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Fixed Point Theory in Kasahara Spaces

Ph.D. Thesis

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Contents

Introduction

Fixed point theory becomes, in the last decades, not only a field with a huge development, but also a strong tool for solving various problems arising in different fields of pure and applied mathematics. A central element of the metric fixed point theory is the Banach-Caccioppoli Contraction Principle. Today we have many generalizations of this result, which were given in all kind of generalized metric spaces. If we carefully examine their proofs, one can see that the metric properties, in particular part of the axioms of the metric, are not all the time essential. Therefore the following problem arises: *In which general spaces contractive type fixed point theorems hold ?*

This problem has been studied since 1975 by a distinguished mathematician Shouro Kasahara, professor at the Kobe University. By following the work of Maurice Fréchet [42] which has introduced the structure of L-space, Kasahara has endowed this structure with a functional d which is not necessarily a metric. Therefore he has defined a more general space: the d-complete L-space. By using this notion, Kasahara has extended Maia's theorem, published in 1968 in [84], a well-known fixed point result given in a set endowed with two metrics. We mention here some other authors which have given fixed point theorems in a set with two metrics: V. Berinde [10], S. Iyer [57], A. Petruşel and I.A. Rus [102], R. Precup [105], I.A. Rus [118], I.A. Rus, A.S. Mureşan and V. Mureşan [122], B. Rzepecki [129], L.M. Saliga [130].

In a number of papers [66]-[70] Kasahara constructed a fixed point theory in d-complete L-spaces. T.L. Hicks [47] and T.L. Hicks - B.E. Rhoades [49] gave some fixed point theorems in a d-complete topological space. Other results in these directions were given by V.G. Angelov [3], J. Daneš [22], K. Iséki [55], L. Guran [45], P.Q. Khanh [75].

However, the notion of d-complete L-space was, in some sense, difficult to be used. Hence, by following the work of Kasahara and the results given by the mathematicians which have been already mentioned above, Ioan A. Rus has defined in 2010 the notions of *Kasahara space*, *generalized Kasahara space* and *large Kasahara space*. His work [121] contains also fixed point theorems and research problems concerning Kasahara spaces. Some solutions regarding the formulated research problems can be found in our thesis.

This thesis is divided into three chapters, each chapter containing several sections.

Chapter 1: Preliminaries.

In this chapter we present the basic notions and results which are further considered in the next chapters of this work, allowing us to present the results of this thesis. This chapter contains the following sections:

$\diamond$ *L-spaces* in which we recall the notion of L-space. Several examples of L-spaces are also given in this section.

$\diamond$ *Generalized metric spaces* in which the notions of *distance functional* and *G-metric* defined on a nonempty set X are recalled. The connexion between generalized metric spaces and L-spaces is also discussed. Our contribution in this section is a solution given for the Problem 1.2.1, by studying the cases when a distance functional induces a structure of L-space.

$\diamond$ *Partial metric spaces* in which we recall the notion of partial metric as a particular case of generalized metric. Several examples of partial metric spaces are also presented. We give also the notions regarding the convergence induced by the quasimetric q_p and the metric d_p, both this functional being obtained from a partial metric p. Our contributions to this section are Remark 1.3.3 and Remark 1.3.4.

$\diamond$ *w-distance on a metric space* (X, d). In this section we recall the notion of w-distance and we give some examples regarding this notion. The convergence with respect to a w-distance and the connexion with L-spaces are also discussed. Our contributions to this section are Definition 1.4.2; Remarks 1.4.1, 1.4.2 and 1.4.3; Lemma 1.4.2.

$\diamond$ *τ-distance on a metric space* (X, d). The notion of τ-distance as well as some examples concerning this notion are given in this section. The connexion between τ-distance and w-distance is also recalled. Our contribution in this section is Lemma 1.5.4.

$\diamond$ *Kasahara spaces.* The notions of Kasahara space, generalized Kasahara space and large Kasahara space are recalled in this section together with some useful examples. Our contributions in this section are some solutions to the Problems 1.6.1, 1.6.2 and 1.6.3, posed by I.A. Rus in [121].

$\diamond$ *Operators on Kasahara spaces.* In this section we consider the Kasahara space $(X, \to, d)$ where $d : X \times X \to \mathbb{R}_+$ is a functional. We define the continuity and the closeness for a self-mapping f with respect to $\to$, we give metric conditions for f with respect to d and present some generalized contractions in this sense. Finally, we define the well-posed fixed point problem and the limit shadowing property for f with respect to d. The case of multivalued operators defined on Kasahara spaces is also studied.

Chapter 2: Generalized contractions on Kasahara spaces.

$\diamond$ In the first section of this chapter we develop the theory of some well-known fixed point results as the Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, the Caristi-Browder and Matkowski type theorems. Our results are given for single-valued generalized contractions in the context of a Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We present also some extensions of our results in generalized and large Kasahara spaces.

Our contributions in this section are: Theorem 2.1.2 which is a fixed point theory in Kasahara space extending and complementing the Banach-Caccioppoli's Contraction Principle; Theorem 2.1.3 which is a generalization of Theorem 2.1.2 by replacing the α-contractions with Rakotch operators; Theorem 2.1.5 which is a fixed point theory in Kasahara space extending and complementing the Graphic Contraction Principle; Theorem 2.1.7 and Theorem 2.1.9 which are fixed point theories in Kasahara spaces, extending Caristi and Matkowski type theorems; Theorem 2.1.11 which is a local fixed point result for Zamfirescu operators given in Kasahara spaces, extending and generalizing Krasnoselskii's local fixed point theorem; Theorem 2.1.12 which is a homotopy result given as application of the global variant of Theorem 2.1.11 in large Kasahara spaces; Theorems 2.1.13, 2.1.14, 2.1.15 and 2.1.16 which are fixed point results in generalized Kasahara spaces ($d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$) for α-contractions, graphic contractions, φ-contractions and Caristi operators; Theorem 2.1.17 which is a Maia fixed point result in generalized Kasahara spaces ($d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$); Theorem 2.1.18 which is an application of Theorem 2.1.17, regarding the existence and uniqueness of solution of a Cauchy problem; Theorem 2.1.22 which is a fixed point theory for the local variant of Banach-Caccioppoli's Contraction Principle, given in large Kasahara spaces (d is a w-distance); Theorems 2.1.23, 2.1.24, 2.1.25 and 2.1.26 which are given in large Kasahara spaces (d is perturbed by an increasing, subadditive and continuous function φ), extending and complementing Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, the Caristi-Browder and Matkowski type theorems; Corollary 2.1.1 which is the global variant of Theorem 2.1.11, extending Theorem 1 given by T. Zamfirescu in [150]; Corollary 2.1.2 which is a generalization of Theorem 2.1.11; Corollaries 2.1.3 and 2.1.4 which are fixed point result for single-valued operators satisfying a certain contractive type condition in generalized Kasahara spaces ($d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$); Lemmas 2.1.2, 2.1.4; Definitions 2.1.8, 2.1.9, 2.1.10, 2.1.11; Remarks 2.1.2, 2.1.20 and Examples 2.1.2, 2.1.4. Most of the results presented in the first section are included in the following papers: A.-D. Filip [35], [36]; A.-D. Filip and A. Petruşel [40].

$\diamond$ In the second section, the connexion between the Maia type theorems and the fixed point theorems in Kasahara spaces is presented. Some fixed point theorems of Maia type for single-valued operators in a set endowed with two metrics are also given.

Our contributions in this section are: Theorem 2.2.4 which is a fixed point result given for almost contractions defined on a set endowed with two vector-valued metrics, extending and generalizing Maia's fixed point theorem; Remark 2.2.6 which express the connection between the fixed point result given in Kasahara spaces and the fixed point result of Maia type. Our Theorem 2.2.4 is included in the paper A.-D. Filip and A. Petruşel [39].

$\diamond$ In the third section, we introduce a new notion: *Kasahara space with respect to an operator* and we give in this setting several applications regarding the existence and uniqueness of solutions for integral and differential equations.

Our contributions in this section are: Theorem 2.3.1 which is a fixed point theory in Kasahara spaces with respect to an operator, extending and complementing Banach-Caccioppoli's Contraction Principle; Theorem 2.3.2 which is a fixed point theory in Kasahara spaces with respect to an operator, extending and complementing the Graphic Contraction Principle; Theorem 2.3.3 which is an application of Theorem 2.3.1 regarding the existence and uniqueness of solution for integral equations; Theorems 2.3.4 and 2.3.5 which are also applications of Theorem 2.3.1 regarding the existence and uniqueness of solution for boundary value problems; Definition 2.3.1; Remarks 2.3.1, 2.3.2 and 2.3.3; Examples 2.3.1 and 2.3.2. All of the contributions are included in the paper A.-D. Filip [34].

Chapter 3: Multivalued generalized contractions on Kasahara spaces.

$\diamond$ In the first section of this chapter, we present some fixed point theorems for multivalued generalized contractions in Kasahara spaces, generalized Kasahara spaces and large Kasahara spaces.

Our contributions in this section are: Theorem 3.1.2 which extends Nadler's fixed point theorem (Nadler [94]) from complete metric spaces to Kasahara spaces; Theorem 3.1.3 which generalizes Theorem 3.1.2 by replacing multivalued α-contractions with multivalued Rakotch operators; Theorem 3.1.4 given as a fixed point theory for Theorem 3.1.2; Theorem 3.1.5 which is a strict fixed point result, similar to Theorem 3.1.4; Theorem 3.1.6 which is a fixed point result for multivalued φ-contractions, extending a corresponding result given on complete metric spaces by R. Wegrzyk in [148]; Theorem 3.1.7 which is a fixed point result for multivalued Caristi operators, extending a similar result given on complete metric spaces by N. Mizoguchi and W. Takahashi in [89]; Theorem 3.1.8, a fixed point theorem for multivalued (θ, L)-weak contractions which extends Theorem 3 given by M. Berinde and V. Berinde in [8]; Theorem 3.1.9 and Theorem 3.1.10 which extend the well known fixed point results for multivalued Kannan and Reich operators, from the context of complete metric spaces to the context of Kasahara spaces; Theorem 3.1.11 which is a similar local fixed point result to Theorem 2.1.11, but for multivalued Zamfirescu operators; Theorem 3.1.12 which extends Theorem 3.1.11 to generalized Kasahara spaces ($d(x,y) \in \mathbb{R}^m_+$); Theorem 3.1.13 given as an application for multivalued Zamfirescu operators in generalized Kasahara spaces, concerning the existence of solutions for semi-linear inclusion systems; Theorem 3.1.14 which is a Perov type fixed point result for multivalued operators; Theorem 3.1.15 and Theorem 3.1.16 extending the Kannan (Theorem 3.1.9) and Reich (Theorem 3.1.10) fixed point results in generalized Kasahara spaces; Theorem 3.1.17 which is a fixed point result for multivalued Zamfirescu operators in large Kasahara spaces; Theorem 3.1.18 which is a data dependence result for multivalued Zamfirescu operators in large Kasahara spaces; Corollaries 3.1.1, 3.1.2; 3.1.3, 3.1.4; Lemmas 3.1.2, 3.1.3, 3.1.4; Definition 3.1.3 and Remarks 3.1.7, 3.1.9, 3.1.10. Most of the results presented in the first section of this chapter are included in the following papers: A.-D. Filip [32], [33], [37].

$\diamond$ In the second section of this chapter, we give some fixed point results of Maia type, in close connexion with the results given for multivalued generalized contractions in Kasahara spaces, presented in the first section of the third chapter.

Our contributions in this section are: Theorem 3.2.2 which is a local fixed point result of Maia type in metric spaces; Theorem 3.2.3 which is a local fixed point result of Maia type in generalized metric spaces ($d(x,y) \in \mathbb{R}^m_+$); Corollaries 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5, 3.2.6 and 3.2.7; Remarks 3.2.1, 3.2.2. The results presented in this section are included in the following papers: A.-D. Filip [31], [32], [33]; A.-D. Filip and A. Petruşel [39].

$\diamond$ In the third section of this chapter, we give the notion of *Kasahara space with respect to a multivalued operator* and we prove two fixed point theorems for multivalued α-contractions in the context of Kasahara spaces with respect to a multivalued operator.

Our contributions in this section are: Theorems 3.3.1 and 3.3.2; Definition 3.3.1 and Example 3.3.1.

The author's contributions included in this thesis are also part of the following scientific papers:

- A.-D. Filip, *On the existence of fixed points for multivalued weak contractions*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, ICTAMI 2009, Alba Iulia, pp. 149-158.

- A.-D. Filip, *Fixed point theorems for multivalued contractions in Kasahara spaces*, Carpathian J. Math., submitted.
- A.-D. Filip, *Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces*, Studia Univ. Babeş-Bolyai Math., 56(2011), no. 3, 19-28.
- A.-D. Filip, *Fixed point theorems in Kasahara spaces with respect to an operator and applications*, Fixed Point Theory, 12(2011), no. 2, 329-340.
- A.-D. Filip, *Fixed point theory in large Kasahara spaces*, Anal. Univ. de Vest, Timişoara, submitted.
- A.-D. Filip, *A note on Zamfirescu's operators in Kasahara spaces*, General Mathematics, submitted.
- A.-D. Filip, *Several fixed point results for multivalued Zamfirescu operators in Kasahara spaces*, JP Journal of Fixed Point Theory and Applications, submitted.
- A.-D. Filip and P.T. Petra, *Fixed point theorems for multivalued weak contractions*, Studia Univ. Babeş-Bolyai Math., 54(2009), no. 3, 33-40.
- A.-D. Filip and A. Petruşel, *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory and Applications, 2010, Art. ID 281381, 15 pp.
- A.-D. Filip and A. Petruşel, *Fixed point theorems for operators in generalized Kasahara spaces*, Sci. Math. Jpn., submitted.

A significant part of the original results proved in this thesis were also presented at the following scientific conferences:

- International Conference on Theory and Applications in Mathematics and Informatics (ICTAMI), September 3^{rd}-6^{th}, 2009, Alba Iulia, Romania;
- The 7^{th} International Conference on Applied Mathematics (ICAM7), September 1^{st}-4^{th}, 2010, North University of Baia Mare, Romania;
- International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), July 5^{th}-8^{th}, 2011, Babes-Bolyai University of Cluj-Napoca, Romania;
- The 13^{th} International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), September 26^{th}-29^{th}, 2011, West University of Timişoara, Romania.

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Chapter 1

Preliminaries

The purpose of this chapter is to present the basic notions and results which are further considered in the next chapters of this work, allowing us to present the results of this thesis. In this sense, we recall the notion of L-space, generalized metric, partial metric, w-distance, τ-distance, Kasahara space, generalized Kasahara space and large Kasahara space, giving also their properties and some illustrative examples. The second aim of this chapter is to give some solutions for the Problems 1.6.1, 1.6.2 and 1.6.3, posed by I.A. Rus in [121].

In order to develop the *Preliminaries*, we mention here the references which were taken in view: M. Fréchet [42]; L.M. Blumenthal [12]; M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten [13]; O. Kada, T. Suzuki and W. Takahashi [60]; S. Kasahara [62], [66]; I.A. Rus [117], [119], [121]; I.A. Rus, A. Petruşel and G. Petruşel [124]; T. Suzuki [139], [140].

1.1 L-spaces

In this section we recall the notion of L-space, an abstract space in which works one of the basic tool in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method. On the other hand, the L-space plays a major role in the definition of Kasahara spaces. Some examples of L-spaces are also presented.

The notion of L-space was introduced in 1906 by M. Fréchet in [42] as follows:

Definition 1.1.1. *Let X be a nonempty set. Let*

$$s(X) := \big\{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition, an element $(x_n)_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence and $x = Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we shall write

$$x_n \to x \text{ as } n \to \infty.$$

We denote an L-space by $(X, \to)$.

Remark 1.1.1. *In the work of S. Kasahara, the notion of L-space is regarded as a multivalued convergence structure, more precisely, any convergent sequence can have more than one limit. A significant example in this sense can be found in S. Kasahara [66], Example 2.*

In our thesis, we deal only with L-spaces in Fréchet sense, i.e., any convergent sequence has a unique limit.

We give next some examples of L-spaces.

Example 1.1.1. *L-structures on an ordered set.*

Let $(X, \leq)$ be an ordered set. We consider

(a) $c_1(X) := \big\{(x_n)_{n\in\mathbb{N}} \subset X \mid (x_n)_{n\in\mathbb{N}}$ *is an increasing sequence and there exists* $\sup\limits_{n\in\mathbb{N}} x_n\big\}$.

Let $Lim(x_n)_{n\in\mathbb{N}} = \sup\limits_{n\in\mathbb{N}} x_n$.

If $(x_n)_{n\in\mathbb{N}}$ *is an increasing sequence of* X *and* $\sup\limits_{n\in\mathbb{N}} x_n = x \in X$, *then we denote this by* $x_n \uparrow x$ *as* $n \to \infty$.

(b) $c_2(X) := \big\{(x_n)_{n\in\mathbb{N}} \subset X \mid (x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence and there exists* $\inf\limits_{n\in\mathbb{N}} x_n\big\}$.

Let $Lim(x_n)_{n\in\mathbb{N}} = \inf\limits_{n\in\mathbb{N}} x_n$.

If $(x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of* X *and* $\inf\limits_{n\in\mathbb{N}} x_n = x \in X$, *then we denote this by* $x_n \downarrow x$ *as* $n \to \infty$.

(c) By definition, a sequence $(x_n)_{n\in\mathbb{N}}$ *0-converges to* x *if there exist two sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *such that*

(i) $a_n \uparrow x$ *as* $n \to \infty$ *and* $b_n \downarrow x$ *as* $n \to \infty$;

(ii) $a_n \leq x_n \leq b_n$ *for all* $n \in \mathbb{N}$.

We denote this convergence by $x_n \xrightarrow{0} x$ *as* $n \to \infty$.

The couples $(X, \uparrow)$, $(X, \downarrow)$ *and* $(X, \xrightarrow{0})$ *are L-spaces.*

Example 1.1.2. *Let* $(X, \|\cdot\|)$ *be a Banach space. We denote by* $\xrightarrow{\|\cdot\|}$ *the strong convergence in* X *and by* $\rightharpoonup$ *the weak convergence in* X. *Then* $(X, \xrightarrow{\|\cdot\|})$ *and* $(X, \rightharpoonup)$ *are L-spaces.*

Example 1.1.3. *Let (X, d) and (Y, ρ) be two metric spaces. Let $\mathbb{M}(X, Y)$ be the set of all operators from X to Y. We denote by $\overset{p}{\to}$ the pointwise convergence on $\mathbb{M}(X, Y)$, by $\overset{unif.}{\to}$ the uniform convergence on $\mathbb{M}(X, Y)$.*

By definition (see M. Angrisani and M. Clavelli [4]), a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{M}(X, Y)$ converges with continuity to f if

$$x_n \overset{d}{\to} x \text{ as } n \to \infty \Rightarrow f_n(x_n) \text{ converges to } f(x) \text{ as } n \to \infty.$$

We denote by $\overset{cont.}{\to}$ this convergence.

Then $(\mathbb{M}(X, Y), \overset{p}{\to})$, $(\mathbb{M}(X, Y), \overset{unif.}{\to})$ and $(\mathbb{M}(X, Y), \overset{cont.}{\to})$ are L-spaces.

Example 1.1.4. *In general, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense (i.e. $d(x, y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e. $d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details in this sense, we have the paper of I.A. Rus [117] and the references therein.*

1.2 Generalized metric spaces

In this section we deal with the notions of distance functional and G-metric defined on a nonempty set X, both notions being used in the definition of generalized metric space. The connexion between L-spaces and generalized metric spaces is also discussed.

By a generalized metric on a given nonempty set X, we mean:

1°. A functional $d : X \times X \to \mathbb{R}_+$ (also called *distance functional*) which satisfies some axioms. The following axioms appear in the definitions of several types of generalized metrics:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (i_1) $d(x, x) = 0$, for all $x \in X$;
- (i_2) $d(x, y) = 0$ implies $x = y$;
- (i_3) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
- (i_4) $d(x, y) = d(y, x) = 0$ implies $x = y$;
- (i_5) $d(x, x) = d(y, y) = d(x, y)$ if and only if $x = y$;
- (i_6) $d(x, x) \le d(x, y)$, for all $x, y \in X$;
- (i_7) $d(y, y) \le d(x, y)$, for all $x, y \in X$;
- (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(ii_1) $d(x,y) \leq cd(y,x)$, for all $x,y \in X$, with $c > 0$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$;

(iii_1) $d(x,y) \leq d(x,z) + d(y,z)$, for all $x,y,z \in X$;

(iii_2) $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, for all $x,y,z \in X$;

(iii_3) for all $\varepsilon > 0$, $d(x,z) \leq \varepsilon$, $d(z,y) \leq \varepsilon$ imply $d(x,y) \leq \varepsilon$;

(iii_4) $d(x,y) \leq b[d(x,z) + d(z,y)]$, for all $x,y,z \in X$, with $b > 1$;

(iii_5) $d(x,y) \leq a\max\{d(x,z), d(z,y)\}$, for all $x,y,z \in X$, with $a > 1$;

(iii_6) $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$, for all $x,y,z \in X$.

By definition, d is a:

- ⊙ *premetric* (or *quasi-pseudometric*) if d satisfies $(i_1) + (iii)$;
- ⊙ *pseudometric* if d satisfies $(i_1) + (ii) + (iii)$;
- ⊙ *quasimetric* (or *halfmetric*) if d satisfies $(i_3) + (iii)$;
- ⊙ *dislocated metric* (or *d-metric*) if d satisfies $(i_4) + (ii) + (iii)$;
- ⊙ *semimetric* if d satisfies $(i) + (ii)$;
- ⊙ *symmetric* if d satisfies $(i_2) + (ii)$;
- ⊙ *ultrametric* if d satisfies $(i) + (ii) + (iii_2)$ or $(i) + (ii) + (iii_3)$;
- ⊙ *quasiultrametric* if d satisfies $(i) + (ii_1) + (iii_5)$;
- ⊙ *b-metric* if d satisfies $(i) + (ii) + (iii_4)$;
- ⊙ *partial metric* if d satisfies $(i_5) + (i_6) + (ii) + (iii_6)$.

2°. A functional $d : X \times X \to (G, +, \leq, \overset{G}{\to})$ (also called G-metric) satisfying the following axioms:

(i) $d(x,y) \geq 0$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$,

where the structure $(G, +, \leq, \overset{G}{\to})$ is an ordered L-group [1].

[1] Let $(G,+)$ be a group, $\leq$ be a partial order relation on G and $\overset{G}{\to}$ be an L-space structure on G. By definition, $(G, +, \leq, \overset{G}{\to})$ is an ordered L-group if the following axioms are satisfied:

(1) $x_n \to x$ and $y_n \to y$ as $n \to \infty$ imply $x_n + y_n \to x + y$ as $n \to \infty$;

(2) $x_n \to x$, $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$ imply $x \leq y$;

(3) $x \leq y$ and $u \leq v$ imply $x + u \leq y + v$.

More consideration on ordered L-groups can be found in the work of I.A. Rus, A. Petruşel and G. Petruşel [124], p.79 .

Remark 1.2.1. *For the definitions mentioned in* 1° *and for the mathematics on a generalized metric space, we have the work of M. Fréchet [42], F. Hausdorff [46], L.M. Blumenthal [12], K. Kunen and J.F. Vaughan [80], J. Dugundji [26], J.L. Kelley [71], C.E. Aull and R. Lowen [6], R. Engelking [30], M.A. Khamsi and W.A. Kirk [73], R. Kopperman [78], J.L. Reilly [108], M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten [13] and I.A. Rus [119].*

Remark 1.2.2. *In the case of* G*-metric, there are several papers with fixed point results in the case when* $G = \mathbb{R}_+$*;* $G = \mathbb{R}$*;* $G = \mathbb{R}^m_+$*;* $G = \mathbb{R}_+ \cup \{+\infty\}$*;* $G = K$*, where* K *is a cone in an ordered Banach space;* $G = E$*, where* E *is an ordered linear space with a notion of linear convergence. The works of C.E. Aull and R. Lowen [6], L.M. Blumenthal [12], R. Engelking [30], M. Fréchet [42], W.A. Kirkand and B. Sims [77], K. Kunen and J.F. Vaughan [80], A.I. Perov [99], I.A. Rus [112],[115], I.A. Rus, A. Petruşel and G. Petruşel [123], E. De Pascale, G. Marino and P. Pietramala [23], M. Frigon [43], H.-P. A. Künzi and V. Vajner [81], S.G. Matthews [88], S.J. O'Neill [97], P.P. Zabrejko [149] are relevant in this sense.*

As we know, if (X, d) is a metric space, then $(X, \overset{d}{\to})$ is an L-space, where $\overset{d}{\to}$ is the convergence induced by the metric d on X. The convergence structure $\overset{d}{\to}$ is defined as follows: if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $x \in X$ then

$$x_n \overset{d}{\to} x \text{ as } n \to \infty \Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0. \tag{1.2.1}$$

The same statements hold in the case when d is a G-metric on X. In this case, the couple (X, d) is called a generalized metric space and $(X, \overset{d}{\to})$ is an L-space. But what happens when d is a distance functional? Clearly the couple (X, d) is called also generalized metric space, but is $(X, \overset{d}{\to})$ an L-space? The following problem arise:

Problem 1.2.1. *Which of the distance functionals* $d : X \times X \to \mathbb{R}_+$ *induces an* L*-space structure on* X*?*

We will try to give a solution to this problem. By following the works of M. Fréchet [42] and L.M. Blumenthal [12], an L-spaces is an *abstract space*, i.e., an abstract set (denoted X) endowed with a topology (denoted τ). The conditions (i) and (ii) of Definition 1.1.1 of L-space, are establishing the so called *limit-topology* on X.

Another way to introduce a topology in an abstract set X is to assume a convention according to which some certain subsets are called *open*. Such a convention is subjected to a few very simple restrictions as, for example, that the empty set and the whole set be open, and that the union of any collection of open sets be open. Bellow we present the construction of such a topology.

Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ be a distance functional on X.

Notice that for each distance functional d on X, we can construct the set

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\} \tag{1.2.2}$$

for any $x \in X$ and $r > 0$. We call this set the r-ball centered in x.

A subset Y of X is called *open set* in X if for any element $y \in Y$, there exists $r > 0$ such that the r-ball centered in y is included in Y.

In this setting, we define

$$\tau_d := \{Y \subset X \mid \text{for each } y \in Y \text{ there exists } r > 0 \text{ such that } B_d(y, r) \subset Y\} \tag{1.2.3}$$

which is the topology (also called *open-set topology*) generated by the distance functional d on X. The couple (X, τ_d) is a topological space.

Let $\overset{\tau_d}{\to}$ be the convergence structure induced by the topology τ_d on X, defined as follows:

$$\begin{aligned} &\text{for any sequence } (x_n)_{n\in\mathbb{N}} \text{ of } X \text{ and } x \in X, \text{ we have } x_n \overset{\tau_d}{\to} x \text{ as } n \to \infty \text{ if and only if} \\ &\text{for any } r\text{-ball } B_d(x, r) \text{ there exists } n_0 \in \mathbb{N} \text{ such that } x_n \in B_d(x, r), \text{ for all } n \geq n_0. \end{aligned} \tag{1.2.4}$$

In order to obtain the L-space $(X, \overset{\tau_d}{\to})$, any convergent sequence with respect to $\overset{\tau_d}{\to}$ must have a unique limit in X, i.e., the topological space (X, τ_d) must be a Hausdorff topological space (or T_2 topological space).

In our case τ_d is a Hausdorff topology if and only if the intersection of any two open sets is open and for all $x, y \in X$ with $x \neq y$, there exists two real numbers $r_x > 0$ and $r_y > 0$ such that

$$B_d(x, r_x) \cap B_d(y, r_y) = \emptyset.$$

Another important aspect regarding a distance functional d is its continuity. Let $\overset{d}{\to}$ be the convergence structure induced by d on X (not necessarily defined as in 1.2.1). Then we have the following

Definition 1.2.1. *Let (X, d) be a generalized metric space. Then the distance functional d is said to be continuous at $(x, y) \in X \times X$ if and only if for any two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of X,*

$$x_n \overset{d}{\to} x \text{ and } y_n \overset{d}{\to} y \text{ as } n \to \infty \text{ imply } \lim_{n\to\infty} d(x_n, y_n) = d(x, y). \tag{1.2.5}$$

The distance functional d is continuous on X if and only if it is continuous at each point-pair of X.

There is an important connexion between the triangle inequality (iii) and the uniform continuity of a distance functional d on X. If (iii) is satisfied, then d is uniformly continuous on X which implies further that d is continuous on X. Requiring that a distance functional d to be uniformly continuous on X means that corresponding to an arbitrary positive ε, there is a positive number δ_ε such that for all pairs (a, b), (c, d) of ordered points of X.

$$d(a, c) + d(b, d) < \delta_\varepsilon \text{ shall imply } |d(a, b) - d(c, d)| < \varepsilon. \tag{1.2.6}$$

Since δ_ε may be taken equal to ε, the requirement (1.2.6) will be most simply satisfied if and only if

$$|d(a, b) - d(c, d)| \leq d(a, c) + d(b, d). \tag{1.2.7}$$

But the relation (1.2.7) is equivalent to the triangle inequality (see L.M. Blumenthal [12], p.15).

