economy.

The fundamental idealization made in modeling an economy is the notion of commodity. We suppose that it is possible to classify all the different goods and services in the world economy into a finite number. Let say m commodities, which are available in infinitely divisible units. The commodity space is $\mathbb{R}^m$. A vector $x \in \mathbb{R}^m$ specifies a list of quantities of each commodity. There are commodity vectors that are exchanged or manufactured or consumed in economic activities and not individual commodities. Of course, if $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ it is possible that some quantities $x_i$, $i \in \{1, \cdots, m\}$ to be equal to zero. We will denote by $E$ the set of all available commodities.

A price vector $p$ lists the value of a unit of each commodity and so $p \in \mathbb{R}^m$.

The value of the commodity vector $x$, when on the market acts the price system $p$ is the inner product $p \cdot x = \sum_{i=1}^{m} p_i x_i$.

Let us make now an important remark: the assumption of the existence of only a finite number of distinct commodities can be eliminated. So, it is possible to consider economies with an infinite number of distinct commodities. In this case the commodity space is an infinite-dimensional vector space and the price vector belongs to the dual space of the commodity space. For some references of this topic, see, for example, the book of Aliprantis, Brown and Burkinshaw [1].

The consumers are the main actors of an economy. The ultimate purpose of an economic organization is to provide commodity vectors for final consumption by consumers. We will assume that there exists a finite given number of consumers.

It is quite obviously that not every commodity vector is admissible as a final consumption for a consumer. We will denote by $X \subset \mathbb{R}^m$ the set of all admissible consumption vectors for a given consumer. (or $X_i \subset \mathbb{R}^m$ if we discuss about the consumer $i$) So, $X$ (or $X_i$) is the consumption set. What restrictions can be placed on the consumption set ?

A first restriction is that the admissible consumption vectors are nonnegative.

An alternative restriction is that the consumption set is bounded below. Under this interpretation, negative quantities of a commodity in a final consumption vector mean that the consumer is supplying the commodity as a service. The lower bound puts a limit in the services that a consumer can provide. Also, the lower bound could be interpreted as a minimum requirement of some commodity for the consumer.

In a private ownership economy consumers are also characterized by their initial endowment of commodities. This is an element $w$ (or $w_i$) in the commodity space. These are the resources the consumer owns.

In a market economy, a consumer must purchase his consumption vector at the market prices. The set of all admissible commodity vectors that he can afford at prices $p$, given an income $M$ (or $M_i$) is called the budget set and will be denoted by $A$ (or $A_i$). The budget set can be represented as:

$$A = \{x \in X | p \cdot x \leq M\}.$$

Of course, the budget set can be also empty.

The problem faced by a consumer in a market economy is to choose a consumption vector or a set of them from the budget set. To do this, the consumer must have some criteria for choosing. A first method to formalize the criterion is to assume that the consumer has a utility index, that is a real-valued function $u$ (or $u_i$) defined on the set of consumption vectors. The idea is that a consumer would prefer to consume vector $x$ rather than vector $y$ if $u(x) > u(y)$ and it would be indifferent if $u(x) = u(y)$. A solution to the consumer's problem is to find all the vectors $x$ which maximize $u$ on the budget set, i.e.,

$$\text{Find } x^* \in A \text{ such that } u(x^*) = \max_{x \in A} u(x).$$

This kind of problem is not so easy like it seems. But, if some restrictions are placed on the utility index, for example if the function $u$ is continuous and the budget set $A$ is compact, then from the well-known theorem of Weierstrass, we get that there exist vectors that maximize the value of $u$ over the budget set, and so the proposed problem has at least a solution. Unfortunately, these assumptions on the consumer's criterion are somewhat severe, because we would like that the consumer's preferences to mirror the order properties of real numbers for example, if

$$u(x^1) = u(x^2), u(x^2) = u(x^3), \cdots, u(x^{n-1}) = u(x^n)$$

then

$$u(x^1) = u(x^n),$$

but on the other hand one can easily imagine situations where a consumer is indifferent between $x^1$ and $x^2$, between $x^2$ and $x^3$, etc but not between $x^1$ and $x^n$. Of course, there are weaker assumptions we can make about the preferences. These approaches involve multivalued operators, in order to describe a consumer's preferences. To do this, let us denote by $U(x)$ the set of all consumption vectors which consumer strictly prefer to $x$, i. e.

$$U(x) = \{y \in A | y \text{ is strictly preferred to } x\}, \ x \in A.$$

Obviously, $U : A \multimap A$ and it is called the preference multifunction or the multivalued operator of preferences. For example, in terms of the utility function, we have

$$U(x) = \{y \in A | u(y) > u(x)\}.$$

If we consider the abstract preference multifunction $U$ then a vector $x^* \in A$ is an optimal preference for a given consumer if and only if

$$U(x^*) = \emptyset.$$

Such elements $x^*$ are also called **U-maximal** or simply maximal. It is easy to see that any fixed point result for a multifunction generate an existence result for an $U$-maximal element of the above preference multifunction. Indeed, let us suppose that

$$U : A \to \mathcal{P}(A)$$

is a multivalued operator such that

$$U : A \to P(A)$$

satisfies to a fixed point theorem. If $y \notin U(y)$, for each $y \in A$ then there exists at least one $U$-maximal element of $U$. In order to justify the above assertion, let us suppose by contradiction, that $U(y) \neq \emptyset$, for any $y \in A$. From the fixed point theorem we obtain the existence of an element $x^* \in A$ such that $x^* \in U(x^*)$, which is a contradiction with the hypothesis. Hence, any fixed point result for a multivalued operator is an $U$-maximal existence theorem for the preference multifunction.

On the other hand, if we a preference multifunction defined by the relation:

$$U(x) = \{y \in A | y \text{ is preferred to } x\}, \ x \in A,$$

then a vector $x^* \in A$ is an optimal preference for the consumer if and only if

$$\{x^*\} = U(x^*).$$

Such points are, by definition, strict fixed points of $U$. They are also called end points for the multivalued dynamical system $(A, U)$ generated by the multivalued operator $U$. Hence, any strict fixed point theorem is, in fact, an existence result for an optimal preference.

So, more general the consumer's problem is to find all vectors which are optimal preferences with respect to $U$. The set of solution to a consumer's problem for given price system $p$ is called **the demand set**.

Let us discuss now something about the supplier's problem. This is much simpler, because the suppliers are motivated by profit. Each supplier j has a production set $Y$ (or $Y_j$) of technologically feasible supply vectors. A supply vector $y$ specifies the quantities of each commodity supplied and the amount of each commodity used as an input. Inputs are denoted by negative quantities and outputs by positive ones. The profit (net income) associated with a supply vector $y$ at prices $p$ is just $p \cdot y = \sum_{i=1}^{m} p_i y_i$. The supplier's problem is then to choose an element $y$ from the set of technologically feasible supply vectors which maximizes the associated profit. As in the consumer's problem, there may be no solution, as it may pay to increase the outputs and inputs indefinitely at ever increasing profits. The set of all solutions of the supplier's problem is called **the supply set**.

Thus, for a given price vector $p$, there is a set of supply vectors $y_j$, for each supplier $j$ (determined by maximizing the profit) and a set of demand vectors $x_i$, for each consumer $i$ (determined by preference optimality).

More precisely, let:

$(s_1)$ $\{y_1^1, y_1^2, \cdots, y_1^s\}$

$(s_2)$ $\{y_2^1, y_2^2, \cdots, y_2^s\}$

$\cdots$

$(s_l)$ $\{y_l^1, y_l^2, \cdots, y_l^s\}$

be the supply vectors for the suppliers $1, 2, \cdots l$, where $y_i^j \in \mathbb{R}^m$, for $i \in \{1, 2, \cdots, l\}$ and $j \in \{1, 2, \cdots, s\}$.