We analyze the following particular cases, when d is a:

(1) *premetric* (or *quasi-pseudometric*) i.e. d satisfies

(i_1) $d(x,x)=0$, for all $x \in X$;

(iii) $d(x,y) \leq d(x,z)+d(z,y)$, for all $x,y,z \in X$.

The pair (X,d) is called a premetric space. Notice that the topology induced by a premetric is not necessarily a Hausdorff topology. Indeed, let us consider the following

Example 1.2.1. *Let $X=\{0,1\}$ and $d: X \times X \to \mathbb{R}_+$ be defined as follows:*

$$d(0,0)=d(1,1)=0,$$

$$d(0,1)=1 \text{ and } d(1,0)=0.$$

Then d is a premetric on X. By following L.A. Steen and J.A. Seebach Jr. [136], p.47, we may define an excluded point topology on X by declaring open, in addition to X itself, all sets which do not include a given point $p \in X$. Since X has just two points, the excluded point topology generated by d on X is called also the Sierpinski topology, which is not a Hausdorff topology.

In [62] S. Kasahara shows that in some additional conditions, a premetric induces an L-space structure on X.

Since for a premetric d we have no symmetry, let us consider the functional $d^*: X \times X \to \mathbb{R}_+$, defined by

$$d^*(x,y)=d(y,x), \text{ for all } x,y \in X.$$

If d is a premetric on X, then d^* is also a premetric (the so called *dual premetric* of d). Hence, the couples (X,d) and (X,d^*) are premetric spaces.

We consider now the premetric space (X,d).

A sequence $(x_n)_{n\in\mathbb{N}}$ of X is called r-Cauchy (*right*-Cauchy) sequence if and only if for each $\varepsilon>0$, there exists $n_0 \in \mathbb{N}$ such that for any $m,n \in \mathbb{N}$ with $m \geq n \geq n_0$, we have $d(x_m,x_n)<\varepsilon$. Notice that an r-Cauchy sequence may have more than one limit.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a premetric space (X,d) is r-convergent (*right*-convergent) to $x \in X$ and we denote this by

$$x_n \stackrel{d}{\to} x \text{ as } n \to \infty$$

if and only if for each $\varepsilon>0$, there exists a positive integer n_ε such that $d(x,x_n)<\varepsilon$, whenever $n \geq n_\varepsilon$, i.e. $\lim\limits_{n\to\infty} d(x,x_n)=0$.

In order to have a unique limit for any r-convergent sequence on a premetric space (X,d) we use the notion of r-separated premetric space. We say that (X,d) is an r-separated premetric space if every sequence in X is r-convergent to at most one point of X. Notice also that if (X,d) is an r-separated premetric space, then $d(x,y)=0 \Rightarrow x=y$, for all $x,y \in X$ (an important axiom used to prove that a certain topology is Hausdorff).

Similarly, we define the l-Cauchy ($left$-Cauchy) sequence, l-convergent ($left$-convergent) sequence and l-separateness property for the premetric space (X, d^*).

We conclude that if (X, d) is an r-separated premetric space, then $(X, \stackrel{\tau_d}{\rightarrow})$ is an L-space, where $\stackrel{\tau_d}{\rightarrow}$ is defined by

$$x_n \stackrel{\tau_d}{\rightarrow} x \text{ as } n \to \infty \text{ if and only if } x_n \stackrel{d}{\rightarrow} x \text{ as } n \to \infty.$$

Similarly, if (X, d) is an l-separated premetric space, then $(X, \stackrel{\tau_{d^*}}{\rightarrow})$ is an L-space, where $\stackrel{\tau_{d^*}}{\rightarrow}$ is defined by

$$x_n \stackrel{\tau_{d^*}}{\rightarrow} x \text{ as } n \to \infty \text{ if and only if } x_n \stackrel{d^*}{\rightarrow} x \text{ as } n \to \infty.$$

An example of a separated premetric space is given bellow.

Example 1.2.2 (S. Kasahara [62],[66])**.** *Let B be a nonempty bounded star-shaped convex subset of a Hausdorff topological linear space (X, τ). (B is said to be star-shaped if $\lambda B \subset B$ for every $\lambda \in [0, 1]$).*

Let $M := \{(x, y) \in X \times X \mid x - y \in \lambda B,$ for some $\lambda > 0\}$.

Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \begin{cases} \inf\{\lambda > 0 \mid x - y \in \lambda B\}, & \text{for all } (x, y) \in M \\ 0, & \text{otherwise.} \end{cases}$$

Then (X, d) is an r-separated and l-separated premetric space.

(**2**) *pseudometric* (sometimes, the term of *gauge* is used instead of *pseudometric*), i.e. d satisfies

(i_1) $d(x, x) = 0$, for all $x \in X$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Let us notice that a *pseudometric* is a premetric which satisfies in addition the axiom of symmetry (ii). In this case we do not need to define the two type of convergences presented in the premetric case.

The couple (X, d) is a pseudometric space. The r-balls defined in (1.2.2) together with the empty set, form the basis of the topology τ_d in X. This topology is Hausdorff if and only if the pseudometric d satisfies in addition the axiom

(i_2) $d(x, y) = 0$ implies $x = y$.

But in this case the pseudometric d becomes a metric, and the topology τ_d is the topology induced by the metric d.

Hence, we conclude that in general, a pseudometric on X, does not induce an L-space structure on X. In this sense, we present bellow the following

Example 1.2.3. *Let $X = \mathbb{R}^2$ and consider the function $d : X \times X \to \mathbb{R}_+$ defined by*

$$d(x,y) = |x_1 - y_1|, \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in X.$$

Then $d(x,x) = |x_1 - x_1| = 0$, $d(x,y) = |x_1 - y_1| = |y_1 - x_1| = d(y,x)$ and the triangle inequality follows from the triangle inequality of $\mathbb{R}$. Hence, d is a pseudometric.

But d is not a metric since we can find two distinct points $x, y \in \mathbb{R}^2$, $x \neq y$ such that $d(x,y) = 0$. Indeed, choose $x = (2,3)$ and $y = (2,4)$ in $\mathbb{R}^2$. Then $d(x,y) = |2-2| = 0$. In this case, we get that $d(x,y) < r_x$ for any $r_x > 0$, i.e., $y \in B_d(x,r)$. On the other hand, since the symmetry holds for d, we have similarly that $d(y,x) < r_y$ for any $r_y > 0$, i.e., $x \in B_d(y,r)$. So any open ball containing x contains also y and conversely. Hence, no open set can separate the two distinct points x, y. The topology τ_d induced by d on X is not a Hausdorff topology and thus, $(X, \stackrel{\tau_d}{\to})$ is not an L-space.

(**3**) *quasimetric* (or *halfmetric*) i.e. d satisfies

- (i_3) $d(x,y) = d(y,x) = 0$ if and only if $x = y$;
- (iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

In this case, the couple (X,d) is a quasimetric space.

Since $d(x,y) = d(y,x)$ does not hold for any $x, y \in X$, by following M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten [13], we can define two types of Cauchy sequences and convergence on the quasimetric space (X,d).

A sequence $(x_n)_{n\in\mathbb{N}}$ is r-Cauchy (*right*-Cauchy) in the quasimetric space (X,d) if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \leq \varepsilon$, for any $n, m \in \mathbb{N}$ with $n \geq m \geq n_0$.

A sequence $(x_n)_{n\in\mathbb{N}}$ is l-Cauchy (*left*-Cauchy) in the quasimetric space (X,d) if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$, for any $n, m \in \mathbb{N}$ with $n \geq m \geq n_0$.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a quasimetric space (X,d) and $x \in X$ then we denote and define the r-convergence and l-convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x as follows:

$$x_n \stackrel{r.d}{\to} x \text{ as } n \to \infty \Leftrightarrow \lim_{n\to\infty} d(x, x_n) = 0$$
$$x_n \stackrel{l.d}{\to} x \text{ as } n \to \infty \Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$$

where $\stackrel{r.d}{\to}$ denotes the r-convergence and respectively $\stackrel{l.d}{\to}$ denotes the l-convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x.

Notice that Cauchy sequences may have more than one limit, but in the case of quasimetric spaces, the uniqueness of the limit is assured by the axiom (i_3).

If (X,d) is a quasimetric space, then the couples $(X, \stackrel{r.d}{\to})$ and $(X, \stackrel{l.d}{\to})$ are L-spaces.

Some examples of quasimetrics spaces are given bellow.

Example 1.2.4 (M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten [13]). *Let $X = \mathbb{R}_+$ and $d : X \times X \to \mathbb{R}_+$ be a functional defined by*

$$d(x,y) = \begin{cases} 0, & \text{if } x \geq y \\ y - x, & \text{if } x < y \end{cases}$$

for all $x, y \in X$. Then (X, d) is a quasimetric space.

Example 1.2.5 (D. Doitchinov [25])**.** *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+$ be a functional defined by*

$$d(x,y) = \begin{cases} |x-y|, & \text{if } y \neq 0 \text{ or } x = y = 0 \\ 1, & \text{if } y = 0 \text{ and } 0 < x \leq 1 \end{cases}$$

for all $x, y \in X$. Then (X, d) is a quasimetric space.

(**4**) *dislocated metric* (or *d-metric*) i.e. d satisfies

- (i_4) $d(x,y) = d(y,x) = 0$ implies $x = y$;
- (ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;
- (iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

On dislocated metric spaces we have the work of P. Hitzler and A.K. Seda [50]. If (X, d) is a dislocated metric spaces, then $(X, \overset{d}{\to})$ is an L-space, where $\overset{d}{\to}$ is defined by (1.2.1).

(**5**) *semimetric* i.e. d satisfies

- (i) $d(x,y) = 0$ if and only if $x = y$;
- (ii) $d(x,y) = d(y,x)$, for all $x, y \in X$.

In general, if d is a semimetric on X, we define the convergence induced by d, for a sequence $(x_n)_{n\in\mathbb{N}}$ of X to an element $x \in X$, as in (1.2.1).

On semimetric spaces, we have the work of L.M. Blumenthal [12] where it is proved that if (X, d) is a semimetric space with d continuous (see Definition 1.2.1) then

- $B_d(x,r)$ is an open set, for all $x \in X$ and $r > 0$;
- (X, τ_d) is a Hausdorff topological space.

Hence, if d is a continuous semimetric on X, then $(X, \overset{d}{\to})$ is an L-space.

In general, a semimetric is not continuous (see L.M. Blumenthal [12] p.9).

Notice also that since d is a semimetric on X, the well-known triangle inequality (iii) is not necessarily satisfied. An example of a semimetric is given bellow.

Example 1.2.6. *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+$ be defined by*

$$d(x,y) = (x-y)^2 \text{ for all } x, y \in X.$$

In this case d is a uniformly continuous semimetric on X, but not a metric. (By choosing $x = \frac{3}{4}$, $y = 0$ and $z = \frac{1}{2}$, the triangle inequality does not hold).

(**6**) *symmetric* i.e. d satisfies

- (i_2) $d(x,y) = 0$ implies $x = y$;

(*ii*) $d(x,y) = d(y,x)$, for all $x, y \in X$.

If d is a symmetric on X, then (X,d) is a symmetric space.

The definitions regarding the convergence structure $\stackrel{d}{\to}$ and the continuity of a symmetric d are the same as in the semimetric case. Notice that any semimetric is a symmetric, but not conversely and in general a symmetric is not necessarily continuous. An example is given in this sense:

Example 1.2.7. *Let $X = [0,1]$ and let $d : X \times X \to \mathbb{R}_+$ defined by*

$$d(x,y) = \begin{cases} (x-y)^2, & \text{if } x \neq 1 \text{ or } y \neq 1 \\ \sqrt{2}, & \text{otherwise} \end{cases}$$

for all $x, y \in X$. In this case d is a symmetric, but not a semimetric, since for $x = y = 1$ we have $d(1,1) = \sqrt{2} \neq 0$.

On the other hand d is not continuous on X. Indeed, by choosing the sequences $(x_n)_{n\in\mathbb{N}} \subset X$, $x_n = 1 - \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $(y_n)_{n\in\mathbb{N}} \subset X$, $y_n = \frac{n^2}{n^2+1}$ for all $n \in \mathbb{N}$ we have that $x_n \stackrel{d}{\to} 1$ and $y_n \stackrel{d}{\to} 1$ as $n \to \infty$. But $d(1,1) \neq \lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

As in the semimetric case, it can be shown that if d is continuous, then the r-ball $B_d(x,r)$ defined as in (1.2.2) is open for any $x \in X$ and $r > 0$. More than that, the topology τ_d defined as in (1.2.3) is a Hausdorff topology. Hence, if (X,d) is a symmetric space with d continuous, then $(X, \stackrel{d}{\to})$ is an L-space.

(**7**) *ultrametric* i.e. d satisfies

(*i*) $d(x,y) = 0$ if and only if $x = y$;

(*ii*) $d(x,y) = d(y,x)$, for all $x, y \in X$;

together with

(*iii*$_2$) $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, for all $x, y, z \in X$,

or

(*iii*$_3$) for all $\varepsilon > 0$, $d(x,z) \leq \varepsilon$, $d(z,y) \leq \varepsilon$ imply $d(x,y) \leq \varepsilon$.

In this case, the couple (X,d) is called ultrametric space. Notice that since (*i*) and (*ii*) are satisfied by d, the ultrametric space (X,d) is also a semimetric space. Regarding the axioms (*iii*$_2$), (*iii*$_3$) and the triangle inequality (*iii*), we have the implications (*iii*$_2$) $\Rightarrow$ (*iii*) $\Rightarrow$ (*iii*$_3$).

We conclude that if (X,d) is an ultrametric space, then

- if d satisfies (*iii*$_2$), then (X,d) is a metric space and $(X, \stackrel{d}{\to})$ is an L-space
- if d satisfies (*iii*$_3$) and d is continuous, then $(X, \stackrel{d}{\to})$ is an L-space

where the convergence structure $\xrightarrow{d}$ is defined as in (1.2.1).

An example of ultrametric which is not continuous on X is the following

Example 1.2.8. *Let X be a nonempty set. Then the discrete metric $d : X \times X \to \mathbb{R}_+$, defined by*

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

for all $x, y \in X$ is an ultrametric on X. Recall that the discrete metric induces the discrete topology on X, which is a Hausdorff topology. Clearly (X, d) is a metric space and $(X, \xrightarrow{d})$ is an L-space.

(**8**) *quasiultrametric* i.e. d satisfies

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii_1) $d(x,y) \leq c d(y,x)$, for all $x, y \in X$, with $c > 0$;

(iii_5) $d(x,y) \leq \alpha \max\{d(x,z), d(z,y)\}$, for all $x, y, z \in X$, with $\alpha > 1$.

In the quasiultrametric space (X, d), we consider the convergence structure $\xrightarrow{d}$ defined as in (1.2.1). On the other hand, let $\rho : X \times X \to \mathbb{R}_+$ be a functional defined by $\rho(x,y) = d(y,x)$, for all $x, y \in X$. Notice that by (ii_1) we have the following implication

$$\lim_{n\to\infty} \rho(x_n, x) = 0 \Rightarrow x_n \xrightarrow{d} x \text{ as } n \to \infty.$$

Open question: if (X, d) is a quasiultrametric space, is $(X, \xrightarrow{d})$ an L-space?

(**9**) *b-metric* i.e. d satisfies

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(iii_4) $d(x,y) \leq b[d(x,z) + d(z,y)]$, for all $x, y, z \in X$, with $b > 1$.

On *b-metric* spaces we have the work of M.-F. Bota [15]. If (X, d) is a *b-metric* space then the convergence structure $\xrightarrow{d}$ is defined as in (1.2.1). On the other hand any *b-metric* space is also a semimetric space, but notice also that a *b-metric* is not necessarily continuous[2] on X. An example of b-metric, which is not continuous is presented bellow.

Example 1.2.9. *Let $b \in \mathbb{R}$, $b > 1$. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+$ be defined by*

$$d(x,y) = \begin{cases} b(x-y)^2, & \text{if } (x,y) \in X \times X \setminus \{(0,1),(1,0)\} \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

[2]For continuity, see Definition 1.2.1.

for all $x, y \in X$. Then d is a b-metric, but it is not continuous on X.

Indeed, by choosing two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of X, for example $x_n = 1 - \frac{1}{n+1}$ and $y_n = \frac{1}{n^2+1}$ for all $n \in \mathbb{N}$ we have

$$\lim_{n\to\infty} d(x_n, 1) = \lim_{n\to\infty} b\left(\frac{1}{n+1}\right)^2 = 0 \ \Rightarrow \ x_n \xrightarrow{d} 1 \ \text{as } n \to \infty$$
$$\lim_{n\to\infty} d(y_n, 0) = \lim_{n\to\infty} b\left(\frac{1}{n^2+1}\right)^2 = 0 \ \Rightarrow \ y_n \xrightarrow{d} 0 \ \text{as } n \to \infty$$

but on the other hand

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} b\left(1 - \frac{1}{n+1} - \frac{1}{n^2+1}\right)^2 = b \neq \frac{1}{2} = d(1,0),$$

since $b > 1$. Hence d is not continuous in $(1,0)$.

If (X, d) is a b-metric space and d is continuous on X, then $(X, \xrightarrow{d})$ is an L-space.

(**10**) *partial-metric*

We present this case in the next section.

1.3 Partial metric spaces

The notion of partial metric was introduced by S.G. Matthews in [87] as follows:

Definition 1.3.1. *Let X be a nonempty set. A functional $p : X \times X \to \mathbb{R}_+$ is a partial metric on X if p satisfies the following conditions:*

(p_1) $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$;

(p_2) $p(x, x) \leq p(x, y)$, *for all* $x, y \in X$;

(p_3) $p(x, y) = p(y, x)$, *for all* $x, y \in X$;

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, *for all* $x, y, z \in X$.

The couple (X, p), where X is a nonempty set and p is a partial metric on X, is called a partial metric space.

Some examples of partial metric spaces are presented bellow:

Example 1.3.1. *Let (X, d) be a metric space. Then (X, d) is a partial metric space.*

Example 1.3.2. *Let $X = \mathbb{R}$ and $p : X \times X \to \mathbb{R}_+$ defined by $p(x, y) = \max\{0, x, y\}$, for all $x, y \in X$. Then (X, p) is a partial metric space.*

Example 1.3.3. *Let Y be a set and $X := Y^\infty$ - the set of all finite and infinite sequences in Y. Let $\eta : Y^\infty \times Y^\infty \to \mathbb{R}_+ \cup \{+\infty\}$ be defined by*

$$\eta(x,y) := \begin{cases} \sup\{n \in \mathbb{N} \mid x(k) = y(k),\ k \leq n\}, & \text{if } x(0) = y(0) \\ 0, & \text{if } x(0) \neq y(0). \end{cases}$$

Let $p : X \times X \to \mathbb{R}_+$ be defined by $p(x,y) := 2^{-\eta(x,y)}$, for $x, y \in X$. Then (X,p) is a partial metric space.

Example 1.3.4. *Let $X := \{[a,b] \mid a, b \in \mathbb{R}_+,\ a \leq b\}$ and $p : X \times X \to \mathbb{R}_+$ be the functional defined by*

$$p([a,b],[c,d]) := \max\{b,d\} - \min\{a,c\},\ \text{for all } [a,b],[c,d] \in X.$$

Then (X,p) is a partial metric space.

Example 1.3.5 (generic example)**.** *By definition, a quasimetric space (X,d) is weightable if and only if there exists $w : X \to \mathbb{R}_+$ such that*

$$d(x,y) + w(x) = d(y,x) + w(y),\ \text{for all } x, y \in X.$$

If (X,d,w) is a weighted quasimetric space, then the functional $p : X \times X \to \mathbb{R}_+$, defined by

$$p(x,y) := d(x,y) + w(x),\ \text{for all } x, y \in X$$

is a partial metric on X.

By following the papers of S.G. Matthews [87], [88], we can construct a quasimetric and a metric starting from a partial metric. The following lemma in relevant in this sense.

Lemma 1.3.1. *Let (X,p) be a partial metric space.*

(1) *the functional $q_p : X \times X \to \mathbb{R}_+$ defined by*

$$q_p(x,y) := p(x,y) - p(x,x),\ \text{for all } x, y \in X$$

is a quasimetric on X.

(2) *the functional $d_p : X \times X \to \mathbb{R}_+$ defined by*

$$d_p(x,y) := 2p(x,y) - p(x,x) - p(y,y),\ \text{for all } x, y \in X$$

is a metric on X.

Due to Lemma 1.3.1 some elementary remarks are arising.

Remark 1.3.1. *Notice that we have a connexion between the functionals q_p and d_p given by*

$$d_p(x,y) := q_p(x,y) + q_p(y,x),\ \text{for all } x, y \in X.$$

Remark 1.3.2. *Since q_p is a quasimetric on X and d_p is a metric on X, we have that (X, q_p) is a quasimetric space and (X, d_p) is a metric space, both obtained from the partial metric space (X, p).*

In the sequel, we analyze the convergence structures induced by the functionals p, q_p and d_p on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and x be an arbitrary element of X.

(i) In the partial metric space (X, p), let $\xrightarrow{p}$ be the convergence structure induced by p on X. Then $\xrightarrow{p}$ is defined as follows

$$x_n \xrightarrow{p} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

(ii) In the quasimetric space (X, q_p), since we do not have symmetry for q_p, (i.e., $q_p(x, y) = q_p(y, x)$ is not necessary true for all $x, y \in X$) we have two type of convergences induced by q_p on X. Let $\overset{r.q_p}{\to}$ be the $right$-convergence structure induced by q_p on X, respectively $\overset{l.q_p}{\to}$ be the $left$-convergence structure induced by q_p on X. Then

$$x_n \overset{r.q_p}{\to} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} q_p(x, x_n) = 0$$
$$x_n \overset{l.q_p}{\to} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} q_p(x_n, x) = 0.$$

(iii) In the metric space (X, d_p), let $\xrightarrow{d_p}$ be the convergence structure induced by d_p on X. We have

$$x_n \xrightarrow{d_p} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} d_p(x_n, x) = 0.$$

Remark 1.3.3. *On a partial metric space (X, p), between the convergence structures $\xrightarrow{p}$, $\overset{r.q_p}{\to}$, $\overset{l.q_p}{\to}$ and $\xrightarrow{d_p}$, we can establish the following connexions:*

$$\begin{aligned} x_n \xrightarrow{p} x \textit{ as } n \to \infty &\textit{ if and only if } x_n \xrightarrow{d_p} x \textit{ as } n \to \infty \\ &\textit{ if and only if } x_n \overset{r.q_p}{\to} x \textit{ and } x_n \overset{l.q_p}{\to} x \textit{ as } n \to \infty. \end{aligned}$$

Remark 1.3.4. *The couples $(X, \xrightarrow{p})$,$(X, \overset{r.q_p}{\to})$,$(X, \overset{l.q_p}{\to})$ and $(X, \xrightarrow{d_p})$ are L-spaces.*

The notion of Cauchy sequence in a partial metric space is also introduced by S.G. Matthews in [87] as follows:

Definition 1.3.2. *Let (X, p) be a partial metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X if and only if there exists the limit*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} p(x_n, x_m).$$

Also we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,p) if and only if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d_p). In addition, (X,p) is a complete partial metric space if and only if (X,d_p) is a complete metric space.

Remark 1.3.5. *For the convergence structure $\overset{d_p}{\to}$ in a partial metric space (X,p), by definition, the following statement holds:*

$$x_n \overset{d_p}{\to} x \text{ as } n\to\infty \text{ if and only if } p(x,x)=\lim_{n\to\infty} p(x_n,x)=\lim_{\substack{n\to\infty\\ m\to\infty}} p(x_n,x_m).$$

Remark 1.3.6. *For more considerations on partial metric spaces and applications, see S.G. Matthews [87], [88], H.-P. A. Künzi and V. Vajner [81], M. Fitting [41], R. Kopperman, S. Matthews and H. Pajoohesh [79], S.J. O'Neill [97], S. Romaguera and M. Schellekens [109], A.K. Seda [134], S. Oltra and O. Valero [96], I.A. Rus [119].*

1.4 w-distance on a metric space (X,d)

Another generalized metric is the so called w-distance. We present in this section its definition, properties and some examples.

In 1996, O. Kada, T. Suzuki and W. Takahashi [60] introduced the concept of w-distance on a metric space, gave some examples, properties of w-distance and they improved the Caristi's fixed point theorem [18], Ekeland's ε-Variational Principle [28] and the nonconvex minimization theorem according to Takahashi [143]. Finally, using the concept of w-distance, they proved a fixed point theorem in a complete metric space, that generalizes the fixed point theorems of P.V. Subrahmanyam [137], R. Kannan [61] and L.B. Čirič [20].

In the same year, T. Suzuki and W. Takahashi [141] gave other properties for w-distance and by using this notion, they proved a fixed point theorem for set valued mappings on complete metric spaces, which is related to Nadler's fixed point theorem [94] and Edelstein's theorem [27]. Finally, they gave a characterization of metric completeness.

In 1997, T. Suzuki [138] gave several new properties for w-distance, which generalize some of the properties mentioned in [60]. He proved also several fixed point theorems which are generalizations of the Banach's contraction principle and Kannan's fixed point theorem.

Let us recall now the notion of w-distance.

Definition 1.4.1 (O. Kada, T. Suzuki and W. Takahashi [60])**.** *Let (X,d) be a metric space. Then a function $p: X\times X\to\mathbb{R}_+$ is called a w-distance on X if the following conditions are satisfied:*

(w_1) $p(x,z)\le p(x,y)+p(y,z)$, *for all* $x,y,z\in X$;

(w_2) for any $x\in X$, $p(x,\cdot): X\to\mathbb{R}_+$ *is lower semicontinuous*

(w_3) for any $\varepsilon>0$, *there exists* $\delta>0$ *such that* $p(z,x)\le\delta$ *and* $p(z,y)\le\delta$ *imply* $d(x,y)\le\varepsilon$.

Some examples of w-distances on metric spaces are presented bellow:

Example 1.4.1. *Let (X,d) be a metric space. Then the metric d is a w-distance on (X,d).*

Example 1.4.2. *Let (X,d) be a metric space and let $f : X \to X$ be a continuous operator. Then the functional $p : X \times X \to \mathbb{R}_+$ defined by $p(x,y) := \max\{d(f(x),y), d(f(x),f(y))\}$, for every $x,y \in X$, is a w-distance on (X,d).*

Example 1.4.3. *Let $(\mathbb{R}, |\cdot|)$ be a metric space and let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous operator such that*

$$\inf_{x\in\mathbb{R}} \int_x^{x+\tau} f(s)ds > 0, \text{ for any } \tau > 0.$$

Then the functional $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by $p(x,y) := \left| \int_x^y f(s)ds \right|$, for every $x,y \in \mathbb{R}$, is a w-distance on $\mathbb{R}$.

Example 1.4.4. *Let (X,d) be a metric space. Let $M \subset X$ be a bounded and closed set. Assume that M contains at least two points and let λ be a constant such that $diamM \leq \lambda$, where $diamM := \sup\{d(x,y) \mid x,y \in M\}$ is the diameter of M. Them the functional $p : X \times X \to \mathbb{R}_+$ defined by*

$$p(x,y) := \begin{cases} d(x,y), & \text{if } x,y \in M \\ \lambda, & \text{if } x \notin M \text{ or } y \notin M \end{cases}$$

for all $x,y \in X$, is a w-distance on (X,d).

Example 1.4.5. *Let X be a normed linear space with norm $\|\cdot\|$. Then the functional $p : X \times X \to \mathbb{R}_+$, defined by $p(x,y) = \|x\| + \|y\|$, for all $x,y \in X$, is a w-distance on X.*

By following O. Kada, T. Suzuki and W. Takahashi [60] and T. Suzuki [138], we have the following result regarding convergent and Cauchy sequences in the metric space (X,d) endowed with the w-distance p:

Lemma 1.4.1. *Let (X,d) be a metric space and p be a w-distance on X. Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be sequences in X and let $x,y,z \in X$. Then the following statements hold:*

- *(i) if $\lim\limits_{n\to\infty} p(x_n,y) = 0$ and $\lim\limits_{n\to\infty} p(x_n,z) = 0$, then $y = z$.*
  In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.
- *(ii) if $\lim\limits_{n\to\infty} p(x_n,y_n) = 0$ and $\lim\limits_{n\to\infty} p(x_n,z) = 0$, then $(y_n)_{n\in\mathbb{N}}$ converges to z in (X,d).*
- *(iii) if $\lim\limits_{n\to\infty} p(x_n,y_n) = 0$ and $\lim\limits_{n\to\infty} p(x_n,z_n) = 0$, then $\lim\limits_{n\to\infty} d(y_n,z_n) = 0$.*
- *(iv) if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $p(x_n,x_m) \leq \alpha_n$ for all $m,n \in \mathbb{N}$ with $m > n$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).*
- *(v) if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $p(y,x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).*

Regarding the connexion between w-distances and L-spaces, the following problem arise:

Problem 1.4.1. *If (X, d) is a metric space and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X, would it be possible to define a convergence structure with respect to p on X?*

In order to give a solution to this problem, notice first that p is not symmetric on X. Indeed (see T. Suzuki [138]), by considering $X = [0, 1] \subset \mathbb{R}$ be the metric space endowed with the usual metric and $p : X \times X \to \mathbb{R}_+$ be defined by

$$p(x, y) = \begin{cases} y - x, & \text{if } x \le y \\ 3x - 3y, & \text{if } x > y \end{cases}$$

we get that p is a w-distance on X which is not symmetric. On the other hand, if p is a w-distance on X, then $p^* : X \times X \to \mathbb{R}_+$, defined by $p^*(x, y) = p(y, x)$, for any $x, y \in X$ is not necessarily a w-distance on X.

However, we can define a type of convergence and Cauchy property for a sequence with respect to p as follows:

Definition 1.4.2. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}_+$ be a w-distance on X. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

(1) *$(x_n)_{n\in\mathbb{N}}$ is convergent (left-convergent) to x with respect to p and we denote this by*

$$x_n \xrightarrow{p} x \text{ as } n \to \infty$$

if and only if for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $p(x_n, x) < \varepsilon$ for any $n \in \mathbb{N}$ with $n \ge n_\varepsilon$.

(2) *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy (left-Cauchy) sequence in X with respect to p if and only if for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $p(x_n, x_m) < \varepsilon$ for any $m, n \in \mathbb{N}$ with $m > n \ge n_\varepsilon$.*

Remark 1.4.1. *The convergence structure $\xrightarrow{p}$ defined in the Definition 1.4.2 can be put in an equivalent form as follows*

$$x_n \xrightarrow{p} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} p(x_n, x) = 0.$$

Remark 1.4.2. *In the context of Definition 1.4.2, any convergent sequence with respect to p has a unique limit.*

Indeed, let $y, z \in X$ be arbitrary. If $x_n \xrightarrow{p} y$ and $x_n \xrightarrow{p} z$ as $n \to \infty$, then by Lemma 1.4.1, item (i), we obtain that $y = z$.

Remark 1.4.3. *If (X, d) is a metric space and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X, then the couple $(X, \xrightarrow{p})$ is an L-space.*

Lemma 1.4.2. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}_+$ be a w-distance on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.*

Proof. Let $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p, there exists $n_\varepsilon \in \mathbb{N}$ such that $p(x_n, x_m) < \varepsilon$ for all $m, n \in \mathbb{N}$ with $m > n \geq n_\varepsilon$. Let $k \in \mathbb{N}$ such that $k > m$. We get that $p(x_n, x_k) < \varepsilon$ for all $k, n \in \mathbb{N}$ with $k > m > n \geq n_\varepsilon$.

Let $\varepsilon' > 0$ and choose $\delta := \varepsilon$. Then there exists $\delta > 0$ such that $p(x_n, x_m) < \delta$ and $p(x_n, x_k) < \delta$. By Definition 1.4.1, item (w_3), we get that $d(x_m, x_k) < \varepsilon'$, for all $k, m \in \mathbb{N}$ with $k > m \geq n_\varepsilon$. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. $\square$

Remark 1.4.4. *More considerations on w-distances can be found in the papers of O. Kada, T. Suzuki and W. Takahashi [60], T. Suzuki [138], L. Guran [45] and the references therein.*

1.5 τ-distance on a metric space (X,d)

In [139], T. Suzuki introduces the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance (see D. Tataru [144]). He also give generalizations for Banach's contraction principle, Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem of Takahashi.

Definition 1.5.1. *Let (X,d) be a metric space. A functional $p : X \times X \to \mathbb{R}_+$ is called a τ-distance on X if there exists an operator $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and the following are satisfied:*

(τ_1) *$p(x,z) \leq p(x,y) + p(y,z)$, for all $x, y, z \in X$;*

(τ_2) *$\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its second variable;*

(τ_3) *$\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \sup\limits_{m\geq n} \eta(z_n, p(z_n, x_m)) = 0$ imply $p(w,x) \leq \liminf\limits_{n\to\infty} p(w, x_n)$, for all $w \in X$;*

(τ_4) *$\lim\limits_{n\to\infty} \sup\limits_{m\geq n} p(x_n, y_m) = 0$ and $\lim\limits_{n\to\infty} \eta(x_n, t_n) = 0$ imply $\lim\limits_{n\to\infty} \eta(y_n, t_n) = 0$;*

(τ_5) *$\lim\limits_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim\limits_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.*

We may replace (τ_2) by

(τ_2)$'$ *$\inf\limits_{t>0} \eta(x,t) = 0$ for all $x \in X$, and η is nondecreasing in its second variable.*

We present bellow some examples regarding τ-distances. More examples can be found in the papers of T. Suzuki [139] and [140].

Example 1.5.1. *Let p be a w-distance on a metric space (X,d). Then p is a τ-distance on (X,d).*

Example 1.5.2. *Let (X, d) be a metric space, let p be a w-distance on X, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that*

$$\int_0^\infty \frac{1}{1+\varphi(s)} ds = \infty$$

and let $x_0 \in X$ be fixed. Then the function $q : X \times X \to \mathbb{R}_+$, defined by

$$q(x,y) = \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)} \frac{ds}{1+\varphi(s)}$$

for all $x, y \in X$, is a τ-distance on X.

Example 1.5.3. *Let (X, d) be a metric space and let p be a τ-distance on X. Let $f : X \to X$ be an operator satisfying the following condition:*

$$\lim_{n\to\infty} x_n = y \text{ and } \lim_{n\to\infty} f(x_n) = y \text{ imply } f(y) = y.$$

Then a function $q : X \times X \to \mathbb{R}_+$, defined by $q(x, y) = \max\{p(f(x), f(y)), p(f(x), y)\}$, for all $x, y \in X$, is also a τ-distance on X.

Example 1.5.4. *Let p be a τ-distance on a metric space X and let c be a positive real number. Then a functional $q : X \times X \to \mathbb{R}_+$, defined by $q(x, y) = c \cdot p(x, y)$, for all $x, y \in X$, is also a τ-distance on X.*

In [139] T. Suzuki introduces also the notion of Cauchy sequence with respect to the τ-distance p on a metric space (X, d), as follows:

Definition 1.5.2. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}_+$ be a τ-distance on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p (also called p-Cauchy sequence) if there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (τ_2)-(τ_5) and a sequence $(z_n)_{n\in\mathbb{N}}$ of X such that*

$$\lim_{n\to\infty} \sup_{m\geq n} \eta(z_n, p(z_n, x_m)) = 0.$$

In addition, we have the following lemmas (see Lemma 2 and Lemma 3 in [139]) regarding Cauchy sequences with respect to a τ-distance p.

Lemma 1.5.1. *Let (X, d) be a metric space and let p be a τ-distance on X. If a sequence $(x_n)_{n\in\mathbb{N}}$ of X satisfies $\lim_{n\to\infty} p(z, x_n) = 0$ for some $z \in X$, then $(x_n)_{n\in\mathbb{N}}$ is a p-Cauchy sequence. Moreover, if a sequence $(y_n)_{n\in\mathbb{N}}$ of X also satisfies $\lim_{n\to\infty} p(z, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. In particular, for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

Lemma 1.5.2. *Let (X, d) be a metric space and let p be a τ-distance on X. If a sequence $(x_n)_{n\in\mathbb{N}}$ of X satisfies $\lim_{n\to\infty} \sup_{m\geq n} p(x_n, x_m) = 0$ then $(x_n)_{n\in\mathbb{N}}$ is a p-Cauchy sequence. Moreover, if a sequence $(y_n)_{n\in\mathbb{N}}$ of X satisfies $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $(y_n)_{n\in\mathbb{N}}$ is also a p-Cauchy sequence and $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

In a metric space (X, d) endowed with a τ-distance p, we have the following connexion between the functionals p and d regarding the Cauchy sequences in X:

Lemma 1.5.3 (T. Suzuki, [139]). *Let (X, d) be a metric space and let p be a τ-distance on X. If $(x_n)_{n\in\mathbb{N}}$ is a p-Cauchy sequence, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Moreover, if $(y_n)_{n\in\mathbb{N}}$ is a sequence satisfying

$$\lim_{n\to\infty} \sup_{m\geq n} p(x_n, y_m) = 0$$

then $(y_n)_{n\in\mathbb{N}}$ is also p-Cauchy sequence and $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Due to the above result, the following one holds:

Lemma 1.5.4. *Let (X, d) be a complete metric space and let p be a τ-distance on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p, then $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in (X, d).*

Between a w-distance and a τ-distance on a metric space, we have the following relations expressed by (see [139]):

Proposition 1.5.1. *Let p be a w-distance on a metric space X. Then p is also a τ-distance on X.*

Proposition 1.5.2. *Let X be a compact metric space, let p be a τ-distance on X and $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $(\tau_2)'$, (τ_4) and (τ_5). If p is lower semicontinuous in its second variable and η is continuous in its first variable, then p is a w-distance on X.*

Remark 1.5.1. *More considerations on τ-distances and fixed point results, can be found in the work of T. Suzuki [139], [140] and L. Guran [45].*

1.6 Kasahara spaces

Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ be a functional. Let $\to$ be a convergence structure on X. By following S. Kasahara [66], the L-space $(X, \to)$ is called d-complete if any sequence $(x_n)_{n\in\mathbb{N}}$ in X, with $\sum\limits_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty$, converges in $(X, \to)$.

In a number of papers [66]-[70] S. Kasahara constructs a fixed point theory in such spaces. T.L. Hicks [47] and T.L. Hicks - B.E. Rhoades [49] give some fixed point theorems in a d-complete topological space. Other results in these directions were given by V.G. Angelov [3], J. Daneš [22], K. Iséki [55], L. Guran [45], P.Q. Khanh [75]. On the other hand, some authors give some fixed point theorems in a set with two metrics: M.G. Maia [84], V. Berinde [10], R. Precup [105], A. Petruşel and I.A. Rus [102], I.A. Rus [118], B. Rzepecki [129], L.M. Saliga [130], S. Iyer [57], I.A. Rus, A. Petruşel and G. Petruşel ([124], pp. 39-40).

We recall the notions of Kasahara space, generalized Kasahara space and large Kasahara space which were introduced by I.A. Rus in [121]:

Definition 1.6.1 (Kasahara space). *Let $(X, \to)$ be an L-space and $d : X \times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if we have the following compatibility condition between $\to$ and d:*

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty \ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \tag{1.6.1}$$

Definition 1.6.2 (Generalized Kasahara space). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \overset{G}{\to})$ be an L-space ordered semigroup with the unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a generalized Kasahara space if and only if we have the following compatibility condition between $\to$ and d_G:*

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) < +\infty \ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \tag{1.6.2}$$

Notice that by the inequality with the symbol $+\infty$ in the compatibility condition (1.6.2), we mean that the series $\sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1})$ is bounded in $(G, \leq)$.

Definition 1.6.3 (Large Kasahara space). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \overset{G}{\to})$ be an L-space ordered semigroup with the unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a large Kasahara space if and only if we have the following compatibility condition between $\to$ and d_G:*

$$\begin{aligned} x_n \in X,\ (x_n)_{n\in\mathbb{N}} &\text{ a Cauchy sequence (in a certain sense) with respect to } d_G \\ &\text{implies that } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \end{aligned} \tag{1.6.3}$$

Some examples of Kasahara spaces are presented in the sequel.

Example 1.6.1 (The trivial Kasahara space). *Let (X, d) be a complete metric space. Let $\overset{d}{\to}$ be the convergence structure induced by the metric d on X. Then $(X, \overset{d}{\to}, d)$ is a Kasahara space.*

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X such that $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty$. Then for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq n_\varepsilon$,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \varepsilon.$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and since (X, d) is complete, we get that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in the metric space (X, d).

Example 1.6.2 (I.A. Rus [121]). *Let (X, ρ) be a complete semimetric space, where $\rho : X \times X \to \mathbb{R}_+$ is continuous. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.*

Example 1.6.3 (I.A. Rus [121]). *Let (X, ρ) be a complete quasimetric space where $\rho : X \times X \to \mathbb{R}_+$. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.*

Example 1.6.4 (I.A. Rus [121]). *Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by*

$$d_\lambda(x, y) := \begin{cases} \rho(x, y), & \text{if } x \neq x_0 \text{ and } y \neq x_0 \\ \lambda, & \text{if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \overset{\rho}{\to}, d_\lambda)$ is a generalized Kasahara space.

Example 1.6.5 (I.A. Rus [121]). *Let (X, ρ) be a complete partial metric space. Then $(X, \overset{\rho}{\to}, d_\rho)$ is a large Kasahara space, where $d_\rho : X \times X \to \mathbb{R}_+$ is defined by*

$$d_\rho(x, y) := \rho(x, y) + \rho(y, x) - \rho(x, x) - \rho(y, y)$$

for all $x, y \in X$.

We present next some solutions for the following problems which were formulated by I.A. Rus in [121]:

Problem 1.6.1. *Give relevant examples of Kasahara spaces.*

We present bellow some relevant examples of Kasahara spaces:

Example 1.6.6. *Let $X = \mathbb{R}$ and $\to$ be the convergence structure induced on X by the usual metric $d : X \times X \to \mathbb{R}_+$, $d(x, y) = |x - y|$, for all $x, y \in X$. Then $(X, \to, d)$ is a trivial Kasahara space.*

Example 1.6.7. *Let $X = \mathbb{R}^m$ and $\to$ be the convergence structure induced on X by the euclidean distance $d : X \times X \to \mathbb{R}_+$,*

$$d(x, y) = \left(\sum_{i=1}^{m} (x_i - y_i)^2 \right)^{\frac{1}{2}}, \text{ for all } x = (x_1, \ldots, x_m),\ y = (y_1, \ldots, y_m) \in X.$$

Then $(X, \to, d)$ is a trivial Kasahara space.

Example 1.6.8. *Let $X = C([a, b]) = \{f : [a, b] \to \mathbb{R}_+ \mid f \text{ is continuous } \}$ and let $d : X \times X \to \mathbb{R}_+$, be defined by*

$$d(f, g) = \|f(x) - g(x)\| = \sup_{x \in [a,b]} |f(x) - g(x)|, \text{ for all } f, g \in X.$$

Let $\overset{d}{\to}$ be the convergence structure induced by d on X. Then $(X, \overset{d}{\to}, d)$ is a trivial Kasahara space.

Example 1.6.9 (S. Kasahara [66]). *Let X denote the closed interval $[0,1]$ and $\to$ be the usual convergence structure on $\mathbb{R}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by*

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 1, & \text{otherwise .} \end{cases}$$

Then $(X, \to, d)$ is a Kasahara space.

Example 1.6.10. *Let $I \subset \mathbb{R}_+^*$ be an interval and let*

$$X_I := \{\alpha^n \mid \alpha \in I,\ n \in \mathbb{N}\}.$$

Let $d : X_I \times X_I \to \mathbb{R}_+$ be defined by

$$d(x,y) = \begin{cases} x+y, & \text{if } x, y \in X_I \\ 1, & \text{otherwise .} \end{cases}$$

Then $(X_I, \to, d)$ is a Kasahara space, where $\to$ is considered the usual convergence structure on $\mathbb{R}$.

Example 1.6.11. *Let $X = \mathbb{R}$ and $\rho : X \times X \to \mathbb{R}_+$ be defined by $\rho(x,y) = |x-y|$, for all $x, y \in X$. We denote by $\xrightarrow{\rho}$ the usual convergence structure induced by ρ on X.*

Let $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ be defined by

$$d(x,y) = \begin{cases} \rho(x,y), & \text{if } x \neq 0 \text{ and } y \neq 0 \\ +\infty, & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Then $(X, \xrightarrow{\rho}, d)$ is a generalized Kasahara space.

Problem 1.6.2. *Let p be a w-distance on a complete metric space (X,d). In which conditions $(X, \xrightarrow{d}, p)$ is a large Kasahara space?*

As we know, by Definition 1.6.3, the triple $(X, \xrightarrow{d}, p)$ is a large Kasahara space if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ which is Cauchy (in a certain sense) with respect to p, we get that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in (X,d).

Since (X,d) is complete, in order to have the convergence for the sequence $(x_n)_{n\in\mathbb{N}}$, it is sufficient to show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).

In the sequel, we present some cases when the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to an w-distance p, implying further that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).

(1) By Definition 1.4.2, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p if and only if for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $p(x_n, x_m) < \varepsilon$ for any $m, n \in \mathbb{N}$ with $m > n \geq n_\varepsilon$.

In this case, applying further the Lemma 1.4.2, we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense: there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

(2_a) $\lim\limits_{n\to\infty} \alpha_n = 0$

(2_b) $p(x_n, x_m) \leq \alpha_n,$ for all $m, n \in \mathbb{N}$, with $m > n$.

In this case, applying further the Lemma 1.4.1, item (iv), we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d).

(3) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense: there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

(3_a) $\lim\limits_{n\to\infty} \alpha_n = 0$

(3_b) $p(y, x_n) \leq \alpha_n,$ for all $n \in \mathbb{N}$ and $y \in X$.

In this case, applying further the Lemma 1.4.1, item (v), we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d).

(4) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense:

$$\sum_{n\in\mathbb{N}} p(x_n, x_{n+1}) < \infty.$$

In this case, if there exists a constant $c > 0$ such that $d(x, y) \leq c \cdot p(x, y)$, for all $x, y \in X$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d).

(5) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense: there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (τ_2)-(τ_5) of Definition 1.5.1 and a sequence $(z_n)_{n\in\mathbb{N}}$ of X such that

$$\lim_{n\to\infty} \sup_{m\geq n} \eta(z_n, p(z_n, x_m)) = 0.$$

In this case, since p is a w-distance on X, by Proposition 1.5.1 we get that p is a τ-distance on X and hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p as a τ-distance on X. By Lemma 1.5.3 we get further that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d).

Problem 1.6.3. *Let p be a τ-distance on the complete metric space (X, d). In which conditions $(X, \xrightarrow{d}, p)$ is a large Kasahara space?*

As we have mentioned in the solution given for the Problem 1.6.2, the triple $(X, \xrightarrow{d}, p)$ is a large Kasahara space if and only if for each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ with respect to the τ-distance p on X, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in the metric space (X, d).

We present the following three cases:

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the sense of Definition 1.5.2.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense (see Lemma 1.5.1):
$$\lim_{n\to\infty} p(z, x_n) = 0 \text{ for some } z \in X.$$

(3) The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p in the following sense (see Lemma 1.5.2):
$$\lim_{n\to\infty} \sup_{m\geq n} p(x_n, x_m) = 0.$$

In all three cases, by Lemma 1.5.4, it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in the metric space (X, d).

1.7 Operators on Kasahara spaces

In this section we consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We define the continuity and the closeness properties for self-mappings $f : X \to X$ with respect to $\to$ and we give metric conditions for f with respect to d, by presenting some generalized contractions in this sense. Finally, we define the well-posed fixed point problem and the limit shadowing property for f with respect to d. By a similar way, the case of multivalued operators defined on Kasahara spaces is also presented.

- **Single-valued operators.**

Let X be a nonempty set and $f : X \to X$ be an operator. We denote by:

$$F_f := \{x \in X \mid x = f(x)\} \text{ - the fixed point set of } f.$$

$$f^0 := 1_X,\ f^1 := f,\ f^{n+1} := f \circ f^n,\ n \in \mathbb{N} \text{ - the iterate operators of } f$$
$$\text{(by } 1_X \text{ we understand the identity operator).}$$

Definition 1.7.1. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $f : X \to X$ be an operator. Let $x, y \in X$. Then*

(1) *f is continuous on x with respect to $\to$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ of X*
$$x_n \to x \text{ as } n \to \infty \text{ implies } f(x_n) \to f(x) \text{ as } n \to \infty.$$

(2) *f has closed graph with respect to $\to$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ of X*
$$x_n \to x \text{ as } n \to \infty \text{ and } f(x_n) \to y \text{ as } n \to \infty \text{ implies } f(x) = y.$$

Remark 1.7.1. *In the context of Definition 1.7.1, we say that f is continuous on X if and only if f is continuous on each point x of X.*

Remark 1.7.2. *Let us consider the following set (also called the graph of f)*

$$Graph(f) := \{(x, y) \in X \times X \mid f(x) = y\}.$$

Then f has closed graph with respect to $\to$ if and only if $Graph(f)$ is closed with respect to $\to$, i.e., for any sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of X satisfying

(i) $x_n \to x \in X$ *as* $n \to \infty$*;*

(ii) $y_n \to y \in X$ *as* $n \to \infty$*;*

(iii) $f(x_n) = y_n$, *for all* $n \in \mathbb{N}$

we get that $f(x) = y$.

In our fixed point results we will use often some metric conditions satisfied by f with respect to d. These metric conditions are included in the following

Definition 1.7.2. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $f : X \to X$ be an operator. Then f is called*

(j) α-contraction if there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \le \alpha d(x, y), \text{ for all } x, y \in X.$$

(jj) α-graphic contraction if there exists an $\alpha \in [0, 1[$ such that

$$d(f(x), f^2(x)) \le \alpha d(x, f(x)), \text{ for all } x \in X.$$

We present next some of the generalized contraction conditions usually imposed upon self-mappings $f : X \to X$ with respect to the functional $d : X \times X \to \mathbb{R}_+$ of a Kasahara space $(X, \to, d)$. We say that f is a

(a) Rakotch operator if there exists a decreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\alpha(t) < 1$ for all $t > 0$ and

$$d(f(x), f(y)) \le \alpha(d(x, y))d(x, y), \text{ for all } x, y \in X.$$

(b) Caristi operator if there exists a functional $\varphi : X \to \mathbb{R}_+$ such that

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)), \text{ for all } x \in X.$$

(c) φ-contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \le \varphi(d(x, y)), \text{ for all } x, y \in X$$

(φ is a comparison function if φ is increasing and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, for all $t \in \mathbb{R}_+$).

(*d*) Kannan operator if there exists $\alpha \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \leq \alpha\big[d(x, f(x)) + d(y, f(y))\big], \text{ for all } x, y \in X.$$

(*e*) Zamfirescu operator if there exist $a, b, c \in \mathbb{R}_+$ with $a < 1$, $b < \frac{1}{2}$ and $c < \frac{1}{2}$ such that for each $x, y \in X$ at least one of the following conditions is true:

(e_1) $d(f(x), f(y)) \leq ad(x, y)$;

(e_2) $d(f(x), f(y)) \leq b\big[d(x, f(x)) + d(y, f(y))\big]$;

(e_3) $d(f(x), f(y)) \leq c\big[d(x, f(y)) + d(y, f(x))\big]$.

The well-posed fixed point problem and the limit shadowing property for an operator f with respect to the functional $d : X \times X \to \mathbb{R}_+$ of a Kasahara space $(X, \to, d)$ are defined bellow.

Definition 1.7.3. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $f : X \to X$ be an operator such that $F_f = \{x^*\}$. The fixed point problem for the operator f is well-posed if for every sequence $(x_n)_{n\in\mathbb{N}}$ of X*

$$d(x_n, f(x_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty \text{ implies } x_n \to x^* \text{ as } n \to \infty.$$

Definition 1.7.4. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. An operator $f : X \to X$ has the limit shadowing property if for every sequence $(x_n)_{n\in\mathbb{N}}$ of X*

$$d(x_{n+1}, f(x_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty \text{ implies the existence of } x \in X \text{ such that}$$
$$d(x_{n+1}, f^{n+1}(x)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

- **Multivalued operators.**

Let X be a nonempty set and let $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$. Let $T : X \to P(X)$ be a multivalued operator. We denote by

$$F_T := \big\{x \in X \mid x \in Tx\big\} \text{ - the fixed point set of } T.$$

$$(SF)_T := \big\{x \in X \mid \{x\} = Tx\big\} \text{ - the strict fixed point set of } T.$$

$$T(Y) := \bigcup_{x\in Y} Tx \text{ for each } Y \subset X.$$

$$T^1(Y) := T(Y),\ T^2(Y) := T(T(Y)),\ \ldots,\ T^n(Y) := T(T^{n-1}(Y)) \text{ - the iterates of } T.$$

$$T^{-1}(Y) := \big\{x \in X \mid Tx \cap Y \neq \emptyset\big\}, \text{ for each } Y \in P(X).$$

Definition 1.7.5. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $T : X \to P(X)$ be a multivalued operator. Then*

(1) *T is continuous on X if and only if T is upper semicontinuous and lower semicontinuous on X.*

   (1_a) *T is upper semicontinuous on X if and only if for each closed set $Y \in P(X)$ w.r.t. $\to$, $T^{-1}(Y)$ is a closed subset of X w.r.t. $\to$.*

   (1_b) *T is lower semicontinuous on X if and only if for each open set $Y \in P(X)$ w.r.t. $\to$, $T^{-1}(Y)$ is an open subset of X w.r.t. $\to$.*

(2) *T has closed graph w.r.t. $\to$ if and only if for every sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of X satisfying*

   (2_a) *$x_n \to x \in X$ as $n \to \infty$;*

   (2_b) *$y_n \to y \in X$ as $n \to \infty$;*

   (2_c) *$y_n \in Tx_n$ for all $n \in \mathbb{N}$,*

   we have $y \in Tx$.

   We denote by

$$Graph(T) := \left\{(x, y) \in X \times X \;\middle|\; y \in Tx\right\} \text{ - the graph of } T.$$

In order to give metric conditions with respect to d, the following functionals need to be defined:

Definition 1.7.6. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. We define:*

i) the gap functional $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\},\ \text{for all } A, B \in P(X).$$

Note that $D(x, B)$, where $x \in X$, will be understood as $D(\{x\}, B)$.

ii) the excess functional $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\rho(A, B) = \sup_{a\in A} D(a, B),\ \text{for all } A, B \in P(X).$$

iii) the generalized Pompeiu-Hausdorff functional $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, by

$$H_d(A, B) = \max\{\rho(A, B), \rho(B, A)\},\ \text{for all } A, B \in P(X).$$

Definition 1.7.7. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $T : X \to P(X)$ be a multivalued operator. Then T is called:*

(j) *α-contraction if there exists $\alpha \in [0,1[$ such that*

$$H_d(Tx,Ty) \leq \alpha d(x,y), \text{ for all } x,y \in X.$$

(jj) *α-graphic contraction if there exists an $\alpha \in [0,1[$ such that*

$$H_d(Tx,Ty) \leq \alpha d(x,y), \text{ for each } (x,y) \in Graph(T).$$

We present next some multivalued generalized contractions with respect to the functional $d : X \times X \to \mathbb{R}_+$ of a Kasahara space $(X, \to, d)$. We say that the multivalued operator $T : X \to P(X)$ is a

(a) *Kannan operator* if there exists $\alpha \in [0, \frac{1}{2}[$ such that

$$H_d(Tx,Ty) \leq \alpha[D(x,Tx) + D(y,Ty)], \text{ for all } x,y \in X.$$

(b) *Reich operator* if there exist $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$ such that

$$H_d(Tx,Ty) \leq \alpha d(x,y) + \beta D(x,Tx) + \gamma D(y,Ty), \text{ for all } x,y \in X.$$

(c) *φ-contraction* if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H_d(Tx,Ty) \leq \varphi(d(x,y)), \text{ for all } x,y \in X$$

(φ is a comparison function if φ is increasing and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, for all $t \in \mathbb{R}_+$).

(d) *Caristi operator* if for all $x \in X$, there exists $y \in Tx$ such that

$$d(x,y) \leq \varphi(x) - \varphi(y).$$

(e) *Zamfirescu operator* if there exist α, β, $\gamma \in \mathbb{R}_+$ with $\alpha < 1$, $\beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}$ such that for each $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that at least one of the following conditions is true:

(e_1) $d(u,v) \leq \alpha d(x,y)$;

(e_2) $d(u,v) \leq \beta[d(x,u) + d(y,v)]$;

(e_3) $d(u,v) \leq \gamma[d(x,v) + d(y,u)]$.

(f) *(θ, L)-weak contraction* if there exist two constants $\theta \in [0,1[$ and $L \geq 0$ such that

$$H_d(Tx,Ty) \leq \theta \cdot d(x,y) + L \cdot D(y,Tx), \text{ for all } x,y \in X.$$

The well-posed fixed point problem and the limit shadowing property for a multivalued operator T are defined bellow.

Definition 1.7.8. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $T : X \to P(X)$ be a multivalued operator.*

- *The fixed point problem for T with respect to D is well-posed if*

  (1_D) $F_T = \{x^*\}$;

  (2_D) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $D(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.*

- *The fixed point problem for T with respect to H_d is well-posed if*

  (1_D) $(SF)_T = \{x^*\}$;

  (2_D) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $H_d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.*

Definition 1.7.9. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional and let $T : X \to P(X)$ be a multivalued operator. Then T has the limit shadowing property if for every sequence $(y_n)_{n\in\mathbb{N}}$ in X with $D(Ty_n, y_{n+1}) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T in X such that $d(x_n, y_{n+1}) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$.*

Chapter 2

Generalized contractions on Kasahara spaces

In this chapter we develop the theory of some important fixed point results as the Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, Caristi-Browder and Matkowski type theorems. Our results are given for single-valued generalized contractions in the context of a Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We present also some extensions of our results in generalized and large Kasahara spaces.

In the sequel, we present the connexion between the Maia type theorems and the fixed point theorems in Kasahara spaces, we introduce a new notion: Kasahara space with respect to an operator and we give in this setting several applications regarding the existence and uniqueness of solutions for integral and differential equations.

The references which were used to develop this chapter are: A.-D. Filip [34], [35], [36]; A.-D. Filip and A. Petruşel [39], [40]; S. Kasahara [66]; M.G. Maia [84]; I.A. Rus [110], [115], [117], [119], [121]; I.A. Rus, A.S. Mureşan and V. Mureşan [122]; I.A. Rus, A. Petruşel and G. Petruşel [124]; M.-A. Şerban [142]; T. Zamfirescu [150].