Let

$(d_1)$ $\{x_1^1, x_1^2, \cdots, y_1^r\}$

$(d_2)$ $\{x_2^1, x_2^2, \cdots, x_2^s\}$

$\cdots$

$(d_n)$ $\{x_n^1, x_n^2, \cdots, x_n^s\}$

be the demand vectors for the consumers $1, 2, \cdots n$, where $x_i^j \in \mathbb{R}^m$, for $i \in \{1, 2, \cdots, r\}$ and $j \in \{1, 2, \cdots, s\}$.

Denote

$$\mathcal{S}(p) := \{\sum_{i=1}^{l} y_i^{t(i)} : t(i) = \bar{1,s}\} - \text{ the supply set}$$

and

$$\mathcal{D}(p) := \{\sum_{j=1}^{n} x_j^{t(j)} : t(j) = \bar{1,r}\} - \text{ the demand set}$$

The excess demand multifunction with respect to a given price system $p$, is defined as the set of sums of demand vectors minus the set of sums of supply vectors (i.e., the demand set minus the supply set) and it is denoted by $E(p)$. Obviously, $E$ is a multivalued operator

$$E : \mathbb{R}_+^m \multimap \mathbb{R}^m, \text{ given by } E(p) := \mathcal{D}(p) - \mathcal{S}(p).$$

The notion of equilibrium that I am now recalling was basically formalized by Leon Walras in 1874.

By definition, a price vector $p^* \in \mathbb{R}^m$ is a Walrasian equilibrium price if

$$0 \in E(p^*).$$

This means that some combinations of supply and demand vectors adds up to zero. We may say that $p^*$ clears the market.

There exists another situation called a Walrasian free disposal equilibrium. That is the following situation: some commodities might be allowed to be in excess supply at equilibrium provided their price is zero. So, the price $p^*$ is a **Walrasian free disposal equilibrium price** if there exists $z \in E(p^*)$ such that $z \leq 0$ and whenever $z_i << 0$ then $p_i^* = 0$.

Of fundamental importance to this approach is a property of the excess demand multifunction known as Walras' law. Shortly, Walras' law says that if the profits of all suppliers are returned to consumers as dividends, then the value at prices $p$ of any excess demand vector must be non-positive. This happens because the value of each consumer's demand must be no more than his income and the sum of all incomes must be the sum of all profits from suppliers. Thus, the value of total supply must be at least as large as the value of total demand. If each consumer spends all his income, then these two values are equal and the value of excess demand multifunction must be zero.

Let us present now briefly an example of how the excess demand multifunction can be expressed. We will consider, for simplicity, the problem of sharing between "n" consumers a commodity bundle $w$, i. e. the supply. So, the problem is to find $n$ commodity bundles $x^i$, such that $\sum_{i=1}^{n} x^i \leq w$. A solution to this problem is called an allocation of w. The solution proposed by Walras and his followers consists in letting price systems play a crucial role. Namely, a consumer $i$ is defined as an automaton associating to every price vector $p$ and every income $r$ (in monetary units) its demand $d_i(p, r)$, which is the commodity bundle that he buys when the price system is $p$ and its income is $r$. So it is assumed that demand operator $d_i$ describes the behavior of the consumer $i$. Let us recall that, neoclassical economists assume that demand operators derive from the maximization of an utility function. But, in what follows, we assume that consumers are just demand operators $d_i(\cdot,\cdot)$ independent of the supply bundle w.

We also assume that an income allocation of the gross income w is given. This means the following: if $p$ is the price vector, the gross income is the value $p \cdot w$ of the supply w. We then assume that gross income $r(p) = p \cdot w$ is allocated among consumers in incomes $r_i(p)$ and hence $r(p) = \sum_{i=1}^{n} r_i(p)$. We must observe that the model does not provide this allocation of income, but assumes that it is given. An example of such an income allocation is supplied by the so-called exchange economies, where the supply $w$ is the sum of $n$ supply bundles $w_i$ brought to the market by n consumers. So, in this case $r(p) = p \cdot w$ and $r_i(p) = p \cdot w_i$ is the income derived by consumer $i$ from its supply bundle $w_i$. In summary, the mechanism we are about to describe depends upon:

1) the description of each consumer i by its demand operator $d_i(\cdot,\cdot)$

2) an allocation $r(p) = \sum_{i=1}^{n} r_i(p)$ of the gross income.