2.1 Fixed point theorems in Kasahara spaces

The aim of this section is to present the theory of some well-known fixed point results in the context of Kasahara spaces. Some of these results are also given in generalized and large Kasahara spaces as follows:

- fixed point theorems for generalized contractions in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional;
- a fixed point theory for the local variant of Banach-Caccioppoli's Contraction Principle in large Kasahara spaces $(X, \overset{d}{\to}, p)$, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X;

- fixed point theorems for generalized contractions in large Kasahara spaces $(X, \xrightarrow{d}, \varphi \circ d)$ which are obtained from complete metric spaces (X, d), by perturbing the metric with an increasing, subadditive and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

We consider first the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. In our results we will use the following notions and notations:

Definition 2.1.1. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $f : X \to X$ be an operator. Then*

(i) f is a Picard operator if and only if $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$;

(ii) f is a weakly Picard operator if and only if the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f;

(iii) if f is a weakly Picard operator, then we define the operator

$$f^\infty : X \to X \text{ by } f^\infty(x) := Lim(f^n(x))_{n \in \mathbb{N}};$$

Remark 2.1.1. *More considerations on Picard operators and weakly Picard operators can be found in the work of I.A. Rus [117], [115], I.A. Rus, A. Petruşel and M.A. Şerban [127].*

We recall also a very useful tool which will help us to prove the uniqueness of a fixed point for a single-valued operator defined on a Kasahara space.

Lemma 2.1.1 (Kasahara's lemma [66])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Then*

$$\text{for all } x, y \in X,\ d(x, y) = d(y, x) = 0 \Rightarrow x = y.$$

Proof. Let $x, y \in X$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X, defined by $x_{2k} = x$ and $x_{2k+1} = y$, for all $k \in \mathbb{N}$. Then

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) = 0.$$

Since $(X, \to, d)$ is a Kasahara space, $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in $(X, \to)$. On the other hand $(x_{2k})_{k \in \mathbb{N}}$ and $(x_{2k+1})_{k \in \mathbb{N}}$ are subsequences of $(x_n)_{n \in \mathbb{N}}$. We deduce that $x = y$. □

We present next our fixed point results and their theory by taking in view some important fixed point principles and theorems which were given in the context of complete metric spaces.

In 1922 S. Banach [7] and in 1930 R. Caccioppoli [17] have given the well-known Contraction Principle as follows

Theorem 2.1.1 (see e.g. I.A. Rus, A. Petruşel and G. Petruşel [124] p.30)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be an α-contraction. Then we have:*

(1) *$F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$;*

(2) *for each* $x \in X$ *the sequence of successive approximations* $(f^n(x))_{n\in\mathbb{N}^*}$ *of* f *starting from* x *converges to* x^**;*

(3) $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for each* $x \in X$*.*

A theory for the Theorem 2.1.1 in the context of Kasahara spaces is given in the following result.

Theorem 2.1.2 (The Contraction Principle). *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a functional. Let* $f : X \to X$ *be an operator. We assume that*

(*i*) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(*ii*) $f : (X, d) \to (X, d)$ *is an* α*-contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

Then the following statements hold:

(1) $F_f = F_{f^n} = \{x_f^*\}$*, for all* $n \in \mathbb{N}^*$ *and* $d(x_f^*, x_f^*) = 0$*;*

(2) $f^n(x) \to x_f^*$ *as* $n \to \infty$*, for all* $x \in X$*, i.e.,* $f : (X, \to) \to (X, \to)$ *is a PO;*

(3) *for all* $x \in X$ *we have,*

  (3.1) $d(f^n(x), x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*;*

  (3.2) $d(x_f^*, f^n(x)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*;*

(4) *if the functional* d *is a quasimetric (i.e.,* $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$ *and* d *satisfies the triangle inequality), then*

  (4.1) $d(x, x_f^*) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*;*

  (4.2) $d(x_f^*, x) \leq \frac{1}{1-\alpha} d(f(x), x)$*, for all* $x \in X$*;*

  (4.3) $d(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*;*

  (4.4) $d(x_f^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$*, for all* $x \in X$*;*

  (4.5) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$ *then* $d(z_n, x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*, i.e., the fixed point problem for the operator* f *is well-posed with respect to* d*;*

  (4.6) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$ *then* $d(z_{n+1}, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*, for all* $z \in X$*, i.e., the operator* f *has the limit shadowing property with respect to* d*;*

  (4.7) *if* $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which* $d(g(x), f(x)) \leq \eta$*, for all* $x \in X$*, then*

$$x_g^* \in F_g \text{ implies } d(x_g^*, x_f^*) \leq \frac{\eta}{1-\alpha}.$$

Proof. (1) & (2). Let $x_0 \in X$. We construct the sequence of successive approximations for f starting from x_0. Let $(x_n)_{n\in\mathbb{N}}$ be this sequence. Hence $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$.

Since f is an α-contraction, we have the following estimations:

$$\begin{aligned} d(f(x_0), f^2(x_0)) &\le \alpha d(x_0, f(x_0)) \\ d(f^2(x_0), f^3(x_0)) &\le \alpha d(f(x_0), f^2(x_0)) \\ &\dots \\ d(f^n(x_0), f^{n+1}(x_0)) &\le \alpha d(f^{n-1}(x_0), f^n(x_0)). \end{aligned}$$

Hence, we can write for all $n \in \mathbb{N}$ that

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\le \alpha d(f^{n-1}(x_0), f^n(x_0)) \le \alpha^2 d(f^{n-2}(x_0), f^{n-1}(x_0)) \\ &\le \dots \le \alpha^n d(x_0, f(x_0)). \end{aligned}$$

Next we estimate

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) \le \sum_{n\in\mathbb{N}} \alpha^n d(x_0, f(x_0)) = \frac{1}{1-\alpha} d(x_0, f(x_0)) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. Hence, there exists an element $x_f^* \in X$ such that $x_n \to x_f^*$ as $n \to \infty$.

Using the fact that $f : (X, \to) \to (X, \to)$ has closed graph, we obtain that $x_f^* \in F_f$. On the other hand $x_f^* = f(x_f^*) = f(f(x_f^*)) = \dots = f^n(x_f^*)$ and thus $x_f^* \in F_{f^n}$.

Next we show the uniqueness of the fixed point x_f^*.

Let $y_f^* \in X$ be another fixed point for the operator f such that $x^* \neq y^*$. Then

$$d(x_f^*, y_f^*) = d(f^n(x_f^*), f^n(y_f^*)) \le \alpha^n d(x_f^*, y_f^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

Similarly, we get that $d(y_f^*, x_f^*) = 0$ and applying Lemma 2.1.1, we conclude that $x_f^* = y_f^*$. Hence f is a PO.

Finally, if $x_f^* \in F_f$ then $d(x_f^*, x_f^*) = 0$.

Indeed, $d(x_f^*, x_f^*) = d(f^n(x_f^*), f^n(x_f^*)) \le \alpha d(f^{n-1}(x_f^*), f^{n-1}(x_f^*)) \le \dots \le \alpha^n d(x_f^*, x_f^*) \overset{\mathbb{R}}{\to} 0$, as $n \to \infty$.

(3). Let $x \in X$. Then by (ii) we have

$$\begin{aligned} d(f^n(x), x_f^*) = d(f^n(x), f^n(x_f^*)) &\le \alpha d(f^{n-1}(x), f^{n-1}(x_f^*)) \\ &\le \dots \le \alpha^n d(x, x_f^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty, \end{aligned}$$

so (3.1) holds. By the same way of proof we obtain (3.2).

(4). Let $x \in X$. Since d satisfies the triangle inequality, we have $d(x, x_f^*) \le d(x, f(x)) + d(f(x), f(x_f^*)) \le d(x, f(x)) + \alpha d(x, x_f^*)$ and hence

$$d(x, x_f^*) \le \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4.1) holds. Similarly we get (4.2).

We prove next (4.3). Using the property (4.1), we have the following estimation

$$d(f^n(x), x_f^*) \leq \frac{1}{1-\alpha} d(f^n(x), f^{n+1}(x)), \text{ for all } x \in X. \tag{2.1.1}$$

On the other hand we have

$$\begin{aligned} d(f^n(x), f^{n+1}(x)) &\leq \alpha d(f^{n-1}(x), f^n(x)) \leq \alpha^2 d(f^{n-2}(x), f^{n-1}(x)) \\ &\leq \ldots \leq \alpha^n d(x, f(x)), \text{ for all } x \in X. \end{aligned} \tag{2.1.2}$$

By (2.1.1) and (2.1.2) we obtain

$$d(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4.3) holds. By a similar procedure we obtain (4.4).

We prove next (4.5). Let $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$. By (4.1) we have

$$d(z_n, x_f^*) \leq \frac{1}{1-\alpha} d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty$$

so (4.5) holds.

(4.6). Let $z \in X$ and $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$. Since $x_f^* \in F_f$, by (ii) and (3.2) we have that

$$d(x_f^*, f^{n+1}(z)) = d(f(x_f^*), f^{n+1}(z)) \leq \alpha d(x_f^*, f^n(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \tag{2.1.3}$$

We need to prove that $d(z_{n+1}, x_f^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$.

We have

$$\begin{aligned} d(z_{n+1}, x_f^*) &\leq d(z_{n+1}, f(z_n)) + d(f(z_n), x_f^*) \leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, x_f^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \alpha^2 d(z_{n-1}, x_f^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \ldots + \alpha^{n+1} d(z_0, x_f^*). \end{aligned}$$

From a Cauchy lemma (see the references in [115], [117] or [128]) we have that

$$d(z_{n+1}, x_f^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \tag{2.1.4}$$

By (2.1.3) and (2.1.4), we obtain

$$d(z_{n+1}, f^{n+1}(z)) \leq d(z_{n+1}, x_f^*) + d(x_f^*, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

Finally, we show (4.7). Let $x_g^* \in F_g$. By (4.1) we have that

$$d(x_g^*, x_f^*) \leq \frac{1}{1-\alpha} d(x_g^*, f(x_g^*)) = \frac{1}{1-\alpha} d(g(x_g^*), f(x_g^*)) \leq \frac{\eta}{1-\alpha}.$$

□

Remark 2.1.2. *Theorem 2.1.2 extends Banach-Caccioppoli's Contraction Principle stated in Theorem 2.1.1 in the sense that instead of the metric space (X, d) we use the Kasahara space $(X, \to, d)$. The functional $d : X \times X \to \mathbb{R}_+$ need not to satisfy all of the axioms of the metric. On the other hand, Theorem 2.1.2 complements the conclusions of Theorem 2.1.1 in the sense that some fixed point problems are considered: well-possedness (item (4.5)), limit shadowing property (item (4.6)), data dependence (item (4.7)).*

Remark 2.1.3. *A generalization of Theorem 2.1.2 can be obtained by using Rakotch operators instead of α-contractions. The following result is suggestive in this sense.*

Theorem 2.1.3. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $f : X \to X$ be an operator and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with the property that $\alpha(t) < 1$, for all $t \in \mathbb{R}_+$. We assume that:*

(i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(ii) $f : (X, d) \to (X, d)$ *is a Rakotch operator, i.e.,*

$$d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y), \text{ for all } x, y \in X.$$

Then the following statements hold

(1) $F_f = F_{f^n} = \{x_f^*\}$, *for all* $n \in \mathbb{N}^*$ *and* $d(x_f^*, x_f^*) = 0$;

(2) $f^n(x) \to x_f^*$ *as* $n \to \infty$, *for all* $x \in X$;

(3) *for all* $x \in X$, *we have*

(3.1) $d(f^n(x), x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$;

(3.2) $d(x_f^*, f^n(x)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$;

(4) *if d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), then*

(4.1) $d(x, x_f^*) \leq \frac{1}{1-\alpha(d(x,x_f^*))} d(x, f(x))$, *for all* $x \in X$;

(4.2) $d(x_f^*, x) \leq \frac{1}{1-\alpha(d(x_f^*,x))} d(f(x), x)$, *for all* $x \in X$;

(4.3) $d(f^n(x), x_f^*) \leq \frac{1}{1-\alpha(d(f^n(x),x_f^*))} \displaystyle\prod_{k=1}^{n} \alpha(d(f^{k-1}(x), f^k(x))) \cdot d(x, f(x))$, *for all* $x \in X$;

(4.4) $d(x_f^*, f^n(x)) \leq \frac{1}{1-\alpha(d(x_f^*,f^n(x)))} \displaystyle\prod_{k=1}^{n} \alpha(d(f^k(x), f^{k-1}(x))) \cdot d(f(x), x)$, *for all* $x \in X$;

(4.5) *if* $(z_n)_{n\in\mathbb{N}}$ *is a sequence of* X, *then*

$$d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty \text{ implies } d(z_n, x_f^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty,$$

i.e., the fixed point problem for the operator f is well-posed with respect to d;

(4.6) *if* $(z_n)_{n\in\mathbb{N}}$ *is a sequence of* X, *then*

$$d(z_{n+1}, f(z_n)) \to 0 \text{ as } n \to \infty \text{ implies } d(z_{n+1}, f^{n+1}(x)) \to 0 \text{ as } n \to \infty,$$

for all $x \in X$, *i.e., the operator* f *has the limit shadowing property with respect to* d;

(4.7) *if* $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which* $d(g(x), f(x)) \leq \eta$, *for all* $x \in X$, *then*

$$x_g^* \in F_g \text{ implies } d(x_g^*, x_f^*) \leq \frac{\eta}{1 - \alpha(d(x_g^*, x_f^*))}.$$

Proof. (1) & (2). Let $x \in X$ and $(f^n(x))_{n\in\mathbb{N}}$ be the sequence of successive approximations for f starting from x. By (ii) we have

$$\begin{aligned} d(f(x), f^2(x)) &\leq \alpha(d(x, f(x))) \cdot d(x, f(x)) \\ d(f^2(x), f^3(x)) &\leq \alpha(d(f(x), f^2(x))) \cdot d(f(x), f^2(x)) \\ &\dots \\ d(f^n(x), f^{n+1}(x)) &\leq \alpha(d(f^{n-1}(x), f^n(x))) \cdot d(f^{n-1}(x), f^n(x)). \end{aligned}$$

Let $\alpha_k = \alpha(d(f^{k-1}(x), f^k(x)))$, for all $k = \overline{1,n}$.

Hence we have

$$\begin{aligned} d(f^n(x), f^{n+1}(x)) &\leq \alpha_n d(f^{n-1}(x), f^n(x)) \leq \alpha_n \alpha_{n-1} d(f^{n-2}(x), f^{n-1}(x)) \\ &\leq \dots \leq \prod_{k=1}^{n} \alpha_k d(x, f(x)). \end{aligned}$$

Now let $\alpha := \max\left\{\alpha_k \mid k = \overline{1,n}\right\}$. Hence $\alpha < 1$ and

$$d(f^n(x), f^{n+1}(x)) \leq \alpha^n d(x, f(x)), \text{ for all } n \in \mathbb{N}. \tag{2.1.5}$$

By following the proof of Theorem 2.1.2, the conclusions follow.

(3). Let $x \in X$. Then by (2.1.5) we have

$$d(f^n(x), x_f^*) = d(f^n(x), f^n(x_f^*)) \leq \alpha^n d(x, x_f^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty$$

so (3.1) holds. Similarly we get (3.2).

(4). Since d satisfies the triangle inequality, we have

$$d(x, x_f^*) \leq d(x, f(x)) + d(f(x), f(x_f^*)) \leq d(x, f(x)) + \alpha(d(x, x_f^*)) \cdot d(x, x_f^*), \text{ for all } x \in X$$

and hence we get (4.1). By a similar procedure we obtain (4.2).

We show (4.3). In (4.1) we take $x := f^n(x)$. Then we have

$$d(f^n(x), x_f^*) \leq \frac{1}{1 - \alpha(d(f^n(x), x_f^*))} d(f^n(x), f^{n+1}(x)). \tag{2.1.6}$$

On the other hand, for all $x \in X$,

$$d(f^n(x), f^{n+1}(x)) \leq \prod_{k=1}^{n} \alpha(d(f^{k-1}(x), f^k(x))) \cdot d(x, f(x)). \tag{2.1.7}$$

By (2.1.6) and (2.1.7) we get (4.3). Similarly we obtain (4.4).

Next we show (4.5). Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in X such that $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$. By (4.1) we have

$$d(z_n, x_f^*) \leq \frac{1}{1-\alpha(d(z_n, x_f^*))} d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

(4.6). Let $z \in X$ and $(z_n)_{n\in\mathbb{N}}$ be a sequence of X such that $d(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$. Since $x_f^* \in F_f$, by (ii) and (3.2) we have that

$$d(x_f^*, f^{n+1}(z)) = d(f(x_f^*), f^{n+1}(z)) \leq \alpha(d(x_f^*, f^n(z)))d(x_f^*, f^n(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

We need to prove that $d(z_{n+1}, x_f^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$.

We have

$$\begin{aligned} d(z_{n+1}, x_f^*) &\leq d(z_{n+1}, f(z_n)) + d(f(z_n), x_f^*) \leq d(z_{n+1}, f(z_n)) + \alpha(d(z_n, x_f^*))d(z_n, x_f^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha(d(z_n, x_f^*))d(z_n, f(z_{n-1})) + \alpha(d(z_n, x_f^*))\alpha(d(z_{n-1}, x_f^*))d(z_{n-1}, x_f^*). \end{aligned}$$

Let $\alpha_k = \alpha(d(z_k, x_f^*))$ for all $k = \overline{0,n}$ and $\alpha := \max\{\alpha_k \mid k = \overline{0,n}\}$. Then $\alpha < 1$ and we get

$$d(z_{n+1}, x_f^*) \leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \ldots + \alpha^{n+1} d(z_0, x_f^*).$$

We follow the proof of Theorem 2.1.2, item (4.6).

In order to show (4.7), let $x_g^* \in F_g$. By (4.1) we have

$$\begin{aligned} d(x_g^*, x_f^*) &\leq \frac{1}{1-\alpha(d(x_g^*, x_f^*))} d(x_g^*, f(x_g^*)) \\ &= \frac{1}{1-\alpha(d(x_g^*, x_f^*))} d(g(x_g^*), f(x_g^*)) \leq \frac{\eta}{1-\alpha(d(x_g^*, x_f^*))} \end{aligned}$$

so (4.7) holds. □

Another important fixed point result is the Graphic Contraction Principle which was given by I.A. Rus in 1972, S. Kasahara in 1975 (see [63]), T.L. Hicks and B.E. Rhoades in 1979 (see [48]) as follows:

Theorem 2.1.4 (see e.g. I.A. Rus, A. Petruşel and G. Petruşel [124] p.35)**.** *Let (X, d) be a complete metric space, $f : X \to X$ be an operator and $\alpha \in [0, 1[$. We suppose that*

(i) $d(f^2(x), f(x)) \leq \alpha d(x, f(x))$, *for all* $x \in X$;

(*ii*) *the operator* f *has closed graph.*

Then

(1) $F_f \neq \emptyset$*;*

(2) $f^n(x) \to f^\infty(x)$ *as* $n \to \infty$ *and* $f^\infty(x) \in F_f$ *for all* $x \in X$*;*

(3) $d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$ *for all* $x \in X$.

A theory for Theorem 2.1.4 in Kasahara spaces is presented in the sequel.

Theorem 2.1.5 (The Graphic Contraction Principle). *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a functional. Let* $f : X \to X$ *be an operator. We assume that*

(*i*) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(*ii*) $f : (X, d) \to (X, d)$ *is an* α*-graphic contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then the following statements hold:

(1) $F_f \neq \emptyset$*;*

(2) $f^n(x) \to f^\infty(x) \in F_f$ *as* $n \to \infty$*, for all* $x \in X$*, i.e.,* $f : (X, \to) \to (X, \to)$ *is a weakly Picard operator;*

(3) $d(x^*, x^*) = 0$*, for all* $x^* \in F_f$*;*

(4) *if* d *is a quasimetric (i.e.,* $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$ *and* d *satisfies the triangle inequality) and* d *is continuous with respect to* $\to$*, then*

(4.1) $d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*,*

(4.2) *Let* $g : X \to X$ *be an operator. If there exists* $c > 0$ *such that*

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)), \text{ for all } x \in X \tag{2.1.8}$$

and for all $x \in X$ *and some* $\eta > 0$*,*

$$\max\{d(g(x), f(x)), d(f(x), g(x))\} \leq \eta \tag{2.1.9}$$

then

$$H_d(F_f, F_g) \leq \max\left\{\frac{1}{1-\alpha}, c\right\}\eta,$$

where H_d *stands for the Pompeiu-Hausdorff functional (see [51]).*

Proof. (1) & (2). Let $x \in X$ and consider the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from x. Since f is an α-graphic contraction, we deduce that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha d(f^{n-1}(x), f^n(x)), \text{ for all } n \in \mathbb{N}.$$

By the proof of Theorem 2.1.2 we get that $(f^n(x))_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \to)$. By (i) it follows that its limit, denoted by $f^\infty(x)$, is a fixed point of f. So $F_f \neq \emptyset$.

(3). Let $x^* \in F_f$. Then by (ii) we have

$$\begin{aligned} d(x^*, x^*) = d(f^n(x^*), f^{n+1}(x^*)) &\leq \alpha d(f^{n-1}(x^*), f^n(x^*)) \leq \alpha^2 d(f^{n-2}(x^*), f^{n-1}(x^*)) \\ &\leq \ldots \leq \alpha^n d(x^*, f(x^*)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \end{aligned}$$

(4). Let $x \in X$. Then

$$\begin{aligned} d(x, f^\infty(x)) &\leq d(x, f^n(x)) + d(f^n(x), f^\infty(x)) \\ &\leq d(x, f(x)) + d(f(x), f^2(x)) + \ldots + d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^\infty(x)) \\ &\leq (1 + \alpha + \ldots + \alpha^{n-1}) d(x, f(x)) + d(f^n(x), f^\infty(x)) \\ &\leq \frac{1}{1-\alpha} d(x, f(x)) + d(f^n(x), f^\infty(x)), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By letting $n \to \infty$ and then using (3), we obtain

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for each } x \in X,$$

so (4.1) holds.

We show next (4.2).

Let $x \in F_f$ and $y \in F_g$. Since g satisfies (2.1.8) and (2.1.9), we have

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)) = c \cdot d(f(x), g(x)) \leq c\eta.$$

Since $g^\infty(x) \in F_g$ we have

$$\inf_{y \in F_g} d(x, y) \leq d(x, g^\infty(x)) \leq c\eta$$

and by taking the supremum over $x \in F_f$, we obtain

$$\sup_{x \in F_f} \inf_{y \in F_g} d(x, y) \leq c\eta. \tag{2.1.10}$$

On the other hand, since f satisfies (4.1), we have

$$d(y, f^\infty(y)) \leq \frac{1}{1-\alpha} d(y, f(y)) = \frac{1}{1-\alpha} d(g(y), f(y)) \leq \frac{\eta}{1-\alpha}.$$

Since $f^\infty(y) \in F_f$ we have

$$\inf_{x \in F_f} d(y, x) \leq d(y, f^\infty(y)) \leq \frac{\eta}{1-\alpha}$$

and by taking the supremum over $y \in F_g$, we obtain

$$\sup_{y\in F_g}\inf_{x\in F_f} d(y,x) \le \frac{\eta}{1-\alpha}. \tag{2.1.11}$$

By (2.1.10) and (2.1.11) we get

$$\begin{aligned} H_d(F_f,F_g) &:= \max\left\{\sup_{x\in F_f}\inf_{y\in F_g} d(x,y), \sup_{y\in F_g}\inf_{x\in F_f} d(y,x)\right\} \\ &\le \max\left\{c\eta, \frac{\eta}{1-\alpha}\right\} = \max\left\{\frac{1}{1-\alpha}, c\right\}\eta. \end{aligned}$$

□

Remark 2.1.4. *Notice that the Graphic Contraction Principle stated in Theorem 2.1.5 extends Theorem 2.1.4 since the metric space is replaced by a Kasahara space. On the other hand, the conclusions of Theorem 2.1.4 are complemented by a data dependence result stated in Theorem 2.1.5, item (4.2).*

In 1976 J. Caristi [18] and F.E. Browder [16] have given the following fixed point result:

Theorem 2.1.6 (see e.g. I.A. Rus, A. Petruşel and G. Petruşel [124] p.35)**.** *Let (X,d) be a complete metric space, $f: X \to X$ be an operator and $\varphi: X \to \mathbb{R}_+$ be a functional. We suppose that:*

(i) $d(x,f(x)) \le \varphi(x) - \varphi(f(x))$, for all $x \in X$;

(ii) the operator f has closed graph.

Then

(1) $F_f \neq \emptyset$;

(2) $f^n(x) \to f^\infty(x)$ as $n \to \infty$ and $f^\infty(x) \in F_f$, for all $x \in X$;

(3) if there exists an $\alpha \in \mathbb{R}_+^*$ such that $\varphi(x) \le \alpha d(x,f(x))$, then

$$d(x,f^\infty(x)) \le \alpha d(x,f(x)), \text{ for all } x \in X.$$

The corresponding theory extending the fixed point Theorem 2.1.6 to Kasahara spaces is presented in the sequel.

Theorem 2.1.7 (Caristi-Browder type theorem). *Let $(X,\to,d)$ be a Kasahara space, where $d: X\times X \to \mathbb{R}_+$ is a functional. Let $f: X \to X$ be an operator and $\varphi: X \to \mathbb{R}_+$ be a functional. We assume that*

(i) $f:(X,\to) \to (X,\to)$ has closed graph;

(*ii*) $f : (X, d) \to (X, d)$ *is a Caristi operator, i.e.,*

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)), \text{ for all } x \in X.$$

Then the following statements hold:

(1) $F_f \neq \emptyset$.

(2) $f^n(x) \to f^\infty(x) \in F_f$ *as* $n \to \infty$, *for all* $x \in X$, *i.e.,* $f : (X, \to) \to (X, \to)$ *is a weakly Picard operator;*

(3) $d(x^*, x^*) = 0$, *for all* $x^* \in F_f$*;*

(4) *if* d *is a quasimetric (i.e.,* $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$ *and* d *satisfies the triangle inequality), continuous with respect to* $\to$ *and if there exists an* $\alpha \in \mathbb{R}_+^*$ *such that* $\varphi(x) \le \alpha d(x, f(x))$*, then*

$$d(x, f^\infty(x)) \le \alpha d(x, f(x)), \text{ for all } x \in X. \tag{2.1.12}$$

Proof. (1) & (2). Let $x \in X$. We consider $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x)$ for all $n \in \mathbb{N}$, the sequence of successive approximations for f starting from $x_0 = x$. By (*ii*) we have

$$\begin{aligned} d(x, f(x)) &\le \varphi(x) - \varphi(f(x)) \\ d(f(x), f^2(x)) &\le \varphi(f(x)) - \varphi(f^2(x)) \\ &\cdots \\ d(f^n(x), f^{n+1}(x)) &\le \varphi(f^n(x)) - \varphi(f^{n+1}(x)) \end{aligned}$$

Hence, the following estimations hold

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \le \varphi(x) - \varphi(f^{n+1}(x)) \le \varphi(x) < \infty$$

and since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. Hence, there exists an element $f^\infty(x) \in X$ such that $f^n(x) \to f^\infty(x)$ as $n \to \infty$.

By (*i*), we get that $f^\infty(x) \in F_f$.

(3). Let $x^* \in F_f$. Then we have

$$0 \le d(x^*, x^*) = d(x^*, f(x^*)) \le \varphi(x^*) - \varphi(f(x^*)) = 0.$$

(4). Let $x \in X$, Then we have

$$d(x, f^n(x)) \le \sum_{k=0}^{n-1} d(f^k(x), f^{k+1}(x)) \le \varphi(x) \le \alpha d(x, f(x))$$

and by letting $n \to \infty$ we obtain (2.1.12). $\square$

We recall also another fixed point result which was given by Matkowski in 1975 (see [86]).

Theorem 2.1.8 (see e.g. I.A. Rus, A. Petruşel and G. Petruşel [124] p.31). *Let (X, d) be a complete metric space and $f : X \to X$ be a φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

Then we have:

(1) $F_f = F_{f^n} = \{x^*\}$, *for each* $n \in \mathbb{N}^*$;

(2) *for each $x \in X$ the sequence of successive approximations $f^n(x)$ of f starting from x converges to x^*.*

In Kasahara spaces, Theorem 2.1.8 has the following correspondent result.

Theorem 2.1.9 (Matkowski type theorem). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $f : X \to X$ be an operator and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function. We assume that*

(*i*) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(*ii*) $f : (X, d) \to (X, d)$ *is a φ-contraction, i.e.,*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X;$$

(*iii*) $\sum\limits_{n\in\mathbb{N}} \varphi^n(t) < \infty$, *for all* $t \in \mathbb{R}_+$.

Then the following statements hold:

(1) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$;

(2) $f^n(x) \to x^*$ *as* $n \to \infty$, *for all* $x \in X$;

(3) $d(x^*, x^*) = 0$.

Proof. (1) & (2). Let $x \in X$. We consider the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from x. By (*ii*) we have

$$d(f(x), f^2(x)) \leq \varphi(d(x, f(x)))$$

$$d(f^2(x), f^3(x)) \leq \varphi(d(f(x), f^2(x))) \leq \varphi^2(d(x, f(x)))$$

By induction after $n \in \mathbb{N}$ we get that

$$d(f^n(x), f^{n+1}(x)) \leq \varphi^n(d(x, f(x))), \text{ for all } n \in \mathbb{N}.$$

Hence, we can estimate

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} \varphi^n(d(x, f(x))) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. Hence, there exists an element $x^* \in X$ such that $f^n(x) \to x^*$ as $n \to \infty$ and by (i) we get that $x^* \in F_f$. Since we used the sequence of successive approximations in order to prove that x^* is a fixed point for f, we get further that $x^* \in F_{f^n}$.

We show next the uniqueness of the fixed point x^* for f.

Let $y^* \in F_f$ be another fixer point for f such that $x^* \neq y^*$. Then

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq \varphi(d(f^{n-1}(x^*), f^{n-1}(y^*))) \leq \ldots \leq \varphi^n(d(x^*, y^*)) \stackrel{\mathbb{R}}{\to} 0.$$

Similarly we have $d(y^*, x^*) = 0$. By Lemma 2.1.1, we get $x^* = y^*$.

(3). Let $x^* \in F_f$. Then

$$0 \leq d(x^*, x^*) = d(f^n(x^*), f^n(x^*)) \leq \varphi^n(d(x^*, x^*)) \stackrel{\mathbb{R}}{\to} 0$$

and the conclusion follows. □

- We give next some fixed point results concerning single-valued Zamfirescu operators.

In 1972, T. Zamfirescu gives in [150] several fixed point theorems for single-valued mappings of contractive type in metric spaces, obtaining generalizations for Banach-Caccioppoli's contraction principle, Kannan's, Edelstein's and Singh's theorems. We give local and global similar results for Zamfirescu operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators.

We define first the single-valued Zamfirescu's operator in Kasahara spaces.

Definition 2.1.2. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. The mapping $f : X \to X$ is called Zamfirescu operator if there exist α, β, $\gamma \in \mathbb{R}_+$ with $\alpha < 1$, $\beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}$ such that for each $x, y \in X$ at least one of the following conditions is true:*

(1_z) $d(f(x), f(y)) \leq \alpha d(x, y)$;

(2_z) $d(f(x), f(y)) \leq \beta[d(x, f(x)) + d(y, f(y))]$;

(3_z) $d(f(x), f(y)) \leq \gamma[d(x, f(y)) + d(y, f(x))]$.

Remark 2.1.5. *In our fixed point results we will consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric, i.e.,*

(1) $d(x,x) = 0$, *for all* $x \in X$;

(2) $d(x,z) \leq d(x,y) + d(y,z)$, *for all* $x, y \in X$.

We also will consider the following notion and notation.

Definition 2.1.3. *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a premetric. Then*

$$\tilde{B}(x_0, r) := \{x \in X \mid d(x_0, x) \leq r\}$$

is the right closed ball centered in $x_0 \in X$ *with radius* $r \in \mathbb{R}_+$.

Remark 2.1.6. *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a premetric. Let* $x_0 \in X$ *and* $r \in \mathbb{R}_+$. *If* d *is continuous on* X *with respect to the second argument, then the right closed ball* $\tilde{B}(x_0, r)$ *is a closed set in* X *with respect to* $\to$, *i.e., for any sequence* $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, r)$, *with* $z_n \to z \in X$, *as* $n \to \infty$, *we get that* $z \in \tilde{B}(x_0, r)$.

We present our first main local fixed point result which extends and generalizes Krasnoselskii's theorem.

Theorem 2.1.10 (Krasnoselskii (see e.g. [44]))**.** *Let* (X, d) *be a complete metric space. Let* $x_0 \in X$, $r \in \mathbb{R}_+$ *and* $f : \tilde{B}(x_0, r) \to X$ *be an operator. We assume that*

(*i*) *there exists* $\alpha \in [0, 1[$ *such that* $d(f(x), f(y)) \leq \alpha d(x, y)$, *for all* $x, y \in X$;

(*ii*) $d(x_0, f(x_0)) < (1 - \alpha)r$.

Then f *has at least one fixed point in* $\tilde{B}(x_0, r)$.

Our main result is the following.

Theorem 2.1.11 (A.-D. Filip [36])**.** *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a premetric. Let* $x_0 \in X$, $r \in \mathbb{R}_+$ *and* $f : \tilde{B}(x_0, r) \to X$ *be a Zamfirescu operator. We assume that:*

(*i*) *Graph*(f) *is closed in* $X \times X$ *with respect to* $\to$;

(*ii*) $d(x_0, f(x_0)) \leq (1 - \delta)r$, *where* $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(*iii*) d *is continuous with respect to the second argument.*

Then:

(1°) f *has at least one fixed point in* $\tilde{B}(x_0, r)$ *and* $f^n(x_0) \to x^* \in F_f$, *as* $n \to \infty$.

(2°) *the following estimation holds:*

$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{2.1.13}$$

where $x^* \in F_f$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* f *starting from* x_0.

Proof. (1°). Let $n \in \mathbb{N}$. We take

$$x = f^n(x_0) \text{ and } y = f^{n+1}(x_0).$$

If $x = y$ then $x \in F_f$. We suppose that $x \neq y$ and we take into account the Definition 2.1.2. If for this two points, condition (1_z) is satisfied, then we have

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \alpha d(f^n(x_0), f^{n+1}(x_0)).$$

If for x, y condition (2_z) is satisfied, then we have

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \beta[d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0))],$$

or equivalent

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \frac{\beta}{1-\beta} d(f^n(x_0), f^{n+1}(x_0)) \leq \delta d(f^n(x_0), f^{n+1}(x_0)).$$

In case condition (3_z) is satisfied, we have

$$\begin{aligned} d(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq \gamma[d(f^n(x_0), f^{n+2}(x_0)) + d(f^{n+1}(x_0), f^{n+1}(x_0))] \\ &\leq \gamma[d(f^n(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^{n+2}(x_0)] \end{aligned}$$

or equivalent

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \frac{\gamma}{1-\gamma} d(f^n(x_0), f^{n+1}(x_0)) \leq \delta d(f^n(x_0), f^{n+1}(x_0)).$$

In both three cases, we get that

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \delta d(f^n(x_0), f^{n+1}(x_0)), \text{for all } n \in \mathbb{N},$$

or, by using the sequence of successive approximations for f, we have

$$d(x_{n+1}, x_{n+2}) \leq \delta d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

By induction, we get that

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n) \leq \ldots \leq \delta^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}. \quad (2.1.14)$$

We show next that $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, r)$.

By (ii), since $d(x_0, x_1) = d(x_0, f(x_0)) \leq (1-\delta)r \leq r$, we have already that $x_1 \in \tilde{B}(x_0, r)$. Further, we have

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \leq d(x_0, x_1) + \delta d(x_0, x_1) \\ &\leq (1-\delta)r + \delta(1-\delta)r \leq (1-\delta^2)r \leq r, \end{aligned}$$

so, $x_2 \in \tilde{B}(x_0, r)$. By the same way of proof, we get

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \leq (1 + \delta + \delta^2) d(x_0, x_1) \\ &\leq (1 + \delta + \delta^2)(1 - \delta) r = (1 - \delta^3) r \leq r, \text{ so } x_3 \in \tilde{B}(x_0, r). \end{aligned}$$

By induction, we get that $x_n \in \tilde{B}(x_0, r)$, for all $n \in \mathbb{N}$.

Next, by (2.1.14) we have the following estimations

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n \in \mathbb{N}} \delta^n d(x_0, x_1) = \frac{1}{1 - \delta} d(x_0, x_1) \leq r < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, by (iii) we get that $(\tilde{B}(x_0, r), \to, d)$ is also a Kasahara space. Hence, the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent in $\tilde{B}(x_0, r)$, so there exists an element $x^* \in \tilde{B}(x_0, r)$ such that $x_n \to x^*$, as $n \to \infty$.

Knowing that $Graph(f)$ is closed in $X \times X$ with respect to $\to$, we get that $x^* \in F_f$.

(2°). Let $p \in \mathbb{N}$, $p \geq 1$. Then, by (2.1.14) we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \ldots + \delta^{n+p-1} d(x_0, x_1) \\ &\leq \delta^n \big(1 + \delta + \ldots + \delta^{p-1} + \ldots\big) d(x_0, x_1) = \frac{\delta^n}{1 - \delta} d(x_0, x_1) \leq \delta^n r. \end{aligned}$$

By letting $p \to \infty$, we get the estimation (2.1.13). □

A global variant for Theorem 2.1.11 is given bellow.

Corollary 2.1.1 (A.-D. Filip [36]). *Let $(X, \to, d)$ be a Kasahara space where $d : X \times X \to \mathbb{R}_+$ is a premetric, continuous with respect to the second argument. Let $f : X \to X$ be a Zamfirescu operator, having closed graph with respect to $\to$. Then*

(1°) *f has at least one fixed point in X and $f^n(x_0) \to x^* \in F_f$, as $n \to \infty$;*

(2°) *the following estimation holds:*

$$d(x_n, x^*) \leq \frac{\delta^n}{1 - \delta} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $x^* \in F_f$ and $(x_n)_{n \in \mathbb{N}}$ is the sequence of successive approximations for f starting from x_0.

Proof. Fix $x_0 \in X$ and choose $r \in \mathbb{R}_+$ such that $d(x_0, f(x_0)) \leq (1 - \delta) r$. The conclusions follow from Theorem 2.1.11. □

Remark 2.1.7. *Regarding the Corollary 2.1.1, notice that the functional d need not to be a premetric in order to prove the existence of fixed points for an operator $f : X \to X$ satisfying one of the conditions (1_z) or (2_z) from the Definition 2.1.2. However, the functional d must be at least a premetric in the case when f satisfies condition (3_z).*

Remark 2.1.8. *The global fixed point result given in Corollary 2.1.1 generalizes Banach-Caccioppoli's contraction principle stated in Theorem 2.1.1 since Zamfirescu's operators are used instead of contractions. Corollary 2.1.1 extends also Theorem 1 given by T. Zamfirescu in [150] since the metric space is replaced by a Kasahara space, where the functional* $d : X \times X \to \mathbb{R}_+$ *is not necessarily a metric. Maia's fixed point theorem (see Theorem 1 in M.G. Maia [84]) is also extended and generalized by Corollary 2.1.1 in the sense that the set* X *endowed with two metrics is replaced by a Kasahara space. On the other hand, Zamfirescu's operators are used instead of contractions.*

The following result is a generalization of Theorem 2.1.11.

Corollary 2.1.2 (A.-D. Filip [36]). *Let* $(X, \to, d)$ *be a Kasahara space with* $d : X \times X \to \mathbb{R}_+$ *a premetric. Let* $x_0 \in X$, $r \in \mathbb{R}_+$ *and* $f : \tilde{B}(x_0, r) \to X$ *be an operator. We consider the following functions:*

$$\alpha : \mathbb{R}_+ \to [0, 1[\text{ with } \limsup_{s \to t^+} \alpha(s) < 1, \text{ for all } t \in \mathbb{R}_+;$$
$$\beta : \mathbb{R}_+^2 \to [0, \frac{1}{2}[\text{ with } \limsup_{s \to t^+} \beta(s) < \frac{1}{2}, \text{ for all } t \in \mathbb{R}_+^2;$$
$$\gamma : \mathbb{R}_+ \to [0, \frac{1}{2}[\text{ with } \limsup_{s \to t^+} \gamma(s) < \frac{1}{2}, \text{ for all } t \in \mathbb{R}_+.$$

We assume that:

(i) $Graph(f)$ *is closed in* $X \times X$ *with respect to* $\to$*;*

(ii) f *satisfies one of the following conditions*

$(1'_z)$ $d(f(x), f(y)) \leq \alpha\big(d(x, y)\big)d(x, y)$*;*
$(2'_z)$ $d(f(x), f(y)) \leq \beta\big(d(x, f(x)), d(y, f(y))\big)[d(x, f(x)) + d(y, f(y))]$*;*
$(3'_z)$ $d(f(x), f(y)) \leq \gamma\big(d(x, f(y))\big)[d(x, f(y)) + d(y, f(x))]$

for all $x, y \in \tilde{B}(x_0, r)$*;*

(iii) $d(x_0, f(x_0)) \leq (1 - \delta)r$*, where*

$$\delta := \max\left\{\alpha_M, \frac{\beta_M}{1 - \beta_M}, \frac{\gamma_M}{1 - \gamma_M}\right\}, \text{ with}$$
$$\alpha_M := \max\left\{\alpha(d(f^k(x), f^{k+1}(x))) \mid k \in \mathbb{N}\right\};$$
$$\beta_M := \max\left\{\beta(d(f^k(x), f^{k+1}(x)), d(f^{k+1}(x), f^{k+2}(x))) \mid k \in \mathbb{N}\right\};$$
$$\gamma_M := \max\left\{\gamma(d(f^k(x), f^{k+2}(x))) \mid k \in \mathbb{N}\right\}$$

and $(f^k(x))_{k \in \mathbb{N}}$ *is the sequence of successive approximations for* f *starting from* $x \in X$*;*

(iv) d *is continuous on* X *with respect to the second argument.*

Then the following statements hold:

(1°) *f has at least one fixed point in $\tilde{B}(x_0, r)$ and $f^n(x_0) \to x^* \in F_f$, as $n \to \infty$.*

(2°) *the relation* (2.1.13) *holds.*

Proof. We follow the proof of Theorem 2.1.11. □

Remark 2.1.9. *An extension of our fixed point results to large Kasahara spaces can be made. In order to obtain a large Kasahara space from the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric, we need to define a certain notion of Cauchy sequence with respect to the premetric d. We must take also into account the fact that d is not symmetric.*

Definition 2.1.4. *Let (X, d) be a premetric space with $d : X \times X \to \mathbb{R}_+$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is a right-Cauchy sequence with respect to d if and only if*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0,$$

i.e., for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for every $m, n \in \mathbb{N}$ with $m \geq n \geq k$.

The following notion of large Kasahara space arises.

Definition 2.1.5 (A.-D. Filip [36])**.** *Let $(X, \to)$ be an L-space. Let $d : X \times X \to \mathbb{R}_+$ be a premetric on X. The triple $(X, \to, d)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d holds:*

$$\text{if } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0 \text{ then } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Remark 2.1.10 (A.-D. Filip [36])**.** *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 2.1.5. Then $(X, \to, d)$ is a Kasahara space.*

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty$.

By following S. Kasahara (see [66]), for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $m > n \geq k$ we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \varepsilon.$$

Hence $\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0$ and since $(X, \to, d)$ is a large Kasahara space, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. The conclusion follows from Definition 1.6.1.

Remark 2.1.11. *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 2.1.5. Then Theorem 2.1.11 and Corollaries 2.1.1 and 2.1.2 hold.*

As application of Theorem 2.1.11 in large Kasahara spaces in the sense of Definition 2.1.5, we present a homotopy result which extends some similar homotopy results given on a set endowed with two metrics by A. Chiş in [19].

In our application, the following notion need to be defined.

Definition 2.1.6. *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 2.1.5. A subset U of X is an open set with respect to d if there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$, $x_0 \in U$ such that $B(x_0, r) \subset U$.*

Theorem 2.1.12 (A.-D. Filip [36])**.** *Let $(X, \overset{\rho}{\to}, d)$ be a large Kasahara space in the sense of Definition 2.1.5, where $\rho : X \times X \to \mathbb{R}_+$ is a complete metric on X, $\overset{\rho}{\to}$ is the convergence structure induced by ρ on X and $d : X \times X \to \mathbb{R}_+$ is a continuous premetric on X.*

Let $Q \subset X$ be a closed set with respect to ρ. Let $U \subset X$ be an open set with respect to d and assume that $U \subset Q$.

Suppose $H : Q \times [0, 1] \to X$ satisfies the following properties:

(i) $x \neq H(x, \lambda)$ for all $x \in Q \setminus U$ and all $\lambda \in [0, 1]$;

(ii) for all $\lambda \in [0, 1]$ and $x, y \in Q$, there exist $\alpha \in [0, 1[$ and $\beta \in [0, \frac{1}{2}[$ such that one of the following conditions hold:

(ii_1) $d(H(x, \lambda), H(y, \lambda)) \leq \alpha d(x, y)$;

(ii_2) $d(H(x, \lambda), H(y, \lambda)) \leq \beta[d(x, H(x, \lambda)) + d(y, H(y, \lambda))]$;

(iii) $H(x, \lambda)$ is continuous in λ with respect to d, uniformly for $x \in Q$;

(iv) H is uniformly continuous from $U \times [0, 1]$ endowed with the metric d on U into (X, ρ);

(v) H is continuous from $Q \times [0, 1]$ endowed with the metric ρ on Q into (X, ρ).

In addition, assume that H_0 has a fixed point. Then for each $\lambda \in [0, 1]$ we have that H_λ has a fixed point $x_\lambda \in U$. (here $H_\lambda(\cdot) = H(\cdot, \lambda)$)

Proof. Let $A := \big\{\lambda \in [0, 1] \mid \text{there exists } x \in U \text{ such that } x = H(x, \lambda)\big\}$.

Since H_0 has a fixed point and (i) holds, we have that $0 \in A$ so the set A is nonempty. We will show that A is open and closed in $[0, 1]$ and so, by the connectedness of $[0, 1]$, we will have $A = [0, 1]$ and the proof will be complete.

First we show that A is closed in $[0, 1]$.

Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in A with $\lambda_k \to \lambda \in [0, 1[$ as $k \to \infty$. By the definition of A, for each $k \in \mathbb{N}$, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned} d(x_k, x_j) &= d(H(x_k, \lambda_k), H(x_j, \lambda_j)) \\ &\leq d(H(x_k, \lambda_k), H(x_k, \lambda)) \\ &\quad + d(H(x_k, \lambda), H(x_j, \lambda)) \\ &\quad + d(H(x_j, \lambda), H(x_j, \lambda_j)) \end{aligned} \tag{2.1.15}$$

◇ If H satisfies (ii_1) then by (2.1.15) we get

$$\begin{aligned} &d(x_k, x_j) \leq d(H(x_k, \lambda_k), H(x_k, \lambda)) + \alpha d(x_k, x_j) + d(H(x_j, \lambda), H(x_j, \lambda_j)) \\ &\Leftrightarrow (1 - \alpha) d(x_k, x_j) \leq d(H(x_k, \lambda_k), H(x_k, \lambda)) + d(H(x_j, \lambda), H(x_j, \lambda_j)) \end{aligned}$$

◇ If H satisfies (ii_2) then by (2.1.15) we have

$$\begin{aligned}
d(x_k, x_j) &\leq d(H(x_k, \lambda_k), H(x_k, \lambda)) + d(H(x_j, \lambda), H(x_j, \lambda_j)) \\
&\quad + \beta[d(x_k, H(x_k, \lambda)) + d(x_j, H(x_j, \lambda))] \\
&= (d(H(x_k, \lambda_k), H(x_k, \lambda)) + d(H(x_j, \lambda), H(x_j, \lambda_j)) \\
&\quad + \beta[d(H(x_k, \lambda_k), H(x_k, \lambda)) + d(H(x_j, \lambda_j), H(x_j, \lambda))].
\end{aligned}$$

By (iii), letting $k, j \to \infty$ we get that the sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since $(X, \overset{\rho}{\to}, d)$ is a large Kasahara space, we get that $(x_k)_{k\in\mathbb{N}}$ is convergent in $(X, \overset{\rho}{\to})$. Moreover, since $Q \subset X$ is a closed set with respect to the complete metric ρ, there exists $x \in Q$ such that $\lim\limits_{k\to\infty} \rho(x_k, x) = 0$.

We show next that $x = H(x, \lambda)$. Indeed, we have

$$\begin{aligned}
\rho(x, H(x, \lambda)) &\leq \rho(x, x_k) + \rho(x_k, H(x, \lambda)) \\
&= \rho(x, x_k) + \rho(H(x_k, \lambda_k), H(x, \lambda)).
\end{aligned}$$

By (v) and letting $k \to \infty$, we have $\rho(x, H(x, \lambda)) = 0$, so $x = H(x, \lambda)$ and by (i) we get that $x \in U$. Hence $\lambda \in A$ and so A is closed in $[0, 1]$.

We show next that A is open in $[0, 1]$.

Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. Since U is open with respect to d, by Definition 2.1.6 there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$ such that $B(x_0, r) \subset U$. By (iii), H is uniformly continuous on $B(x_0, r)$.

Let $\varepsilon = \big(1 - \max\big\{\alpha, \frac{\beta}{1-\beta}\big\}\big)r > 0$. By the uniform continuity of H, there exists $\eta = \eta(r) > 0$ such that for each $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| \leq \eta$ we have $d(H(x, \lambda_0), H(x, \lambda)) < \varepsilon$ for any $x \in B(x_0, r)$. Since this property holds for $x = x_0$, we get

$$d(x_0, H(x_0, \lambda)) = d(H(x_0, \lambda_0), H(x_0, \lambda)) < \big(1 - \max\big\{\alpha, \frac{\beta}{1-\beta}\big\}\big)r$$

for any $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| \leq \eta$.

By (ii), (iv) and (v) together with Theorem 2.1.11 in the context of large Kasahara spaces defined as in Definition 2.1.5, (in this case $\delta := \max\big\{\alpha, \frac{\beta}{1-\beta}\big\}$ and $f = H_\lambda$) we obtain the existence of $x_\lambda \in B(x_0, r)$ such that $x_\lambda = H_\lambda(x_\lambda)$ for any $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| \leq \eta$. Consequently A is open in $[0, 1]$. □

- We present in the sequel some fixed point results given in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional.

Our aim is to give some new fixed point theorems for single-valued operators in a generalized Kasahara space, starting from the results given by S. Kasahara in [66], [63], [64], [67], K. Iséki in [55], [53] or [54] and I.A. Rus in [121]. As an application, an existence and uniqueness theorem for a Cauchy problem is given.

The notion of generalized Kasahara space was given in Definition 1.6.2. We consider $G := \mathbb{R}_+ \cup \{+\infty\}$ and we present first an example of generalized Kasahara space in this setting.

Example 2.1.1 (A.-D. Filip and A. Petruşel [40]). *Let $a > 0$ and $I := [t_0 - a, t_0 + a] \subset \mathbb{R}$. Denote*
$$X := C(I) := \big\{x : I \to \mathbb{R} \mid x \text{ is a continuous function on } I\big\}.$$
Let $\lambda > 0$ and consider $d_\lambda : C(I) \times C(I) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by
$$d_\lambda(x, y) := \max\left\{\frac{1}{|t - t_0|^\lambda}|x(t) - y(t)| : t \in I\right\}, \text{ for } x, y \in C(I). \tag{2.1.16}$$
Notice that d_λ is not necessarily finite for every pair of functions $x, y \in C(I)$. Thus, by following W.A.J. Luxemburg [82], we have that d_λ is a generalized metric on $C(I)$ and
$$\lim_{\substack{n\to\infty\\ m\to\infty}} d_\lambda(x_n, x_m) = 0 \;\Rightarrow\; \text{there exists } x \in C(I) \text{ such that } \lim_{n\to\infty} d_\lambda(x_n, x) = 0. \tag{2.1.17}$$
We also denote by $\rho = \max\{|x(t) - y(t)| : t \in I\}$ the metric of uniform convergence on $C(I)$ and by $\xrightarrow{\rho}$ the convergence structure induced by ρ on $C(I)$.

The triple $(C(I), \xrightarrow{\rho}, d_\lambda)$ is a generalized Kasahara space.

Indeed, let us consider a sequence $(x_n)_{n\in\mathbb{N}} \subset C(I)$ with $\sum_{n\in\mathbb{N}} d_\lambda(x_n, x_{n+1}) < +\infty$. Since $a^{-\lambda}\rho(x, y) \leq d_\lambda(x, y)$, for every $x, y \in C(I)$, we immediately get that
$$\sum_{n\in\mathbb{N}} \rho(x_n, x_{n+1}) \leq a^\lambda \sum_{n\in\mathbb{N}} d_\lambda(x_n, x_{n+1}) < +\infty$$
which implies further that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to ρ and, hence, convergent in $(X, \xrightarrow{\rho})$.

In our results, we will use also the following notions.

Definition 2.1.7 (A.-D. Filip and A. Petruşel [40]). *Let $(X, \to, d)$ be a generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We say that f is a*

- ◇ *Picard operator if*
  1) *$F_f = \{x^*\}$;*
  2) *$f^n(x_0) \to x^*$ as $n \to \infty$, for each $x_0 \in X$ with the property $d(x_0, f(x_0)) < +\infty$.*
- ◇ *weakly Picard operator if*
  1) *$F_f \neq \emptyset$;*
  2) *the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges for each $x_0 \in X$ with $d(x_0, f(x_0)) < +\infty$ and the limit is a fixed point of f.*

Remark 2.1.12. *Kasahara's Lemma 2.1.1 holds also in the case when $(X, \to, d)$ is a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. The lemma is proved in the work of S. Kasahara [66].*

We give next our fixed point results.

Theorem 2.1.13 (A.-D. Filip and A. Petruşel [40]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We assume that*

i) $f : (X, \to) \to (X, \to)$ has closed graph;

ii) there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X, \text{ with } d(x, y) < +\infty;$$

iii) there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.

Then we have:

1) f is a weakly Picard operator;

2) if $d(x^*, y^*) < +\infty$, for all $x^*, y^* \in F_f$ then f is a Picard operator;

3) if $d(x, x) = 0$, for all $x \in X$ then $d(x^*, f(x^*)) < +\infty$, for all $x^* \in F_f$;

4) if $x \in X$ and $x^* \in F_f$ such that $d(x, x^*) < +\infty$, then

$$d(f^n(x), x^*) \to 0 \text{ as } n \to \infty;$$

5) if $d(x_0, x^*) < +\infty$, for all $x^* \in F_f$ and

$$d(f^k(x_0), x^*) \leq d(f^k(x_0), f^{k+1}(x_0)) + d(f^{k+1}(x_0), x^*), \text{ for all } k \in \mathbb{N},$$

then

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, f(x_0)).$$

Proof. 1) & 2). We follow the method given by Kasahara in [66].

For $x_0 \in X$, we construct the sequence of successive approximations for f starting from x_0. This sequence has the elements x_0, $f(x_0)$, $f^2(x_0)$, ..., i.e. x_0, $f(x_0)$, $f(f(x_0))$, ... which are all in X. Recursively, this sequence is defined by $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$, i.e. $x_{n+1} = f^{n+1}(x_0)$, for all $n \in \mathbb{N}$.

By *ii*) we have

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \alpha d(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \alpha^n d(x_0, f(x_0)).$$

Then, we get

$$\sum_{n \in \mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) \leq \frac{1}{1-\alpha} d(x_0, f(x_0)) < +\infty.$$

Since $(X, \to, d)$ is a generalized Kasahara space, $(f^n(x_0))_{n \in \mathbb{N}}$ is a convergent sequence in $(X, \to)$. Thus, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to +\infty$.

Since f has closed graph, we get that $x^* \in F_f$, i.e. $F_f \neq \emptyset$.

Let $x^*, y^* \in F_f$ such that $d(x^*, y^*) < +\infty$. Then

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq \alpha^n d(x^*, y^*) \overset{\mathbb{R}}{\to} 0, \text{ as } n \to +\infty.$$

Similarly, $d(y^*, x^*) \overset{\mathbb{R}}{\to} 0$, as $n \to +\infty$.

So we get that $d(x^*, y^*) = d(y^*, x^*) = 0$ and by Lemma 2.1.1 we obtain $x^* = y^*$.

3). Let $x^* \in F_f$. Then $d(x^*, f(x^*)) = d(x^*, x^*) = 0 < +\infty$.

4). Let $x \in X$ and $x^* \in F_f$. By *ii)* we have

$$\begin{aligned} d(f^n(x), x^*) &= d(f^n(x), f^n(x^*)) \leq \alpha d(f^{n-1}(x), f^{n-1}(x^*)) \\ &\leq \ldots \leq \alpha^n d(x, x^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \end{aligned}$$

5). Let $x^* \in F_f$ such that $d(x_0, x^*) < +\infty$. Then the following estimations hold

$$\begin{aligned} d(x_0, x^*) &\leq d(x_0, f(x_0)) + d(f(x_0), x^*) \\ &\leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) + d(f^2(x_0), x^*) \leq \ldots \\ &\leq d(x_0, f(x_0)) + \ldots + d(f^{n-1}(x_0), f^n(x_0)) + d(f^n(x_0), x^*) \\ &\leq (1 + \alpha + \ldots + \alpha^{n-1}) d(x_0, f(x_0)) + d(f^n(x_0), f^n(x^*)) \\ &\leq \frac{1 - \alpha^n}{1 - \alpha} d(x_0, f(x_0)) + \alpha^n d(x_0, x^*). \end{aligned}$$

By letting $n \to \infty$, the conclusion follows. □

Theorem 2.1.14 (A.-D. Filip and A. Petruşel [40]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We suppose that:*

i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

ii) $f : (X, d) \to (X, d)$ *is a graphic contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X \text{ with } d(x, f(x)) < +\infty;$$

iii) there exists $x_0 \in X$ *such that* $d(x_0, f(x_0)) < +\infty$.

Then the following statements hold

1) f *is a weakly Picard operator;*

2) if $d(x_0, f^n(x_0)) < +\infty$, *for all* $n \in \mathbb{N}$ *and*

$$d(f^k(x_0), f^n(x_0)) \leq d(f^k(x_0), f^{k+1}(x_0)) + d(f^{k+1}(x_0), f^n(x_0)), \text{ for all } k \in \mathbb{N},$$

then

$$d(x_0, f^n(x_0)) \leq \frac{1}{1 - \alpha} d(x_0, f(x_0)).$$

In addition, if d is continuous with respect to the second argument and $d(x_0, x^*) < +\infty$, then
$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, f(x_0)).$$

Proof. 1). We follow the method of Theorem 1, given by Kasahara in [66].