The mechanism works if and only if demand balances supply, i. e. if and only if

$$\sum_{i=1}^{n} d_i(p, r_i(p)) \leq w. \qquad (*)$$

A solution $p^*$ to this problem is a Walrasian equilibrium price.

There is no doubt that Adam Smith (1776) is at the origin of what we now call descentralization, i. e. the ability of a complex system, moved by different actions to pursuit of different objectives to achieve an allocation of scarce resources: "Every individual endeavors to employ his capital so that its produce may be of greatest value. He generally neither intends to promote the public security, nor knows how much he is promoting it.

He intends only his own security, only his own gain. And he is in this led by an invisible hand to promote and end which has no part of his intention. By pursuing his own interest, he frequently thus promotes that of society more effectively that when he really intends to promote it". However, Adam Smith did not provide a careful statement of what the invisible hand manipulates, nor a fortiori, a rigorous argument for its existence. We had to wait a century for Leon Walras to recognize that price systems are the elements on which the invisible hand acts and that actions of different agents are guided by those price systems, providing enough information to all the agents for guaranteeing the consistency of their actions with the scarcity of available commodities. (see Aubin and Cellina [4] or Aubin [3], for more comments and details.)

Hence, if Adam Smith's invisible hand does provide a Walras equilibrium $p^*$, then the consumers i are led to demand commodities $d_i(p^*, r_i(p^*))$, that permits to share w according to the desire of everybody.

So, the task is to solve problem $(*)$.

It is remarkable that a sufficient condition with a clear economic interpretation is the following financial constraint on the behavior of the consumers, the so-called individual Walras law:

$$p \cdot d_i(p, r_i) \leq r_i, \text{ for each } i \in \{1, \cdots, n\}.$$

The individual Walras law forbids consumers to spend more than their incomes.

Another hypothesis which appear is the so-called collective Walras law:

$$\sum_{i=1}^{n} p \cdot d_i(p, r_i) \leq \sum_{i=1}^{n} r_i.$$

This law allows financial transactions among consumers.

Both laws do not involve the supply bundle w. A more general model suppose that the supply is not given, but has to be chosen in a set $X^*$ of available commodity bundles supplied to the market. Thus, the income derived from this set $X^*$ is $r(p) = \sup_{w \in X^*} p \cdot w$. When $X^*$ is reduced to one supply vector w, we fall back to the case we have considered above.

The mechanism is described by:

i) the "n" demand operators $d_i(\cdot, \cdot)$

ii) an income allocation $r(p) = \sum_{i=1}^{n} r_i(p)$, which depends upon $X^*$ via the above formula.

The problem is to find a price $p^*$ (a Walrasian equilibrium), cleaning the market in the sense that:

$$\sum_{i=1}^{n} d_i(p^*, r_i(p^*)) \in X^*.$$

This means that the sum of the demands lies among the set of available supplies. If we define the excess demand multifunction $E$ by:

$$E(p) = \sum_{i=1}^{n} d_i(p^*, r_i(p^*)) - X^*,$$

then a Walrasian equilibrium $p^*$ is a solution of the following inclusion:

$$0 \in E(p^*).$$

Hence, an existence result for the zero-point element of the multivalued operator $E$ (i. e. an element $p^* \in X$ with $0 \in E(p^*)$) is, basically, an existence theorem for a Walrasian equilibrium price of the market.

Of course, there are also many bad points of these models. The first is that the fundamental nature of Walras world is static, while we live in a dynamical environment, where no equilibria have been observed. There exist also several dynamical models built on the ideas of the Walras hypothesis. More precisely, one regard the price system not as a state of a dynamical system whose evolution law is known, but as a control which evolves as an operator of the consumptions according to a feedback law.

## 1.2 Equilibrium price, variational inequalities and the complementarity problem.

A particular case of the above model is when the excess demand multifunction is a singlevalued operator. We will consider now the case when excess demand set is a singleton for each price vector $p$ and the price vectors are non-negative. So, for each price vector $p$, there is a vector $f(p)$ of excess demands for each commodity. We assume that $f$ is continuous. A very important property of market excess demand operator is the individual Walras law. The mathematical statement of Walras' law for this singlevalued case can take either two forms. The strong form of Walras' law is:

$$p \cdot f(p) = 0, \text{ for all p },$$

while the weak form of Walras law replaces the equality by the weak inequality:

$$p \cdot f(p) \leq 0, \text{ for all p }.$$

The economic meaning of Walras' law is that in a closed economy, at most all of everyone's income is spent. To see how the mathematical statement follows from the economic hypothesis, first consider the case of a pure exchange economy. The k-th consumer comes to market with a vector $w_k$ of commodities and leaves with a vector $x_k$ of commodities. If all the consumers face the price vector $p$, then their individual budgets require that $p \cdot x_k \leq p \cdot w_k$, that is they cannot spend more than they earn. In this case, the excess demand operator is: $f(p) = \sum x_k - \sum w_k$, i. e. the sum of total demands minus the sum of total supply. Summing up the individual budget constraints and rearranging terms we obtain that: $\sum p \cdot (x_k - w_k) \leq 0$ or equivalently $p \cdot \sum (x_k - w_k) \leq 0$. Hence we have obtained: $p \cdot f(p) \leq 0$, the weak form of Walras law. The strong form obtains if each consumer spends all his income.

The case of a production economy is similar. The j-th supplier produces a net output vector $y_j$, which yields a net income of $p \cdot y_j$. In a private ownership economy this net income is redistributed to consumers. The new budget constraint form for a consumer is :

$$p \cdot x_k \leq p \cdot w_k + \sum_j \alpha_j^k p \cdot y_j,$$

where $\alpha_j^k$ is consumers' k's share of profits of firm j. Thus $\sum\limits_k \alpha_j^k = 1$, for each j. So, the excess demand operator $f(p) = \sum\limits_k x_k - \sum\limits_k w_k - \sum\limits_j y_j$. Again adding up the budget constraints and rearranging terms yields $p \cdot f(p) \leq 0$. The law remains true even if consumers may borrow from each other, as long as, no borrowing from outside the economy takes place. Also, we can restrict the prices to belong to the standard simplex because both constraints and the profit functions are positively homogeneous in prices. Thus we can normalize prices.

By definition, $p^* \in \mathbb{R}_+^m$ is said to be an equilibrium price if $f(p^*) = 0$. A free disposal equilibrium price is a price vector $p^* \in \mathbb{R}_+^m$ satisfying $f(p^*) \leq 0$.

Let us remark that, if $p^* \in \mathbb{R}_+^m$ is a free disposal equilibrium price and the strong form of Walras law take place (i. e. $p \cdot f(p) = 0$), then $f_i(p^*) < 0$ for some $i$ necessarily implies $p_i^* = 0$, i.e., if a commodity is in excess, then the price must be zero.

A mathematical more general problem is what is known as the nonlinear complementarity problem. The function $f$ is assumed to be continuous and its domain is a closed convex cone $C$ (a set $C$ is a cone if for any $x \in C$ and each $\lambda \in \mathbb{R}_+$ we have that $\lambda x \in C$) in $\mathbb{R}^m$. The problem is:

$$\text{find } p^* \in C \text{ such that } f(p^*) \in C^* \text{ and } p^* \cdot f(p^*) = 0.$$

(Here $C^* := \{y \in \mathbb{R}^m : y \cdot x \leq 0 \text{ for each } x \in C\}$ is the dual of the cone $C$.)

If in particular, $C$ is the non-negative cone $\mathbb{R}^m_+$, then its dual $C^* = \mathbb{R}^m_-$ and so $f(p^*) \in C^*$ becomes $f(p^*) \leq 0$. In this case, since $f(p^*) \leq 0$ can be also writen $p \cdot f(p^*) \leq 0$, for each $p \in \mathbb{R}^m_+$, then we immediately get that $p \cdot f(p^*) \leq p^* \cdot f(p^*) = 0$ and so the problem becomes:

$$\text{find } p^* \in \mathbb{R}^m_+ \text{ such that } p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for each } p \in \mathbb{R}^m_+.$$

Of course, the complementarity problem could be formulated in a more general setting, for example in a Hilbert space or in a dual system of locally convex spaces $(E, E^*)$, see Isac [8].