For $x_0 \in X$, we construct the sequence of successive approximations for f starting from x_0, like in Theorem 2.1.13.

By $ii)$ there exists $\alpha \in [0,1[$ such that

$$\begin{gathered} d(f(x_0), f^2(x_0)) \leq \alpha d(x_0, f(x_0)) \\ d(f^2(x_0), f^3(x_0)) \leq \alpha d(f(x_0), f^2(x_0)) \\ \dots \\ d(f^n(x_0), f^{n+1}(x_0)) \leq \alpha d(f^{n-1}(x_0), f^n(x_0)). \end{gathered}$$

Hence, we get

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leq \alpha d(f^{n-1}(x_0), f^n(x_0)) \leq \alpha^2 d(f^{n-2}(x_0), f^{n-1}(x_0)) \\ &\leq \ldots \leq \alpha^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

It follows that
$$\sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) \leq \frac{1}{1-\alpha} d(x_0, f(x_0)) < +\infty.$$

Hence, since $(X, \to, d)$ is a generalized Kasahara space, $(f^n(x_0))_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \to)$. Thus, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to +\infty$.

Since f has closed graph, we get that $x^* \in F_f$. So $F_f \neq \emptyset$.

2). We have the following estimations

$$\begin{aligned} d(x_0, f^n(x_0)) &\leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) + \ldots + d(f^{n-1}(x_0), f^n(x_0)) \\ &= (1 + \alpha + \ldots + \alpha^{n-1}) d(x_0, f(x_0)) \leq \frac{1}{1-\alpha} d(x_0, f(x_0)). \end{aligned}$$

If d is continuous with respect to the second argument, by letting $n \to \infty$ in the above estimation, the conclusion follows. □

Corollary 2.1.3 (A.-D. Filip and A. Petruşel [40]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We assume that*

1) $f : (X, \to) \to (X, \to)$ has closed graph;

2) One of the following conditions holds:

(a) there exists $\alpha \in [1, 2[$ and $\beta > 0$ such that for all $x, y \in X$ with $d(x, y) < +\infty$ we have
$$d(x, f(x)) + d(y, f(y)) \leq \alpha d(x, y) + \beta d(y, f(x));$$

(b) there exists $\alpha \in [1,3[$ *such that for all* $x, y \in X$ *with* $d(x,y) < +\infty$ *we have*

$$d(x,f(x)) + d(y,f(y)) + d(f(x),f(y)) \leq \alpha d(x,y);$$

(c) there exists $\alpha \in [0,1[$ *such that for all* $x, y \in X$ *with* $d(x,y) < +\infty$ *we have*

$$d(f(x),f(y)) \leq \alpha \max\{d(x,y), d(x,f(x)), d(y,f(y)), d(y,f(x))\};$$

(d) i) there exists $\alpha,\beta \in \mathbb{R}_+$, $\alpha+\beta<1$ *such that for all* $x,y \in X$ *with* $0 \neq d(x,y) < +\infty$, *we have*

$$d(f(x),f(y)) \leq \alpha \frac{d(x,f(x))d(y,f(y))}{d(x,y)} + \beta d(x,y);$$

ii) $d(x,y) = 0 \Rightarrow x = y$, *for all* $x, y \in X$;

3) there exists $x_0 \in X$ *such that* $d(x_0, f(x_0)) < +\infty$;

4) $d(x,x) = 0$, *for all* $x \in X$.

Then f *is a weakly Picard operator.*

Proof. Let $x := x_0 \in X$ and $y = f(x) \in X$.

◇ If f satisfies (a) then we get

$$d(x_0,f(x_0)) + d(f(x_0),f^2(x_0)) \leq \alpha d(x_0,f(x_0)) + \beta d(f(x_0),f(x_0)),$$

or equivalent, $d(f(x_0),f^2(x_0)) \leq \Lambda d(x_0,f(x_0))$, where $\Lambda := \alpha - 1$.

◇ If f satisfies (b) then we get

$$d(x_0,f(x_0)) + d(f(x_0),f^2(x_0)) + d(f(x_0),f^2(x_0)) \leq \alpha d(x_0,f(x_0)),$$

or equivalent, $d(f(x_0),f^2(x_0)) \leq \Lambda d(x_0,f(x_0))$, where $\Lambda := \frac{\alpha-1}{2}$.

◇ If f satisfies (c) then we get

$$d(f(x_0),f^2(x_0)) \leq \alpha \max\{d(x_0,f(x_0)), d(f(x_0),f^2(x_0))\}.$$

If $d(f(x_0),f^2(x_0)) \leq \alpha d(f(x_0),f^2(x_0))$ then we have $1 \leq \alpha$, which is a contradiction. Hence $d(f(x_0),f^2(x_0)) \leq \Lambda d(x_0,f(x_0))$, where $\Lambda := \alpha$.

◇ If f satisfies (d) then we get

$$d(f(x_0),f^2(x_0)) \leq \alpha \frac{d(x_0,f(x_0))d(f(x_0),f^2(x_0))}{d(x_0,f(x_0))} + \beta d(x_0,f(x_0)).$$

If $d(x_0,f(x_0)) = 0$ then by (d), item $ii)$, we obtain $x_0 = f(x_0)$ and the conclusion follows, since the sequence of successive approximations is constant and converges to x_0. If $d(x_0,f(x_0)) \neq 0$, then we have

$$d(f(x_0),f^2(x_0)) \leq \Lambda d(x_0,f(x_0)),$$

where $\Lambda := \frac{\beta}{1-\alpha}$.

The conclusion follows by following the proof of Theorem 2.1.14. $\square$

Next, we present the φ-contraction case with respect to d.

Theorem 2.1.15 (A.-D. Filip and A. Petruşel [40]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function, i.e., φ is increasing and $\varphi^n(t) \to 0$ as $n \to +\infty$, for all $t \in \mathbb{R}_+$. We assume that:*

i) $f : (X, \to) \to (X, \to)$ has closed graph;

ii) f is a φ-contraction, i.e.

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X, \text{ with } d(x, y) < +\infty;$$

iii) $\sum\limits_{n\in\mathbb{N}} \varphi^n(t) < +\infty$, for all $t \in \mathbb{R}_+$;

iv) there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.

Then we have:

(1) f is a weakly Picard operator;

(2) if $d(x^*, y^*) < +\infty$ for all $x^*, y^* \in F_f$, then f is a Picard operator;

(3) if $d(x, x^*) < +\infty$ for all $x \in X$, where $x^* \in F_f$, then

$$d(f^n(x), x^*) \to 0 \text{ as } n \to +\infty.$$

Proof. (1). For $x_0 \in X$, we consider the sequence of successive approximations for f starting from x_0, like in Theorem 2.1.13. By *ii)* and by induction after $n \in \mathbb{N}$, we get that

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \varphi^n(d(x_0, f(x_0))), \text{ for all } n \in \mathbb{N}.$$

Hence, we can estimate

$$\sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n\in\mathbb{N}} \varphi^n(d(x_0, f(x_0))) < +\infty.$$

We get further that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$, since $(X, \to, d)$ is a generalized Kasahara space. Hence, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to \infty$.

Since f has closed graph, we obtain $x^* \in F_f$. So $F_f \neq \emptyset$.

(2). Let $x^*, y^* \in F_f$ and we assume that $d(x^*, y^*) < +\infty$. Then

$$0 \leq d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq \varphi^n(d(x^*, y^*)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

By a similar way of proof, we get that $d(y^*, x^*) = 0$.

By Lemma 2.1.1, since $d(x^*, y^*) = d(y^*, x^*) = 0$, we get $x^* = y^*$. So $F_f = \{x^*\}$.

(3). Let $x \in X$ and $x^* \in F_f$ such that $d(x, x^*) < +\infty$. Hence we have

$$d(f^n(x), x^*) = d(f^n(x), f^n(x^*)) \leq \varphi^n(d(x, x^*)) \to 0 \text{ as } n \to \infty.$$

So $d(f^n(x), x^*) \to 0$ as $n \to \infty$. $\square$

A Caristi-Browder result is shown in the next theorem.

Theorem 2.1.16 (A.-D. Filip and A. Petruşel [40])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a functional. We assume that*

i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

ii) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, *for all* $x \in X$;

iii) there exists $x_0 \in X$ *such that* $\varphi(f^n(x_0)) < +\infty$, *for all* $n \in \mathbb{N}$.

Then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to a fixed point* x^* *of* f.

Proof. We construct the sequence of successive approximations for f starting from $x_0 \in X$, as in Theorem 2.1.13. Denote by $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$. Then

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)).$$

We will prove that the series $\sum\limits_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0))$ is convergent.

For this purpose we need to show that the sequence of partials sums is convergent in $\mathbb{R}_+$.

Denote $s_n = \sum\limits_{k=0}^{n} d(f^k(x_0), f^{k+1}(x_0))$.

Then $s_{n+1} - s_n = d(f^{n+1}(x_0), f^{n+2}(x_0)) \geq 0$, for all $n \in \mathbb{N}$.

Moreover $s_n = \sum\limits_{k=0}^{n} d(f^k(x_0), f^{k+1}(x_0)) \leq \varphi(x_0)$.

Hence $(s_n)_{n\in\mathbb{N}}$ is an upper bounded and increasing sequence in $\mathbb{R}_+$, i.e. $(s_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}_+$. We have $\sum\limits_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) < +\infty$.

Since $(X, \to, d)$ be a generalized Kasahara space, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$, i.e. there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to +\infty$.

Since f has closed graph, we get that $x^* \in F_f$, so $F_f \neq \emptyset$. □

Corollary 2.1.4 (A.-D. Filip and A. Petruşel [40])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a functional. We assume that*

i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

ii) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, *for all* $x \in X$;

iii) $\varphi(f^n(x)) < +\infty$, *for all* $x \in X$ *and* $n \in \mathbb{N}$.

Then the following statements hold:

(1) f *is a weakly Picard operator;*

(2) $d(x^*, x^*) = 0$, *for all* $x^* \in F_f$.

Proof. (1). Let $x \in X$. By Theorem 2.1.16, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to a fixed point x^* of f. By *ii*) and *iii*), $d(x, f(x)) < +\infty$. Since x was arbitrary chosen, we get the conclusion.

(2). Let $x^* \in F_f$. Then we have

$$0 \le d(x^*, x^*) = d(x^*, f(x^*)) \le \varphi(x^*) - \varphi(f(x^*)) = 0.$$

□

A fixed point theorem of Maia type in generalized Kasahara spaces is presented bellow.

Theorem 2.1.17 (A.-D. Filip and A. Petruşel [40]). *Let* $(X, \stackrel{\rho}{\to}, d)$ *be a generalized Kasahara space, where* $\rho : X \times X \to \mathbb{R}_+$ *is a complete generalized metric on* X, $\stackrel{\rho}{\to}$ *is the convergence structure induced by* ρ *on* X *and* $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ *is a functional. Let* $f : X \to X$ *be an operator. We assume that:*

i) $f : (X, \stackrel{\rho}{\to}) \to (X, \stackrel{\rho}{\to})$ *has closed graph;*

ii) there exists $\theta \in [0, 1[$ *such that*

$$d(f(x), f(y)) \le \theta \cdot d(x, y), \text{ for all } x, y \in X \text{ with } d(x, y) < +\infty;$$

iii) there exists $x_0 \in X$ *such that* $d(x_0, f(x_0)) < +\infty$.

Then

(1) f *is a weakly Picard operator in* $(X, \stackrel{\rho}{\to})$.

(2) if $d(x^*, y^*) < +\infty$ *for all* $x^*, y^* \in F_f$, *then* f *is a Picard operator.*

(3) if there exists $c > 0$ *such that*

$$\rho(x, y) \le c \cdot d(x, y), \text{ for all } x, y \in X, \text{ with } d(x, y) < +\infty$$

then for every $x^* \in F_f$, *the following estimation holds*

$$\rho(f^n(x_0), x^*) \le c \cdot \frac{\theta^n}{1-\theta} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

Proof. (1). For $x_0 \in X$ let us consider the sequence of successive approximations for f starting from x_0, as in Theorem 2.1.13.

By *ii*) and *iii*) we get that there exists $\theta \in [0, 1[$ such that

$$d(f(x_0), f^2(x_0)) \le \theta \cdot d(x_0, f(x_0)).$$

By induction, we obtain for all $n \in \mathbb{N}$ the following estimations

$$d(f^n(x_0), f^{n+1}(x_0)) \le \theta \cdot d(f^{n-1}(x_0), f^n(x_0)) \le \ldots \le \theta^n \cdot d(x_0, f(x_0)).$$

Hence, the following estimation hold

$$\sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) \leq \frac{1}{1-\theta} d(x_0, f(x_0)) < +\infty.$$

Since $(X, \xrightarrow{\rho}, d)$ is a generalized Kasahara space, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent in $(X \xrightarrow{\rho})$. Thus there exists $x^* \in X$ such that $f^n(x_0) \xrightarrow{\rho} x^*$ as $n \to +\infty$.

By $i)$, we have $x^* \in F_f$.

(2). Let $x^*, y^* \in F_f$ such that $d(x^*, y^*) < +\infty$. By $ii)$, we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \theta \cdot d(x^*, y^*).$$

Since $\theta \in [0, 1[$ we conclude that $d(x^*, y^*) = 0$. By a similar procedure, we obtain $d(y^*, x^*) = 0$. Finally, by Lemma 2.1.1, we get $x^* = y^*$.

(3). By $ii)$, we have for all $n \in \mathbb{N}$ that

$$\rho(f^n(x_0), f^{n+1}(x_0)) \leq c \cdot d(f^n(x_0), f^{n+1}(x_0)) \leq c \cdot \theta^n \cdot d(x_0, f(x_0)).$$

Let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric on X we have that

$$\begin{aligned}\rho(f^n(x_0), f^{n+p}(x_0)) &\leq \sum_{k=n}^{n+p-1} \rho(f^k(x_0), f^{k+1}(x_0)) \leq \sum_{k=n}^{n+p-1} c \cdot \theta^k \cdot d(x_0, f(x_0)) \\ &= c \cdot \theta^n (1 + \theta + \ldots + \theta^{p-1}) \cdot d(x_0, f(x_0)).\end{aligned}$$

So the following estimation holds for all $n, p \in \mathbb{N}$ with $p > 0$

$$\rho(f^n(x_0), f^{n+p}(x_0)) \leq c \cdot \theta^n \frac{1-\theta^p}{1-\theta} d(x_0, f(x_0)).$$

By letting $p \to \infty$ we get the desired estimation. □

Remark 2.1.13. *Theorem 2.1.17 extends Maia's fixed point theorem (see Theorem 1 in M.G. Maia [84]) in the sense that the set X endowed with two metrics ρ and d is replaced by a generalized Kasahara space $(X, \xrightarrow{\rho}, d)$, where $\rho : X \times X \to \mathbb{R}_+$ is a complete generalized metric on X, $\xrightarrow{\rho}$ is the convergence structure induced by ρ on X and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional, not necessarily a metric.*

We give next an application of Theorem 2.1.17.

Theorem 2.1.18 (A.-D. Filip and A. Petruşel [40]). *Let V denote the rectangle: $|t - t_0| \leq a$, $|x - x_0| \leq b$ $(a, b > 0)$ in the (t, x) plane and let $f(t, x)$ be a function of two real variables defined and continuous on V. Let I be the interval defined by $|t - t_0| \leq c := \min(a, \frac{b}{M})$, where $M := \max\{|f(t, x)| \mid (t, x) \in V\}$.*

We assume that for all $(t, x), (t, y) \in V$ we have

(1) $|f(t, x) - f(t, y)| \leq \frac{k}{|t-t_0|}|x - y|$, *for some $k \leq 1$;*

(2) $|f(t,x) - f(t,y)| \leq \Lambda |x-y|^\alpha$, *for some* $\Lambda \in \mathbb{R}_+$ *and some* $\alpha \in [0,1[$;

(3) $k(1-\alpha) < 1$.

Then, the initial value problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{2.1.18}$$

has a unique solution on I.

Proof. Let us consider the space $C(I)$ defined as in Example 2.1.1 and define the operator

$$A : C(I) \to C(I),\ x \mapsto Ax, \text{ given by } Ax(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Then the initial value problem (2.1.18) is equivalent with the fixed point problem $x = Ax$.

Let $p > 1$ such that $pk(1-\alpha) < 1$ (such a p exists since $k(1-\alpha) < 1$) and consider the generalized Kasahara space $(C(I), \xrightarrow{\rho}, d_\lambda)$ presented in the Example 2.1.1, but in the particular case when $\lambda = kp$. Since the operator A is continuous on $C(I)$, A has closed graph with respect to $\xrightarrow{\rho}$.

On the other hand, A is a contraction with respect to d_{pk}.

Indeed, let us consider $x, y \in C(I)$ with $d_{pk}(x,y) < +\infty$. Then, we have

$$\begin{aligned} \big|Ax(t) - Ay(t)\big| &\leq \left| \int_{t_0}^{t} |f(s,x(s)) - f(s,y(s))| ds \right| \leq \left| \int_{t_0}^{t} k|s-t_0|^{-1} |x(s) - y(s)| ds \right| \\ &= \left| \int_{t_0}^{t} k|s-t_0|^{pk-1} |s-t_0|^{-pk} |x(s)-y(s)| ds \right| \\ &\leq d_{pk}(x,y) \left| \int_{t_0}^{t} k|s-t_0|^{pk-1} ds \right| = k\frac{1}{pk} |t-t_0|^{pk} d_{pk}(x,y). \end{aligned}$$

Hence, we obtain

$$\big|Ax(t) - Ay(t)\big| \leq \frac{1}{p} |t-t_0|^{pk} d_{pk}(x,y). \tag{2.1.19}$$

In (2.1.19), by multiplying both members with $|t-t_0|^{-pk}$, we obtain

$$d_{pk}(Ax, Ay) \leq \frac{1}{p} d_{pk}(x,y), \text{ where } \frac{1}{p} \in [0,1[.$$

By Theorem 2.1.17, the operator A has at least one fixed point $x^* \in C(I)$.

Let $x^*, y^* \in C(I)$ with $x^* \neq y^*$ be two solutions for our problem, i.e., two fixed points for A.

Since we are in the particular case $\lambda = pk$, by W.A.J. Luxemburg [82] and taking into account the hypothesis (2), we have

$$|x^*(t) - y^*(t)| \leq \Lambda^{\frac{1}{1-\alpha}} |t-t_0|^{\frac{1}{1-\alpha}}.$$

Thus $d_{pk}(x^*, y^*) < +\infty$ and the uniqueness of the fixed point x^* follws now by Theorem 2.1.17, item *(2)*. □

- We consider next the generalized Kasahara space $(X, \to, d)$, where d is a real vector-valued functional, i.e., $d : X \times X \to \mathbb{R}^n_+$. In this setting, we have some fixed point results given by I.A. Rus in [121].

Theorem 2.1.19 (I.A. Rus [121]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^n_+$ is a functional. Let $f : X \to X$ be an operator. We suppose that:*

(i) *$f : (X, \to) \to (X, \to)$ has closed graph;*

(ii) *$f : (X, d) \to (X, d)$ is a S-contraction, i.e. $d(f(x), f(y)) \leq Sd(x, y)$, for all $x, y \in X$, with S a matrix convergent to zero.*

Then:

(1) *$F_f = \{x^*\}$; $d(x^*, x^*) = 0$;*

(2) *$f^n(x) \to x^*$ as $n \to +\infty$, for all $x \in X$;*

(3) $\diamond$ *$d(f^n(x), x^*) \overset{\mathbb{R}^n}{\to} 0$, as $n \to \infty$, for all $x \in X$;*
$\diamond$ *$d(x^*, f^n(x)) \overset{\mathbb{R}^n}{\to} 0$, as $n \to \infty$, for all $x \in X$;*

(4) *If d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), then:*

(a) $\diamond$ *$d(x, x^*) \leq (I - S)^{-1} d(x, f(x))$, for all $x \in X$;*
$\diamond$ *$d(x^*, x) \leq (I - S)^{-1} d(f(x), x)$, for all $x \in X$;*

(b) *If $g : X \to X$ is such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$d(x^*, y^*) \leq (I - S)^{-1}\eta, \text{ for all } y^* \in F_g.$$

Theorem 2.1.20 (I.A. Rus [121]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^n_+$ is a functional. Let $f : X \to X$ be an operator. We suppose that:*

(i) *$f : (X, \to) \to (X, \to)$ has closed graph;*

(ii) *$f : (X, d) \to (X, d)$ is an S-graphic contraction, i.e.*

$$d(f(x), f^2(x)) \leq Sd(x, f(x)), \text{ for all } x \in X,$$

where S is a matrix convergent to zero.

Then:

(1) *$F_f \neq \emptyset$;*

(2) *$f^n(x) \to x^*(x) \in F_f$, for all $x \in X$, i.e., $f : (X, \to) \to (X, \to)$ is a weakly Picard operator;*

(3) *$d(x^*, x^*) = 0$;*

(4) *if d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), continuous with respect to $\to$, then*

$$d(x, f^{\infty}(x)) \leq (I - S)^{-1} d(x, f(x)), \text{ for all } x \in X,$$

where $f^{\infty}(x) := \lim\limits_{n\to\infty} f^n(x)$ in $(X, \to)$.

Theorem 2.1.21 (I.A. Rus [121])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^n_+$ is a functional. Let $f : X \to X$ be an operator. We suppose that:*

(*i*) *$f : (X, \to) \to (X, \to)$ has closed graph;*

(*ii*) *there exists $\varphi : X \to \mathbb{R}^n_+$ such that $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, for all $x \in X$.*

Then:

(1) *$F_f \neq \emptyset$;*

(2) *$f^n(x) \to f^{\infty}(x) \in F_f$, as $n \to \infty$;*

(3) *$d(x^*, x^*) = 0$, for all $x^* \in F_f$.*

- We present in the sequel a theory for the local variant of Banach-Caccioppoli's Contraction Principle in the context of large Kasahara spaces. To achieve this purpose, some auxiliary notions need to be defined.

Definition 2.1.8. *Let X be a nonempty set and $p : X \times X \to \mathbb{R}_+$ be a w-distance (see Definition 1.4.1) on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

(1) *the convergence structure induced by p on X is denoted by $\xrightarrow{p}$ and it is defined as follows*

$$x_n \xrightarrow{p} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} p(x_n, x) = 0.$$

(2) *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p if and only if there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that*

(2_a) *$\lim\limits_{n\to\infty} \alpha_n = 0$;*

(2_b) *$p(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$.*

By Definition 2.1.8 the following notion of large Kasahara space arises.

Definition 2.1.9. *Let $(X, \to)$ be an L-space. Let $p : X \times X \to \mathbb{R}_+$ be a w-distance on X. The triple $(X, \to, p)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and p holds:*

if $(x_n)_{n\in\mathbb{N}} \subset X$ is a Cauchy sequence with resepct to p in the sense of Definition 2.1.8 then $(x_n)_{n\in\mathbb{N}}$ converges in $(X, \to)$.

Example 2.1.2. *Let (X, d) be a complete metric space and p be a w-distance on X. Then $(X, \xrightarrow{d}, p)$ is a large Kasahara space in the sense of Definition 2.1.9.*

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to p in the sense of Definition 2.1.8. Then by Lemma 1.4.1, item (iv), $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since the metric space (X, d) is complete, $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \xrightarrow{d})$. By Definition 1.6.3 we get the conclusion.

Lemma 2.1.2. *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}_+$ be a w-distance on X. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and*

$$\tilde{B}_p(x_0, r) := \left\{x \in X \mid p(x_0, x) \le r\right\}$$

be the right closed ball centered in x_0 with radius r. Then

(1) *$\tilde{B}_p(x_0, r)$ is a closed set in (X, d);*

(2) *If (X, d) is complete, then $\left(\tilde{B}_p(x_0, r), \xrightarrow{d}, p\right)$ is a large Kasahara space in the sense of Definition 2.1.9.*

Proof. (1). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{B}_p(x_0, r)$ such that $x_n \xrightarrow{d} x$ as $n \to \infty$. Since $p(x_0, x_n) \le r$ for all $n \in \mathbb{N}$ and $p(x_0, \cdot)$ is lower semicontinuous on X, we have

$$p(x_0, x) \le \liminf_{n\to\infty} p(x_0, x_n) \le r.$$

It follows that $x \in \tilde{B}_p(x_0, r)$ and the proof is complete.

(2) By Example 2.1.2 and by (1) the conclusion follows. □

Theorem 2.1.22. *Let $(X, \xrightarrow{d}, p)$ be a large Kasahara space in the sense of Definition 2.1.9, where $\xrightarrow{d}$ is the convergence structure induced by the complete metric $d : X \times X \to \mathbb{R}_+$ on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}_p(x_0, r) \to X$ be an operator such that*

(i) *$f : (\tilde{B}_p(x_0, r), d) \to (X, d)$ has closed graph;*

(ii) *$f : (\tilde{B}_p(x_0, r), p) \to (X, p)$ is an α-contraction on $\tilde{B}_p(x_0, r)$, i.e., there exists $\alpha \in [0, 1[$ such that*

$$p(f(x), f(y)) \le \alpha p(x, y) \text{ for all } x, y \in \tilde{B}_p(x_0, r);$$

(iii) *$p(x_0, f(x_0)) \le (1 - \alpha)r$.*

Then the following statements hold

(1) $F_f = F_{f^n} = \{x_f^*\}$, *for all* $n \in \mathbb{N}^*$ *and* $p(x_f^*, x_f^*) = 0$;

(2) $f^n(x_0) \xrightarrow{d} x_f^* \in \tilde{B}_p(x_0, r)$ *as* $n \to \infty$, *for all* $x \in \tilde{B}_p(x_0, r)$, *i.e.,* $f : (\tilde{B}_p(x_0, r), \xrightarrow{d}) \to (X, \xrightarrow{d})$ *is a Picard operator;*

(3) $\lim\limits_{n\to\infty} p(f^n(x), x_f^*) = 0$, *for all* $x \in \tilde{B}_p(x_0, r)$;

(4) *for all* $x \in \tilde{B}_p(x_0, r)$ *we have:*

(4.1) $p(x, x_f^*) \leq \frac{1}{1-\alpha} p(x, f(x))$;

(4.2) $p(x_f^*, x) \leq \frac{1}{1-\alpha} p(f(x), x)$;

(4.3) $p(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha} p(x, f(x))$;

(4.4) $p(x_f^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} p(f(x), x)$;

(4.5) *if* $g : \tilde{B}_p(x_0, r) \to X$ *has the property that there exists* $\mu > 0$ *for which*

$$p(g(x), f(x)) \leq \mu, \text{ for all } x \in \tilde{B}_p(x_0, r)$$

then

$$x_g^* \in F_g \text{ and } x_g^* \in \tilde{B}_p(x_0, r) \text{ implies } p(x_g^*, x_f^*) \leq \frac{\mu}{1-\alpha}.$$

Proof. (1) & (2). Let $x_0 \in X$ and consider $(f^n(x_0))_{n\in\mathbb{N}}$ be the corresponding sequence of successive approximations of f starting from x_0. By (ii) and (iii) we have

$$p(f(x_0), f^2(x_0)) \leq \alpha p(x_0, f(x_0)) \leq \alpha(1-\alpha)r$$

and since p satisfies the triangle inequality, we get

$$\begin{aligned} p(x_0, f^2(x_0)) &\leq p(x_0, f(x_0)) + p(f(x_0), f^2(x_0)) \leq p(x_0, f(x_0)) + \alpha p(x_0, f(x_0)) \\ &= (1+\alpha) p(x_0, f(x_0)) \leq (1-\alpha^2) r \leq r \Rightarrow f^2(x_0) \in \tilde{B}_p(x_0, r). \end{aligned}$$

By a similar procedure, we have

$$\begin{aligned} p(x_0, f^3(x_0)) &\leq p(x_0, f(x_0)) + p(f(x_0), f^2(x_0)) + p(f^2(x_0), f^3(x_0)) \\ &\leq p(x_0, f(x_0)) + \alpha p(x_0, f(x_0)) + \alpha^2 p(x_0, f(x_0)) \\ &\leq (1+\alpha+\alpha^2)(1-\alpha) r = (1-\alpha^3) r \leq r \Rightarrow f^3(x_0) \in \tilde{B}_p(x_0, r). \end{aligned}$$

By induction we get that $f^n(x_0) \in \tilde{B}_p(x_0, r)$, for all $n \in \mathbb{N}$.

On the other hand, by (ii) and by induction after $n \in \mathbb{N}$, we get that

$$p(f^n(x_0), f^{n+1}(x_0)) \leq \alpha p(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \alpha^n p(x_0, f(x_0)).$$

For all $m, n \in \mathbb{N}$, $m > n$ we have

$$\begin{aligned} p(f^n(x_0), f^m(x_0)) &\leq \sum_{k=n}^{m-1} p(f^k(x_0), f^{k+1}(x_0)) \leq \sum_{k=n}^{m-1} \alpha^k p(x_0, f(x_0)) \\ &\leq \frac{\alpha^n}{1-\alpha} p(x_0, f(x_0)) = \alpha^n r \to 0, \text{ as } n \to \infty. \end{aligned}$$

Since there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, defined by $\alpha_n = \alpha^n r$, for all $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \alpha_n = 0$, by Definition 2.1.8, item (ii), we get that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p in $\tilde{B}_p(x_0, r)$.