So, in both, the price problem and the complementarity problem there is a cone $C$ and a function $f$ defined on $C$ and we are looking for a $p^* \in C$ satisfying $f(p^*) \in C^*$. As we already mentioned above, another way to write the condition $f(p^*) \in C^*$ is the following:

$$p \cdot f(p^*) \leq 0, \text{ for all } p \in C.$$

Since in both problems (in the price problem, on the assumption of the strong Walras' law, while in the complementarity problem, by definition) $p^* \cdot f(p^*) = 0$, we can rewrite this as:

$$p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for all } p \in C.$$

A system of inequalities of the above form is called a system of variational inequalities, because it compares expressions involving $f(p^*)$ and $p^*$ with expressions involving $f(p^*)$ and $p$, where $p$ can be viewed as a variation of $p^*$. The intuition involved in these situation is the following: if a commodity is in excess demand, then its price should be raised and if it in excess supply, then its price should be lowered. This increases the value of demand. Let us say that price $p$ is better than price $p^*$ if $p$ gives a higher value to $p^*$'s excess demand than $p^*$ does. The variational inequalities tell us that we are looking for a maximal element of this binary relation. Of course, a multivalued operator is then involved, namely

$$U(p) = \{q \in C | q \cdot f(p) > p \cdot f(p)\}, \ p \in C,$$

and, as we mentioned above, we are looking for an element $p^* \in C$ such that $U(p^*) = \emptyset$.

If we consider $f : \mathbb{R}^m_+ \to \mathbb{R}^m$ and we denote by (VIP) the variational inequalities problem and by (CP) the complementarity problem then:

$$(VIP) \text{ find } p^* \in \mathbb{R}^m_+ \text{ such that } p \cdot f(p^*) \leq p^* \cdot f(p^*), \text{ for each } p \in \mathbb{R}^m_+.$$

$$(CP) \text{ find } p^* \in \mathbb{R}^m_+ \text{ such that } f(p^*) \in \mathbb{R}^m_- \text{ and } p^* \cdot f(p^*) = 0$$

are equivalent.

Indeed, if $p^* \in \mathbb{R}^m_+$ is a solution of (CP) then $f(p^*) \in \mathbb{R}^m_-$ and $p^* \cdot f(p^*) = 0$. Then $p \cdot f(p^*) \leq 0 = p^* \cdot f(p^*)$, for each $p \in \mathbb{R}^m_+$ and so $p^*$ is a solution of (VIP).

For the reverse implication, let $p^* \in \mathbb{R}^m_+$ is a solution of (VIP). Then $f(p^*) \cdot (p - p^*) \leq 0$, for each $p \in \mathbb{R}^m_+$. By taking $p = 0$ and $p = 2p^*$ in the above relation, we immediately get that

$$f(p^*) \cdot p^* = 0.$$

Next, we need to show now that $f(p^*) \in \mathbb{R}^m_-$. If we suppose by contradiction that here exists $i \in \{1, 2, \cdots, m\}$ such that $f_i(p^*) > 0$ then, by a suitable choice for the vector $p$ (with a large $p_i > 0$) we obtain a contradiction with $f(p^*) \cdot (p - p^*) \leq 0$. This shows that $f_i(p^*) > 0$, for each $i \in \{1, 2, \cdots, m\}$. See also G. Isac [7], pp. 63.

Finally, we would like to point out another (obvious) connection with fixed point theory. Let $H$ be a Hilbert space and $K$ be a convex cone in $H$. Let $f : K \to H$ be an operator defining a complementarity problem. Then, $x^*$ **is a solution for the complementarity problem** if and only if $x^*$ **is a fixed point of the operator** $g := 1_K + f$.

For important contributions in the field of complementarity theory and connections to mathematical economics and variational inequalities theory see Isac [7], [8], Isac, Bulavski, Kalashnikov [9].

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