By Lemma 2.1.2, item (2), $\big(\tilde{B}_p(x_0, r), \xrightarrow{d}, p\big)$ is a large Kasahara space. Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a convergent sequence with respect to d in $\tilde{B}_p(x_0, r)$ and thus, there exists an $x_f^* \in \overline{B}_p(x_0, r)$ such that $f^n(x_0) \xrightarrow{d} x_f^*$ as $n \to \infty$.

By (i) we obtain that $x_f^* \in F_f$. On the other hand $x_f^* = f(x_f^*) = f(f(x_f^*)) = \ldots = f^n(x_f^*)$ and thus $x_f^* \in F_{f^n}$, for all $n \in \mathbb{N}^*$.

Next we show the uniqueness of the fixed point x_f^*.

Let $y_f^* \in X$ be another fixed point for the operator f such that $x^* \neq y^*$. Then for all $x \in X$ we have

$$\begin{aligned} p(f^n(x), x_f^*) = p(f^n(x), f^n(x_f^*)) &\leq \alpha p(f^{n-1}(x), f^{n-1}(x_f^*)) \\ &\leq \ldots \leq \alpha^n p(x, x_f^*) \xrightarrow{\mathbb{R}} 0, \text{ as } n \to \infty; \\ p(f^n(x), y_f^*) = p(f^n(x), f^n(y_f^*)) &\leq \alpha p(f^{n-1}(x), f^{n-1}(y_f^*)) \\ &\leq \ldots \leq \alpha^n p(x, y_f^*) \xrightarrow{\mathbb{R}} 0, \text{ as } n \to \infty. \end{aligned}$$

By Lemma 1.4.1, item (i), we obtain $x_f^* = y_f^*$. Hence f is a Picard operator.

Finally, if $x_f^* \in F_f$ then $p(x_f^*, x_f^*) = 0$. Indeed,

$$\begin{aligned} p(x_f^*, x_f^*) = p(f^n(x_f^*), f^n(x_f^*)) &\leq \alpha p(f^{n-1}(x_f^*), f^{n-1}(x_f^*)) \\ &\leq \ldots \leq \alpha^n p(x_f^*, x_f^*) \xrightarrow{\mathbb{R}} 0, \text{ as } n \to \infty. \end{aligned}$$

(3). Let $x \in \tilde{B}_p(x_0, r)$. Then by (ii) we have

$$\begin{aligned} p(f^n(x), x_f^*) = p(f^n(x), f^n(x_f^*)) &\leq \alpha p(f^{n-1}(x), f^{n-1}(x_f^*)) \\ &\leq \ldots \leq \alpha^n p(x, x_f^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty, \end{aligned}$$

(4). Let $x \in \tilde{B}_p(x_0, r)$. Then

$$\begin{aligned} p(x, x_f^*) \leq p(x, f(x)) + p(f(x), f(x_f^*)) \leq p(x, f(x)) + \alpha p(x, x_f^*) \\ \Rightarrow p(x, x_f^*) \leq \frac{1}{1-\alpha} p(x, f(x)), \text{ so (4.1) holds.} \end{aligned}$$

By a similar way we obtain (4.2).

We show next (4.3). By taking $x := f^n(x)$ in (4.1), we get

$$p(f^n(x), x_f^*) \leq \frac{1}{1-\alpha} p(f^n(x), f^{n+1}(x)) \leq \frac{\alpha}{1-\alpha} p(f^{n-1}(x), f^n(x)) \leq \ldots \leq \frac{\alpha^n}{1-\alpha} p(x, f(x))$$

so (4.3) holds and by a similar procedure, we obtain (4.4).

Finally, let $x_g^* \in F_g$ such that $x_g^* \in \tilde{B}_p(x_0, r)$. Then by taking $x = x_g^*$ in (4.1) we get

$$p(x_g^*, x_f^*) \leq \frac{1}{1-\alpha} p(x_g^*, f(x_g^*)) = \frac{1}{1-\alpha} p(g(x_g^*), f(x_g^*)) \leq \frac{\mu}{1-\alpha}$$

and thus, (4.5) holds. □

- We give next some fixed point theorems in large Kasahara spaces that are obtained from complete metric spaces by perturbing the metric.

Several fixed point theorems were proved in metric spaces with perturbed metric. In this sense we have the works of M.S. Khan, M. Swaleh and S. Sessa [74] ,K.P.R. Sastry and G.V.R. Babu [131], [132], K.P.R. Sastry, G.V.R. Babu and D.N. Rao [133], M.A. Şerban [142]. Our aim is to construct a large Kasahara space starting from a metric space, by perturbing the metric. More precisely, we present some fixed point results (Banach - Caccioppoli contraction principle, graphic contraction principle, Caristi - Browder and Matkowski type theorems) for self operators in a large Kasahara space $(X, \overset{d}{\to}, \rho)$ with $d : X \times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi \circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function.

Let (X, d) be a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. We consider the distance functional $\rho : X \times X \to \mathbb{R}_+$ defined by

$$\rho = \varphi \circ d.$$

Remark 2.1.14. *The distance function ρ is symmetric on X.*

Indeed, for each $x, y \in X$ we have

$$\rho(x, y) = (\varphi \circ d)(x, y) = \varphi(d(x, y)) = \varphi(d(y, x)) = (\varphi \circ d)(y, x) = \rho(y, x).$$

Let $\overset{d}{\to}$ be the convergence structure induced by the metric d on X. The notions of convergence and Cauchy sequence are well-known on metric spaces such as (X, d).

Concerning the distance functional ρ, we will introduce similar notions of convergence and Cauchy sequence as in the case of metric spaces.

Definition 2.1.10. *A sequence $(x_n)_{n\in\mathbb{N}}$ of X is convergent with respect to ρ on X if and only if there exists $x \in X$ such that*

$$\lim_{n\to\infty} \rho(x_n, x) = 0,$$

i.e., for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ we have $\rho(x_n, x) < \varepsilon$.

We shall denote the convergence with respect to ρ by $\xrightarrow{\rho}$. Notice that in this case, the Definition 2.1.10 can be expressed by the following statement:

$$x_n \xrightarrow{\rho} x \text{ as } n \to \infty \Leftrightarrow \lim_{n\to\infty} \rho(x_n, x) = 0.$$

Definition 2.1.11. *A sequence $(x_n)_{n\in\mathbb{N}}$ of X is a Cauchy sequence with respect to ρ if and only if*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \rho(x_n, x_m) = 0,$$

i.e., for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m > n \geq n_\varepsilon$ we have $\rho(x_n, x_m) < \varepsilon$.

Lemma 2.1.3 (M.A. Şerban [142]). *Let (X, d) be a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.*

Lemma 2.1.4 (A.-D. Filip [35]). *Let (X, d) be a complete metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. Let $\rho : X \times X \to \mathbb{R}_+$ defined by $\rho = \varphi \circ d$. Then $(X, \xrightarrow{d}, \rho)$ is a large Kasahara space.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to ρ in X. By Lemma 2.1.3 we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d and since (X, d) is complete, we have that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \xrightarrow{d})$. The conclusion follows from Definition 1.6.3. □

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, subadditive and continuous function.

In the sequel, we consider the large Kasahara space $(X, \xrightarrow{d}, \rho)$, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ is a distance functional defined by $\rho = \varphi \circ d$.

In the above setting, some interesting remarks concerning the distance functional ρ can be made. We recall first the notion of dislocated metric.

Definition 2.1.12 (P. Hitzler and A.K. Seda [50]). *Let X be a nonempty set and $\varrho : X \times X \to \mathbb{R}_+$ be a function. If ϱ satisfies:*

(i) for all $x, y \in X$, if $\varrho(x, y) = 0$, then $x = y$;

(ii) for all $x, y \in X$, $\varrho(x, y) = \varrho(y, x)$;

(iii) for all $x, y, z \in X$, $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$,

then ϱ is a dislocated metric on X.

Example 2.1.3. *Let $\varrho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\varrho(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}_+$. Then ϱ is a dislocated metric on X.*

Remark 2.1.15. *Let X be a nonempty set and $\varrho : X \times X \to \mathbb{R}_+$ be a dislocated metric on X. If $\varrho(x, x) = 0$ for all $x \in X$, then ϱ becomes a metric on X.*

We give next the remarks which can be made upon the distance functional ρ in the large Kasahara space $(X, \overset{d}{\to}, \rho)$ with $d : X \times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi \circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function.

Remark 2.1.16. *Notice that since φ is an increasing subadditive function, the distance function ρ satisfies the triangle inequality (iii) of Definition 2.1.12.*

Indeed, for each $x, y, z \in X$ we have

$$\begin{aligned} \rho(x,y) &= \varphi(d(x,y)) \leq \varphi(d(x,z) + d(z,y)) \\ &\leq \varphi(d(x,z)) + \varphi(d(z,y)) = \rho(x,z) + \rho(z,y). \end{aligned} \tag{2.1.20}$$

Remark 2.1.17. *Notice that*

- *if $\varphi(t) = 0 \Rightarrow t = 0$, for all $t \in \mathbb{R}_+$, then ρ is a dislocated metric on X.*
- *if $\varphi(t) = 0 \Leftrightarrow t = 0$, for all $t \in \mathbb{R}_+$, (i.e., φ is amenable (see P. Corazza [21])) then ρ is a metric on X (see M.A. Şerban [142](Lemma 2.1)).*

We present next our fixed point results which are similar to those given in Kasahara spaces: Banach-Caccioppoli's Contraction Principle (Theorem 2.1.2), Graphic Contraction Principle (Theorem 2.1.5), Caristi-Browder type theorem (Theorem 2.1.7) and Matkowski type theorem (Theorem 2.1.9).

Theorem 2.1.23 (A.-D. Filip [35])**.** *Let $(X, \overset{d}{\to}, \rho)$ be a large Kasahara space with $d : X \times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi \circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function. Let $f : X \to X$ be an operator. We assume that:*

(i) $f : (X, \overset{d}{\to}) \to (X, \overset{d}{\to})$ *has closed graph;*

(ii) $f : (X, \rho) \to (X, \rho)$ *is an α-contraction, i.e., there exists $\alpha \in [0, 1[$ such that*

$$\rho(f(x), f(y)) \leq \alpha \rho(x, y), \text{ for all } x, y \in X;$$

(iii) $\varphi(t) = 0 \Rightarrow t = 0$, *for all $t \in \mathbb{R}_+$.*

Then the following statements hold:

(1) $F_f = F_{f^n} = \{x_f^*\}$, *for all $n \in \mathbb{N}^*$ and $\rho(x_f^*, x_f^*) = 0$;*

(2) $f^n(x) \overset{d}{\to} x_f^*$ *as $n \to \infty$, for all $x \in X$, i.e., $f : (X, \overset{d}{\to}) \to (X, \overset{d}{\to})$ is a PO;*

(3) *for all $x \in X$ we have,*

   (3_a) $\rho(f^n(x), x_f^*) \overset{\mathbb{R}}{\to} 0$ *as $n \to \infty$;*

   (3_b) $\rho(x, x_f^*) \leq \frac{1}{1-\alpha} \rho(x, f(x))$;

(3_c) $\rho(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha}\rho(x, f(x))$, *for all* $n \in \mathbb{N}$;

(4) $(z_n)_{n\in\mathbb{N}} \subset X$, $\rho(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty \Rightarrow \rho(z_n, x_f^*) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$, *i.e., the fixed point problem for the operator* f *is well-posed with respect to* ρ;

(5) $(z_n)_{n\in\mathbb{N}} \subset X$, $\rho(z_{n+1}, f(z_n)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty \Rightarrow \rho(z_{n+1}, f^{n+1}(z)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$, *for all* $z \in X$, *i.e., the operator* f *has the limit shadowing property with respect to* ρ;

(6) *if* $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which*

$$\rho(g(x), f(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$x_g^* \in F_g \text{ implies } \rho(x_g^*, x_f^*) \leq \frac{\eta}{1-\alpha}.$$

Proof. (1) & (2). Let $x \in X$. We construct the sequence of successive approximations for f starting from x. Let $(x_n)_{n\in\mathbb{N}}$ be this sequence, i.e., $x_n = f^n(x)$ for all $n \in \mathbb{N}$.

Since f is an α-contraction with respect to ρ, we have by induction after $n \in \mathbb{N}$ that

$$\rho(f^n(x), f^{n+1}(x)) \leq \alpha^n \rho(x, f(x)). \tag{2.1.21}$$

By Remark 2.1.16, it follows that

$$\begin{aligned}\rho(f^n(x), f^{n+p}(x)) &\leq \sum_{k=n}^{n+p-1} \rho(f^k(x), f^{k+1}(x)) \\ &\leq \sum_{k=n}^{n+p-1} \alpha^k \rho(x, f(x)) \leq \frac{\alpha^n}{1-\alpha}\rho(x, f(x))\end{aligned}$$

for all $x \in X$ and $p \in \mathbb{N}$, $p > n$. Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ.

Since $(X, \xrightarrow{d}, \rho)$ is a large Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \xrightarrow{d})$. Hence, there exists an element $x_f^* \in X$ such that $x_n \xrightarrow{d} x_f^*$ as $n \to \infty$.

Using the fact that $f : (X, \xrightarrow{d}) \to (X, \xrightarrow{d})$ has closed graph, we obtain that $x_f^* \in F_f$. On the other hand $x_f^* = f(x_f^*) = f(f(x_f^*)) = \ldots = f^n(x_f^*)$ and thus $x_f^* \in F_{f^n}$.

Next we show the uniqueness of the fixed point x_f^*.

Let $y_f^* \in X$ be another fixed point for the operator f such that $x^* \neq y^*$. Then

$$\begin{aligned}0 \leq \rho(x_f^*, y_f^*) &= \rho(f^n(x_f^*), f^n(y_f^*)) \leq \alpha\rho(f^{n-1}(x_f^*), f^{n-1}(y_f^*)) \\ &\leq \ldots \leq \alpha^n \rho(x_f^*, y_f^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty.\end{aligned}$$

By (iii), we conclude that $x_f^* = y_f^*$. Hence f is a PO.

Finally, we prove that if $x_f^* \in F_f$ then $\rho(x_f^*, x_f^*) = 0$.

Indeed, $\rho(x_f^*, x_f^*) = \rho(f^n(x_f^*), f^n(x_f^*)) \leq \alpha^n \rho(x_f^*, x_f^*) \xrightarrow{\mathbb{R}} 0$, as $n \to \infty$.

(3_a). Let $x \in X$. Then by (ii) we have

$$\rho(f^n(x), x_f^*) = d(f^n(x), f^n(x_f^*)) \leq \alpha\rho(f^{n-1}(x), f^{n-1}(x_f^*))$$
$$\leq \ldots \leq \alpha^n \rho(x, x_f^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty,$$

so (3_a) holds.

(3_b). Since ρ satisfies the triangle inequality, we have

$$\rho(x, x_f^*) \leq \rho(x, f(x)) + \rho(f(x), f(x_f^*)) \leq \rho(x, f(x)) + \alpha\rho(x, x_f^*)$$

and hence

$$\rho(x, x_f^*) \leq \frac{1}{1-\alpha}\rho(x, f(x)), \text{ for all } x \in X,$$

so (3_b) holds.

(3_c). By (3_b), we have the following estimation

$$\rho(f^n(x), x_f^*) \leq \frac{1}{1-\alpha}\rho(f^n(x), f^{n+1}(x)), \text{ for all } x \in X. \tag{2.1.22}$$

By (2.1.22) and (2.1.21) we obtain

$$\rho(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha}\rho(x, f(x)), \text{ for all } x \in X,$$

so (3_c) holds.

(4). Let $(z_n)_{n\in\mathbb{N}} \subset X$ and we assume that $\rho(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$. By (3_b) we have

$$\rho(z_n, x_f^*) \leq \frac{1}{1-\alpha}\rho(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty$$

so (4) holds.

(5). Let $z \in X$ and $(z_n)_{n\in\mathbb{N}} \subset X$. Since $x_f^* \in F_f$, by (ii), (3_a) and the symmetry of ρ we have that

$$\rho(x_f^*, f^{n+1}(z)) = \rho(f(x_f^*), f^{n+1}(z)) \leq \alpha\rho(x_f^*, f^n(z)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \tag{2.1.23}$$

We need to prove that $\rho(z_{n+1}, x_f^*) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$.

We have

$$\rho(z_{n+1}, x_f^*) \leq \rho(z_{n+1}, f(z_n)) + \rho(f(z_n), x_f^*) \leq \rho(z_{n+1}, f(z_n)) + \alpha\rho(z_n, x_f^*)$$
$$\leq \rho(z_{n+1}, f(z_n)) + \alpha\rho(z_n, f(z_{n-1})) + \alpha^2\rho(z_{n-1}, x_f^*)$$
$$\leq \rho(z_{n+1}, f(z_n)) + \alpha\rho(z_n, f(z_{n-1})) + \ldots + \alpha^{n+1}\rho(z_0, x_f^*).$$

From a Cauchy lemma (see the references in [115], [117] or [128]) we have that

$$\rho(z_{n+1}, x_f^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \tag{2.1.24}$$

By (2.1.23) and (2.1.24), we obtain

$$\rho(z_{n+1}, f^{n+1}(z)) \leq \rho(z_{n+1}, x_f^*) + \rho(x_f^*, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

Finally, we show (5). Let $x_g^* \in F_g$. By (3_b) we have that

$$\rho(x_g^*, x_f^*) \leq \frac{1}{1-\alpha}\rho(x_g^*, f(x_g^*)) = \frac{1}{1-\alpha}\rho(g(x_g^*), f(x_g^*)) \leq \frac{\eta}{1-\alpha}.$$

□

Theorem 2.1.24 (A.-D. Filip [35]). *Let $(X, \overset{d}{\to}, \rho)$ be a large Kasahara space with $d : X \times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi \circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function. Let $f : X \to X$ be an operator. We assume that:*

(i) $f : (X, \overset{d}{\to}) \to (X, \overset{d}{\to})$ *has closed graph;*

(ii) $f : (X, \rho) \to (X, \rho)$ *is an α-graphic contraction, i.e., there exists $\alpha \in [0, 1[$ such that*

$$\rho(f(x), f^2(x)) \leq \alpha\rho(x, f(x)) \text{ for all } x \in X.$$

Then the following statements hold:

(1) $F_f \neq \emptyset$;

(2) $f^n(x) \overset{d}{\to} f^\infty(x) \in F_f$ *as $n \to \infty$, for all $x \in X$, i.e., $f : (X, \overset{d}{\to}) \to (X, \overset{d}{\to})$ is a WPO;*

(3) $\rho(x^*, x^*) = 0$, *for all $x^* \in F_f$;*

(4) $\rho(x, f^\infty(x)) \leq \frac{1}{1-\alpha}\rho(x, f(x))$, *for all $x \in X$,*

(5) *Let $g : X \to X$ be an operator. If there exists $c > 0$ such that*

$$\rho(x, g^\infty(x)) \leq c \cdot \rho(x, g(x)), \text{ for all } x \in X \tag{2.1.25}$$

and for all $x \in X$ and some $\eta > 0$,

$$\rho(f(x), g(x)) \leq \eta \tag{2.1.26}$$

then

$$H_\rho(F_f, F_g) \leq \max\left\{\frac{1}{1-\alpha}, c\right\}\eta,$$

where H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]).

Proof. (1) & (2). Let $x \in X$ and consider the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from x. Since f is a α-graphic contraction with respect to ρ, we deduce that

$$\rho(f^n(x), f^{n+1}(x)) \leq \alpha\rho(f^{n-1}(x), f^n(x)), \text{ for all } n \in \mathbb{N}.$$

By following the proof of Theorem 2.1.23 we get $(f^n(x))_{n\in\mathbb{N}}$ is a convergent sequence in $(X, \xrightarrow{d})$. By (i) it follows that the limit, denoted by $f^\infty(x)$, is a fixed point of f. So $F_f \neq \emptyset$.

(3). Let $x^* \in F_f$. Then by (ii) we have

$$\begin{aligned}\rho(x^*, x^*) = \rho(f^n(x^*), f^{n+1}(x^*)) &\leq \alpha\rho(f^{n-1}(x^*), f^n(x^*)) \leq \alpha^2\rho(f^{n-2}(x^*), f^{n-1}(x^*))\\ &\leq \ldots \leq \alpha^n\rho(x^*, f(x^*)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty.\end{aligned}$$

(4). Let $x \in X$. Then

$$\begin{aligned}\rho(x, f^\infty(x)) &\leq \rho(x, f^n(x)) + \rho(f^n(x), f^\infty(x))\\ &\leq \rho(x, f(x)) + \ldots + \rho(f^{n-1}(x), f^n(x)) + \rho(f^n(x), f^\infty(x))\\ &\leq (1 + \alpha + \ldots + \alpha^{n-1})\rho(x, f(x)) + \rho(f^n(x), f^\infty(x))\\ &\leq \frac{1}{1-\alpha}\rho(x, f(x)) + \rho(f^n(x), f^\infty(x)), \text{ for all } n \in \mathbb{N}.\end{aligned}$$

By letting $n \to \infty$ and then using (3), we obtain

$$\rho(x, f^\infty(x)) \leq \frac{1}{1-\alpha}\rho(x, f(x)), \text{ for each } x \in X,$$

so (4) holds.

We show next (5).

Let $x \in F_f$ and $y \in F_g$. Since g satisfies (2.1.25) and (2.1.26), we have

$$\rho(x, g^\infty(x)) \leq c \cdot \rho(x, g(x)) = c \cdot \rho(f(x), g(x)) \leq c\eta.$$

Since $g^\infty(x) \in F_g$ we have

$$\inf_{y\in F_g} \rho(x, y) \leq \rho(x, g^\infty(x)) \leq c\eta$$

and by taking the supremum over $x \in F_f$, we obtain

$$\sup_{x\in F_f} \inf_{y\in F_g} \rho(x, y) \leq c\eta. \tag{2.1.27}$$

On the other hand, since f satisfies (4), we have

$$\rho(y, f^\infty(y)) \leq \frac{1}{1-\alpha}\rho(y, f(y)) = \frac{1}{1-\alpha}\rho(g(y), f(y)) \leq \frac{\eta}{1-\alpha}.$$

Since $f^\infty(y) \in F_f$ we have

$$\inf_{x\in F_f} \rho(y, x) \leq \rho(y, f^\infty(y)) \leq \frac{\eta}{1-\alpha}$$

and by taking the supremum over $y \in F_g$, we obtain

$$\sup_{y\in F_g}\inf_{x\in F_f}\rho(y,x) \le \frac{\eta}{1-\alpha}. \tag{2.1.28}$$

By (2.1.27) and (2.1.28) we get

$$\begin{aligned} H_\rho(F_f,F_g) &:= \max\left\{\sup_{x\in F_f}\inf_{y\in F_g}\rho(x,y), \sup_{y\in F_g}\inf_{x\in F_f}\rho(y,x)\right\} \\ &\le \max\left\{c\eta, \frac{\eta}{1-\alpha}\right\} = \max\left\{\frac{1}{1-\alpha}, c\right\}\eta. \end{aligned}$$

□

Theorem 2.1.25 (A.-D. Filip [35]). *Let $(X,\xrightarrow{d},\rho)$ be a large Kasahara space with $d : X\times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X\times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi\circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function. Let $f : X \to X$ be an operator and $\psi : X \to \mathbb{R}_+$ be a functional. We assume that:*

(i) *$f : (X,\xrightarrow{d}) \to (X,\xrightarrow{d})$ has closed graph;*

(ii) *$f : (X,\rho) \to (X,\rho)$ is a Caristi operator, i.e.,*

$$\rho(x,f(x)) \le \psi(x) - \psi(f(x)),\ \textit{for all } x\in X.$$

Then the following statements hold:

(1) *$F_f \neq \emptyset$.*

(2) *$f^n(x) \xrightarrow{d} f^\infty(x) \in F_f$ as $n\to\infty$, for all $x\in X$, i.e., $f : (X,\xrightarrow{d}) \to (X,\xrightarrow{d})$ is a WPO;*

(3) *$\rho(x^*,x^*) = 0$, for all $x^*\in F_f$;*

(4) *if there exists an $\alpha\in\mathbb{R}_+^*$ such that $\psi(x)\le\alpha\rho(x,f(x))$, then*

$$\rho(x,f^\infty(x)) \le \alpha\rho(x,f(x)),\ \textit{for all } x\in X. \tag{2.1.29}$$

Proof. (1) & (2). Let $x\in X$. We construct the sequence of successive approximations for f starting from x. Let $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x)$ for all $n\in\mathbb{N}$ be this sequence. By (ii) we have

$$\begin{gathered} \rho(x,f(x)) \le \psi(x) - \psi(f(x)) \\ \rho(f(x),f^2(x)) \le \psi(f(x)) - \psi(f^2(x)) \\ \dots \\ \rho(f^n(x),f^{n+1}(x)) \le \psi(f^n(x)) - \psi(f^{n+1}(x)) \end{gathered}$$

Hence, the following estimations hold

$$\sum_{n\in\mathbb{N}} \rho(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} \rho(f^n(x), f^{n+1}(x)) \leq \psi(x) - \psi(f^{n+1}(x)) \leq \psi(x) < \infty.$$

This implies that $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} \rho(x_n, x_m) = 0$ and since $(X, \stackrel{d}{\to}, \rho)$ is a large Kasahara space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \stackrel{d}{\to})$. Hence, there exists an element $f^\infty(x) \in X$ such that $f^n(x) \stackrel{d}{\to} f^\infty(x)$ as $n \to \infty$.

By (i), we get that $f^\infty(x) \in F_f$. Notice also that since we have used the sequence of successive approximations in order to prove that $f^\infty(x)$ is a fixed point for f, we get further that $f^\infty(x) \in F_{f^n}$.

(3). Let $x^* \in F_f$. Then we have

$$0 \leq \rho(x^*, x^*) = \rho(x^*, f(x^*)) \leq \psi(x^*) - \psi(f(x^*)) = 0.$$

(4). Let $x \in X$, Then we have

$$\rho(x, f^n(x)) \leq \sum_{k=0}^{n-1} \rho(f^k(x), f^{k+1}(x)) \leq \psi(x) \leq \alpha\rho(x, f(x))$$

and by letting $n \to \infty$ we obtain (2.1.29). □

Theorem 2.1.26 (A.-D. Filip [35]). *Let $(X, \stackrel{d}{\to}, \rho)$ be a large Kasahara space with $d : X \times X \to \mathbb{R}_+$ a complete metric on X and $\rho : X \times X \to \mathbb{R}_+$ a distance functional defined by $\rho = \varphi \circ d$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, subadditive and continuous function. Let $f : X \to X$ be an operator and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function, i.e, ψ is increasing and $\lim\limits_{n\to\infty} \psi^n(t) = 0$, for all $t \in \mathbb{R}_+$. We assume that:*

(i) $f : (X, \stackrel{d}{\to}) \to (X, \stackrel{d}{\to})$ *has closed graph;*

(ii) $f : (X, \rho) \to (X, \rho)$ *is a ψ-contraction, i.e.,*

$$\rho(f(x), f(y)) \leq \psi(\rho(x, y)), \text{ for all } x, y \in X;$$

(iii) $\sum\limits_{n\in\mathbb{N}} \psi^n(t) < \infty$, *for all* $t \in \mathbb{R}_+$;

(iv) $\varphi(t) = 0 \Rightarrow t = 0$, *for all* $t \in \mathbb{R}_+$.

Then the following statements hold:

(1) $F_f = F_{f^n} = \{x^*\}$, *for all* $n \in \mathbb{N}^*$;

(2) $f^n(x) \stackrel{d}{\to} x^*$ *as* $n \to \infty$, *for all* $x \in X$;

(3) $\rho(x^*, x^*) = 0$.

Proof. (1) & (2). Let $x \in X$ and consider $(f^n(x))_{n\in\mathbb{N}}$ the sequence of successive approximations for f starting from x. By (ii) we have

$$\rho(f(x), f^2(x)) \leq \psi(\rho(x, f(x)))$$

$$\rho(f^2(x), f^3(x)) \leq \psi(\rho(f(x), f^2(x))) \leq \psi^2(\rho(x, f(x)))$$

By induction after $n \in \mathbb{N}$ we get that

$$\rho(f^n(x), f^{n+1}(x)) \leq \psi^n(\rho(x, f(x))), \text{ for all } n \in \mathbb{N}.$$

Hence, we can estimate

$$\sum_{n\in\mathbb{N}} \rho(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} \psi^n(\rho(x, f(x))) < \infty.$$

This implies that $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} \rho(x_n, x_m) = 0$ and since $(X, \xrightarrow{d}, \rho)$ is a large Kasahara space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \xrightarrow{d})$. Hence, there exists an element $x^* \in X$ such that $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$ and by (i) we get that $x^* \in F_f$. Since we used the sequence of successive approximations in order to prove that x^* is a fixed point for f, we get further that $x^* \in F_{f^n}$.

We show next the uniqueness of the fixed point x^* for f.

Let $y^* \in F_f$ be another fixer point for f such that $x^* \neq y^*$. Then

$$\rho(x^*, y^*) = \rho(f^n(x^*), f^n(y^*)) \leq \psi(\rho(f^{n-1}(x^*), f^{n-1}(y^*))) \leq \ldots \leq \psi^n(\rho(x^*, y^*)) \xrightarrow{\mathbb{R}} 0.$$

By (iv) we get $x^* = y^*$.

(3). Let $x^* \in F_f$. Then

$$0 \leq \rho(x^*, x^*) = \rho(f^n(x^*), f^n(x^*)) \leq \psi^n(\rho(x^*, x^*)) \xrightarrow{\mathbb{R}} 0$$

and the conclusion follows. □

Remark 2.1.18. *The above four results complement and extend some fixed point theorems given by M.S. Khan, M. Swaleh and S. Sessa [74], S.V.R. Naidu [95], M. El Amrani and A.B. Mbarki [29], M.-A. Şerban [142] in the sense that the functional φ which perturbs the metric d is not necessary amenable (see Remark 2.1.17).*

On the other hand, by using large Kasahara spaces $(X, \xrightarrow{d}, \rho)$ in which $\rho = \varphi \circ d$ is not necessary a metric, we extend the Maia's fixed point theorem given in [84](Theorem 1) and other fixed point results for single-valued mappings given in a set endowed with two metrics (see I.A. Rus, A.S. Mureşan and V. Mureşan [122] and the references therein).

Remark 2.1.19. *Particular cases of large Kasahara spaces can be obtained for a given perturbing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. The following example is relevant in this sense.*

Example 2.1.4 (A.-D. Filip [35]). *Let (X, d) be a complete metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by*

$$\varphi(t) = t + \theta(t, u(t)), \text{ for all } t \in \mathbb{R}_+$$

where $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a symmetric function satisfying the triangle inequality and $u : \mathbb{R} \to \mathbb{R}_+$ is a function.

Then $(X, \overset{d}{\to}, \varphi \circ d)$ is a large Kasahara space.

Indeed, let us consider the functional $\varrho : X \times X \to \mathbb{R}_+$ defined by $\varrho = \varphi \circ d$. Then it is easy to verify that

$$\varrho(x, y) = d(x, y) + \theta(d(x, y), u(d(x, y))), \text{ for all } x, y \in X$$

is a dislocated metric on X.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to ϱ. By following P. Hitzler and A.K. Seda [50], for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq n_0$ we have $\varrho(x_n, x_m) < \varepsilon$.

Hence $d(x_n, x_m) \leq d(x_n, x_m) + \theta(d(x_n, x_m), u(d(x_n, x_m))) = \varrho(x_n, x_m) < \varepsilon$ which implies further that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since d is a complete metric on X, we get that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence $(X, \overset{d}{\to})$. The conclusion follows from Definition 1.6.3.

Remark 2.1.20. *If $(X, \overset{d}{\to}, \varrho)$ is a large Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $\varrho : X \times X \to \mathbb{R}_+$ is a continuous dislocated metric on X, then the Theorems 2.1.23, 2.1.24, 2.1.25 and 2.1.26 hold.*

2.2 Maia type fixed point theorems

The aim of this section is to recall the Maia fixed point theorem and some of its versions in order to establish a connexion with fixed point theorems in Kasahara spaces.

The following theorem was given by Maia in 1968:

Theorem 2.2.1 (M.G. Maia, [84]). *Let X be a nonempty subset, d and ρ be two metrics on X and $f : X \to X$ be a mapping. Suppose that:*

(i) $\rho(x, y) \leq d(x, y)$, *for all $x, y \in X$;*

(ii) (X, ρ) *is a complete metric space;*

(iii) $f : (X, \rho) \to (X, \rho)$ *is continuous;*

(iv) $f : (X, d) \to (X, d)$ *is an α-contraction, i.e., there exists $\alpha \in [0, 1[$ such that*

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y), \text{ for all } x, y \in X.$$

Then

(1) $F_f = \{x^*\}$;

(2) $(f^n(x_0))_{n\in\mathbb{N}}$ *converges in* (X,ρ) *to* x^*, *for all* $x_0 \in X$.

In applications we usually use the Rus variant of Maia's Theorem 2.2.1. In this sense, a very useful remark was made by I.A. Rus in [110] (see also [115]).

Remark 2.2.1. *Theorem 2.2.1 remains true if condition* (i) *is replaced by*

(i') *there exists* $c > 0$ *such that* $\rho(f(x), f(y)) \leq c \cdot d(x,y)$, *for all* $x, y \in X$;

Remark 2.2.2. *Some other Maia type results are the fixed point theorems given on a set endowed with two metrics. We recall some of them bellow.*

Theorem 2.2.2 (I.A. Rus, A.S. Mureşan and V. Mureşan [122]). *Let* X *be a nonempty set,* d *and* ρ *be two metrics on* X *and* $f : X \to X$ *be an operator. We suppose that*

(i) *there exists* $c_1 > 0$ *such that*

$$\rho(f(x), f(y)) \leq c_1 d(x,y), \text{ for all } x, y \in X;$$

(ii) (X,ρ) *is a complete metric space;*

(iii) $f : (X,\rho) \to (X,\rho)$ *is with closed graph;*

(iv) *there exists* $\alpha \in [0,1[$ *such that*

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then $f : (X,\rho) \to (X,\rho)$ *is a weakly Picard operator.*
If in addition we suppose that

(v) *there exists* $c_2 > 0$ *such that*

$$d(x,y) \leq c_2 \rho(x,y), \text{ for all } x, y \in X,$$

then $f : (X,\rho) \to (X,\rho)$ *is a c-weakly Picard operator with* $c = 1 + \frac{c_1 c_2}{1-\alpha}$.

- We consider now the case of vector-valued metrics, i.e., $d, \rho : X \times X \to \mathbb{R}^m_+$. In order to give the next Maia type fixed point results, several notions need to be recalled.

We mention that if $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (respectively $\alpha < \beta$), we mean that $\alpha_i \leq \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = \overline{1,m}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$, for all $i = \overline{1,m}$.

Let $x_0 \in X$ and $r = (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m_+$. Then $\tilde{B}_d(x_0, r) := \big\{x \in X \mid d(x_0, x) \leq r\big\}$ is the closed ball centered in x_0 with radius r.

We denote by $\mathcal{M}_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ_m the zero $m \times m$ matrix and by I_m the identity $m \times m$ matrix. If $A = (a_{ij})_{i,j=\overline{1,m}}$, $B = (b_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, then by $A \leq B$ we understand $a_{ij} \leq b_{ij}$, for all $i, j = \overline{1,m}$. The symbol A^τ stands for the transpose of the matrix A. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n \to \Theta_m$ as $n \to +\infty$ (see [146]). Regarding this class of matrices we have the following classical result in matrix analysis.

Theorem 2.2.3 (G. Allaire [2](Lemma 3.3.1, page 55); R. Precup [106]; I.A. Rus [112](page 37); R.S. Varga [146](page 12))**.** *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) *A is convergent to zero;*

(ii) *$A^n \to \Theta_m$ as $n \to +\infty$;*

(iii) *the eigenvalues of A lies in the open unit disc, i.e.,*

$$|\lambda| < 1, \ \text{for all } \lambda \in \mathbb{C} \text{ with } det(A - \lambda I_m) = 0;$$

(iv) *the matrix $I_m - A$ is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

(v) *the matrix $(I_m - A)$ is non-singular and $(I_m - A)^{-1}$ has nonnegative elements;*

(vi) *$A^n q \to 0 \in \mathbb{R}^m$ and $q^\tau A^n \to 0 \in \mathbb{R}^m$ as $n \to +\infty$, for all $q \in \mathbb{R}^m$.*

Remark 2.2.3. *Some examples of matrices convergent to zero are:*

a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, *where $a, b \in \mathbb{R}_+$ and $a + b < 1$;*

b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, *where $a, b \in \mathbb{R}_+$ and $a + b < 1$;*

c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, *where $a, b, c \in \mathbb{R}_+$ and* $\max\{a, c\} < 1$.

Remark 2.2.4. *For more considerations on matrices which converges to zero, see I.A. Rus [112], A.I. Perov [99] and M. Turinici [145].*

Theorem 2.2.4 (A.-D. Filip and A. Petruşel [39])**.** *Let X be a nonempty set and $d, \rho : X \times X \to \mathbb{R}^m_+$ be two generalized metrics on X. Let $f : X \to X$ be an operator. We assume that*

1) there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $\rho(f(x), f(y)) \leq C \cdot d(x, y)$, for all $x, y \in X$;

2) (X, ρ) is a complete generalized metric space;

3) $f : (X, \rho) \to (X, \rho)$ is continuous;

4) $f:(X,d)\to(X,d)$ is an almost contraction, i.e., there exist $A,B\in M_{m,m}(\mathbb{R}_+)$ such that for all $x,y\in X$ one has

$$d(f(x),f(y))\leq Ad(x,y)+Bd(y,f(x)).$$

If the matrix A converges towards zero, then $F_f\neq\emptyset$.

In addition, if the matrix $A+B$ converges to zero, then $F_f=\{x^*\}$.

Proof. We consider the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$ defined recurrently by $x_{n+1}=f(x_n)$, for all $n\in\mathbb{N}$, starting from an arbitrary element $x_0\in X$.

The following statements holds

$$\begin{aligned}
d(x_1,x_2) &= d(f(x_0),f(x_1))\leq Ad(x_0,x_1)+Bd(x_1,f(x_0))=Ad(x_0,x_1)\\
d(x_2,x_3) &= d(f(x_1),f(x_2))\leq Ad(x_1,x_2)+Bd(x_2,f(x_1))\leq A^2d(x_0,x_1)\\
&\dots\\
d(x_n,x_{n+1}) &\leq A^nd(x_0,x_1),\ \text{for all } n\in\mathbb{N},\ n\geq 1.
\end{aligned}$$

Now, let $p\in\mathbb{N}$, $p>0$. We estimate

$$\begin{aligned}
d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{n+p-1},x_{n+p})\\
&\leq A^nd(x_0,x_1)+A^{n+1}d(x_0,x_1)+\ldots+A^{n+p-1}d(x_0,x_1)\\
&\leq A^n(I_m+A+A^2+\ldots+A^{p-1}+\ldots)d(x_0,x_1)\\
&= A^n(I_m-A)^{-1}d(x_0,x_1).
\end{aligned}$$

By letting $n\to\infty$ we obtain that $d(x_n,x_{n+p})\to 0\in\mathbb{R}^m$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.

On the other hand, using the assumption 1), we get

$$\begin{aligned}
\rho(x_n,x_{n+p}) &= \rho(f(x_{n-1}),f(x_{n+p-1}))\leq C\cdot d(x_{n-1},x_{n+p-1})\\
&\leq CA^{n-1}(I_m-A)^{-1}d(x_0,x_1)\to 0\ as\ n\to\infty.
\end{aligned}$$

Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ. Since (X,ρ) is complete, there exists an element $x^*\in X$ such that $x_n\xrightarrow{\rho}x^*$ as $n\to\infty$.

We prove next that $x^*=f(x^*)$, i.e., $F_f\neq\emptyset$.

Indeed, since $x_{n+1}=f(x_n)$, for all $n\in\mathbb{N}$, by letting $n\to\infty$ and by taking into account that f is continuous with respect to ρ, we get that $x^*=f(x^*)$.

The uniqueness of the fixed point x^* is proved bellow.

Let $x^*,y^*\in F_f$ such that $x^*\neq y^*$. We estimate

$$d(x^*,y^*)=d(f(x^*),f(y^*))\leq Ad(x^*,y^*)+Bd(y^*,f(x^*))=(A+B)d(x^*,y^*).$$

Thus, since (I_m-A-B) is a non-singular matrix and $(I_m-A-B)^{-1}$ has nonnegative real elements, we have

$$(I_m-A-B)d(x^*,y^*)\leq 0\Rightarrow d(x^*,y^*)\leq 0\Rightarrow x^*=y^*.$$

□

Remark 2.2.5. *Other fixed point theorems on a set endowed with two metrics can be found in the work of M. Albu [1], V. Berinde [9], B.C. Dhage [24], A.S. Mureşan [92], [90], A.S. Mureşan and V. Mureşan [91], V. Mureşan [93], R. Precup [105], B.K. Ray [107], I.A. Rus [110], [111], [113], B. Rzepecki [129], I.A. Rus, A.S. Mureşan and V. Mureşan [122].*

Remark 2.2.6. *The fixed point theorems in Kasahara spaces are natural generalizations of Maia type fixed point theorems.*

Indeed, we can notice that in Maia's Theorem 2.2.1, (X, ρ) is a complete metric space. Hence ρ induces on X the convergence structure $\xrightarrow{\rho}$. So $(X, \xrightarrow{\rho})$ is an L-space.

On the other hand, X is endowed with another functional $d : X \times X \to \mathbb{R}_+$ which is a metric.

By considering any sequence $(x_n)_{n\in\mathbb{N}}$ in X with

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$$

we get by Maia's Theorem 2.2.1, item (i) that

$$\sum_{n\in\mathbb{N}} \rho(x_n, x_{n+1}) < +\infty$$

which implies further that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X, ρ). Since (X, ρ) is complete, $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ).

By Definition 1.6.1, $(X, \xrightarrow{\rho}, d)$ is a Kasahara space.

Remark 2.2.7. *In order to include Rus' variant of Maia's Theorem 2.2.1 in the field of fixed point theory in Kasahara spaces, a special construction is imposed, which will be presented in the next section.*

2.3 Fixed point theorems in Kasahara spaces with respect to an operator

The aim of this section is to introduce a new notion: Kasahara spaces with respect to an operator. In this setting, some fixed point results are given. We study also the existence and uniqueness of solutions for integral equations and boundary value problems.

Definition 2.3.1 (A.-D. Filip [34])**.** *Let $(X, \to)$ be an L-space, $d : X \times X \to \mathbb{R}_+$ be a functional and $f : X \to X$ be an operator. The triple $(X, \to, d)$ is a Kasahara space with respect to the operator f if and only if*

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty, \text{ for all } x \in X$$

implies that

$$(f^n(x))_{n\in\mathbb{N}} \text{ is convergent in } (X, \to), \text{ for all } x \in X.$$

Remark 2.3.1. *The notion of Kasahara space with respect to an operator generalize the notion of orbital-completeness and the notion of completeness with respect to an operator.*

Remark 2.3.2. *The applications concerning w-distances and τ-distances are also generalized in the context of Kasahara spaces with respect to an operator.*

Remark 2.3.3 (A.-D. Filip [34])**.** *Notice that, in a Kasahara space with respect to an operator, Kasahara's Lemma 2.1.1 need not to be satisfied. Notice also that a Kasahara space is a Kasahara space with respect to an operator, but the reverse implication is false.*

Example 2.3.1 (A.-D. Filip [34])**.** *Let X be a nonempty set, $f : X \to X$ be an operator and $d, \rho : X \times X \to \mathbb{R}_+$ be two functionals. We suppose:*

(i) (X, ρ) is a complete metric space;

(ii) there exists $c > 0$ such that $\rho(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Then $(X, \xrightarrow{\rho}, d)$ is a Kasahara space with respect to f.

Indeed, let $x \in X$ be such that $\sum\limits_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$. Then, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we can write $\rho(f^n(x), f^{n+p}(x)) \leq \sum\limits_{k=n-1}^{n+p-2} \rho(f^{k+1}(x), f^{k+2}(x)) \leq c \sum\limits_{k=n-1}^{n+p-2} d(f^k(x), f^{k+1}(x)) \to 0$ as $n \to +\infty$. Thus, since (X, ρ) is a complete metric space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in (X, ρ). This completes the proof.

Example 2.3.2 (A.-D. Filip [34])**.** *Let*

$$X := C(\overline{\Omega}) := \{x : \overline{\Omega} \to \mathbb{R} \mid x \text{ is a continuous function on } \overline{\Omega}\},$$

where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain.

Let $\xrightarrow{\rho}$ be the convergence structure induced by $\rho : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$, where

$$\rho(x, y) := \|x - y\|_\infty := \sup_{t\in\overline{\Omega}} |x(t) - y(t)|, \text{ for all } x, y \in C(\overline{\Omega}).$$

Let $d : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$ be the functional defined by

$$d(x, y) := \|x - y\|_{L^2(\Omega)} := \left(\int_\Omega |x(t) - y(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } x, y \in C(\overline{\Omega}).$$

We consider the operator $f : C(\overline{\Omega}) \to C(\overline{\Omega})$, defined by

$$f(x)(t) := \int_\Omega K(t, s, x(s))ds$$

where $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$.

We assume that there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L(t, s)|u - v|,$$

for all $t, s \in \overline{\Omega}$ *and* $u, v \in \mathbb{R}$.

Then the triple $(X, \xrightarrow{\rho}, d)$, *i.e.*, $\left(C(\overline{\Omega}), \xrightarrow{\|\cdot\|_\infty}, \|\cdot\|_{L^2(\Omega)}\right)$ *is a Kasahara space with respect to the operator* f.

Indeed, since

$$\rho(f(x), f(y)) \leq \sup_{t\in\overline{\Omega}} \left(\int_\Omega L(t,s)^2 ds \right)^{\frac{1}{2}} \cdot d(x, y),$$

we are in the conditions of Example 2.3.1 and the conclusion follows.

Now we will present some fixed point results for Kasahara spaces with respect to an operator.

Theorem 2.3.1 (A.-D. Filip [34]). *Let* X *be a nonempty set and* $f : X \to X$ *be an operator. Suppose that* $(X, \to, d)$ *is a Kasahara space with respect to* f. *We assume that:*

(i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(ii) $f : (X, d) \to (X, d)$ *is an* α*-contraction;*

(iii) $d(x, y) = d(y, x) = 0 \;\Rightarrow\; x = y$.

Then

(1) $F_f = F_{f^n} = \{x^*\}$ *for all* $n \in \mathbb{N}^*$ *and* $d(x^*, x^*) = 0$.

(2) $f^n(x) \to x^*$ *as* $n \to \infty$, *for all* $x \in X$, *i.e.*, f *is a Picard operator.*

(3) *We have:*

  (3_a) $d(f^n(x), x^*) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$, *for all* $x \in X$;

  (3_b) $d(x^*, f^n(x)) \xrightarrow{\mathbb{R}} 0$, *as* $n \to \infty$, *for all* $x \in X$.

(4) If d *is a quasimetric (i.e.,* $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$ *and* d *satisfies the triangle inequality), then:*

  (4_a) $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x))$, *for all* $x \in X$;

  (4_b) $d(x^*, x) \leq \frac{1}{1-\alpha} d(f(x), x)$, *for all* $x \in X$;

  (4_c) $d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$, *for all* $x \in X$ *and all* $n \in \mathbb{N}$;

  (4_d) $d(x^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$, *for all* $x \in X$ *and all* $n \in \mathbb{N}$;

  (4_e) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$ *then* $d(z_n, x^*) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$, *i.e., the fixed point problem for the operator* f *is well-posed with respect to* d;

  (4_f) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_{n+1}, f(z_n)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$ *then* $d(z_{n+1}, f^{n+1}(z)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$, *for all* $z \in X$, *i.e., the operator* f *has the limit shadowing property with respect to* d;

(4_g) *If* $g : X \to X$ *is an operator such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$d(x^*, y^*) \leq \frac{\eta}{1-\alpha}, \text{ for all } y^* \in F_g.$$

Proof. (1) & (2). Let $x \in X$ and $(f^n(x))_{n\in\mathbb{N}}$ be the sequence of successive approximations of f starting from x.

By (ii) and by induction after $n \in \mathbb{N}^*$ we have that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha^n d(x, f(x)). \tag{2.3.1}$$

It follows that

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} \alpha^n d(x, f(x)) = \frac{1}{1-\alpha} d(x, f(x)) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space with respect to the operator f, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. Hence, there exists an element $x^* \in X$ such that $f^n(x) \to x^*$ as $n \to \infty$.

By (i) we obtain that $x^* \in F_f$. Since $x^* = f(x^*) = f(f(x^*)) = \ldots = f^n(x^*)$ we also conclude that $x^* \in F_{f^n}$.

Next, we show the uniqueness of the fixed point x^*.

Let $y^* \in X$ be another fixed point for the operator f such that $x^* \neq y^*$. Then

$$\begin{aligned} d(x^*, y^*) &= d(f^n(x^*), f^n(y^*)) \leq \alpha d(f^{n-1}(x^*), f^{n-1}(y^*)) \\ &\leq \ldots \leq \alpha^n d(x^*, y^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \end{aligned} \tag{2.3.2}$$

Similarly, we get that $d(y^*, x^*) = 0$. By (iii), we conclude that $x^* = y^*$. Hence f is a Picard operator.

Finally, if $x^* \in F_f$ then we can show that $d(x^*, x^*) = 0$.

Indeed, by (2.3.2), we have

$$d(x^*, x^*) \leq \alpha^n d(x^*, x^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

(3_a). Let $x \in X$. Then by (ii) we have

$$\begin{aligned} d(f^n(x), x^*) = d(f^n(x), f^n(x^*)) &\leq \alpha d(f^{n-1}(x), f^{n-1}(x^*)) \\ &\leq \ldots \leq \alpha^n d(x, x^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty, \end{aligned}$$

so (3_a) holds. By a similar approach we obtain (3_b).

(4_a). Let $x \in X$. Since the functional d satisfies the triangle inequality, we have $d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \alpha d(x, x^*)$ and hence

$$d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4_a) holds. Similarly we get that (4_b) holds.

(4_c). Using the property (4_a), we have for each $n \in \mathbb{N}$ the following estimation

$$d(f^n(x), x^*) \leq \frac{1}{1-\alpha} d(f^n(x), f^{n+1}(x)), \text{ for all } x \in X \tag{2.3.3}$$

By (2.3.3) and (2.3.1) we obtain

$$d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4_c) holds. By a similar procedure we obtain (4_d).

We prove next (4_e). Let $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$. By (4_a) we have

$$d(z_n, x^*) \leq \frac{1}{1-\alpha} d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty$$

so (4_e) holds.

(4_f). Let $z \in X$ and $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$. Since $x^* \in F_f$, by (ii) and (3_b) we have that

$$d(x^*, f^{n+1}(z)) = d(f(x^*), f^{n+1}(z)) \leq \alpha d(x^*, f^n(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \tag{2.3.4}$$

We need to prove that $d(z_{n+1}, x^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$.

We have

$$\begin{aligned} d(z_{n+1}, x^*) &\leq d(z_{n+1}, f(z_n)) + d(f(z_n), x^*) \leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, x^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \alpha^2 d(z_{n-1}, x^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \ldots + \alpha^{n+1} d(z_0, x^*). \end{aligned}$$

From a Cauchy lemma (see the references in [115], [117] or [128]) we have that

$$d(z_{n+1}, x^*) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \tag{2.3.5}$$

By (2.3.4) and (2.3.5), we obtain

$$d(z_{n+1}, f^{n+1}(z)) \leq d(z_{n+1}, x^*) + d(x^*, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty.$$

Finally, we show (4_g). Let $y^* \in F_g$. By (4_b) we have that

$$d(x^*, y^*) \leq \frac{1}{1-\alpha} d(f(y^*), y^*) = \frac{1}{1-\alpha} d(f(y^*), g(y^*)) \leq \frac{\eta}{1-\alpha}.$$

□

Theorem 2.3.2 (A.-D. Filip [34]). *Let X be a nonempty set and $f : X \to X$ be an operator. Suppose that $(X, \to, d)$ is a Kasahara space with respect to f. We assume that:*

(*i*) *$f : (X, \to) \to (X, \to)$ has closed graph;*

(*ii*) *$f : (X, d) \to (X, d)$ is an α-graphic contraction, i.e., there exists $\alpha \in [0, 1[$ such that $d(f(x), f^2(x)) \leq \alpha d(x, f(x))$, for all $x \in X$.*

Then the following statements hold:

(1) *$F_f \neq \emptyset$.*

(2) *$f^n(x) \to f^\infty(x) \in F_f$ as $n \to \infty$, for all $x \in X$, i.e., $f : (X, \to) \to (X, \to)$ is a weakly Picard operator.*

(3) *$d(x^*, x^*) = 0$, for all $x^* \in F_f$.*

(4) *if d satisfies the triangle inequality and d is continuous with respect to $\to$, then*

(4_a) *$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$, for all $x \in X$,*

(4_b) *Let $g : X \to X$ be an operator. If there exists $c > 0$ such that*

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)), \text{ for all } x \in X \tag{2.3.6}$$

and for each $x \in X$, there exists $\eta > 0$ such that

$$\max\{d(g(x), f(x)), d(f(x), g(x))\} \leq \eta, \tag{2.3.7}$$

then

$$H_d(F_f, F_g) \leq \max\left\{\frac{1}{1-\alpha}, c\right\}\eta,$$

where H_d stands for the Pompeiu-Hausdorff functional generated by d (see [51]).

Proof. (1) & (2). Let $x \in X$ and consider the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from x. Since f is an α-graphic contraction, we deduce that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha d(f^{n-1}(x), f^n(x)) \leq \ldots \leq \alpha^n d(x, f(x)), \text{ for all } n \in \mathbb{N}.$$

By the proof of Theorem 2.1.2 we get that $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. By (i) it follows that its limit is a fixed point of f. So $F_f \neq \emptyset$.

(3). Let $x^* \in F_f$. Then by (ii) we have

$$\begin{aligned} d(x^*, x^*) &= d(f^n(x^*), f^{n+1}(x^*)) \leq \alpha d(f^{n-1}(x^*), f^n(x^*)) \\ &\leq \alpha^2 d(f^{n-2}(x^*), f^{n-1}(x^*)) \leq \ldots \leq \alpha^n d(x^*, f(x^*)) \overset{\mathbb{R}}{\to} 0 \text{ as } n \to \infty. \end{aligned}$$

(4). Let $x \in X$. Then

$$\begin{aligned} d(x, f^\infty(x)) &\leq d(x, f^n(x)) + d(f^n(x), f^\infty(x)) \\ &\leq d(x, f(x)) + d(f(x), f^2(x)) + \ldots + d(f^{n-1}(x), f^n(x)) \\ &\qquad + d(f^n(x), f^\infty(x)) \\ &\leq (1 + \alpha + \ldots + \alpha^{n-1}) d(x, f(x)) + d(f^n(x), f^\infty(x)) \\ &\leq \frac{1}{1-\alpha} d(x, f(x)) + d(f^n(x), f^\infty(x)), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By letting $n \to \infty$ and by using (3), we obtain

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for each } x \in X,$$

so (4_a) holds.

We show next (4_b).

Let $x \in F_f$ and $y \in F_g$. Since g satisfies (2.3.6) and (2.3.7), we have

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)) = c \cdot d(f(x), g(x)) \leq c\eta.$$

Since $g^\infty(x) \in F_g$ we have

$$\inf_{y \in F_g} d(x, y) \leq d(x, g^\infty(x)) \leq c\eta$$

and by taking the supremum over $x \in F_f$, we obtain

$$\sup_{x \in F_f} \inf_{y \in F_g} d(x, y) \leq c\eta. \tag{2.3.8}$$

On the other hand, since f satisfies (4_a), we have

$$d(y, f^\infty(y)) \leq \frac{1}{1-\alpha} d(y, f(y)) = \frac{1}{1-\alpha} d(g(y), f(y)) \leq \frac{\eta}{1-\alpha}.$$

Since $f^\infty(y) \in F_f$ we have

$$\inf_{x \in F_f} d(y, x) \leq d(y, f^\infty(y)) \leq \frac{\eta}{1-\alpha}$$

and by taking the supremum over $y \in F_g$, we obtain

$$\sup_{y \in F_g} \inf_{x \in F_f} d(y, x) \leq \frac{\eta}{1-\alpha}. \tag{2.3.9}$$

By (2.3.8) and (2.3.9) we get

$$\begin{aligned} H_d(F_f, F_g) &:= \max\left\{ \sup_{x \in F_f} \inf_{y \in F_g} d(x, y), \sup_{y \in F_g} \inf_{x \in F_f} d(y, x) \right\} \\ &\leq \max\left\{ c\eta, \frac{\eta}{1-\alpha} \right\} = \max\left\{ \frac{1}{1-\alpha}, c \right\} \eta. \end{aligned}$$

$\square$

In what follows, we study the existence and uniqueness for integral equations and boundary value problems.

Theorem 2.3.3 (A.-D. Filip [34]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $K \in C(\overline{\Omega}\times\overline{\Omega}\times\mathbb{R})$ and $g \in C(\overline{\Omega})$. We suppose that:*

(i) $K(t,s,\cdot):\mathbb{R}\to\mathbb{R}$ is increasing, for all $t,s\in\overline{\Omega}$.

(ii) there exists $L \in C(\overline{\Omega}\times\overline{\Omega})$ such that

$$|K(t,s,u)-K(t,s,v)| \leq L(t,s)|u-v|,$$

for all $t,s\in\overline{\Omega}$ and $u,v\in\mathbb{R}$.

(iii) $\displaystyle\int_{\Omega\times\Omega} L(t,s)^2 ds dt < 1$.

Then the integral equation

$$x(t) = \int_{\Omega} K(t,s,x(s))ds + g(t),\ \ t\in\Omega \tag{2.3.10}$$

has a unique solution $x^* \in C(\overline{\Omega})$.

Proof. Let $X = C(\overline{\Omega})$ and $\to := \xrightarrow{\|\cdot\|_\infty}$ be the convergence induced by $\|\cdot\|_\infty$ on X, where $\|x\|_\infty = \sup\limits_{t\in\overline{\Omega}}|x(t)|$, for all $x\in C(\overline{\Omega})$. Let $d: X\times X\to\mathbb{R}_+$ be defined by

$$d(x,y) = \|x-y\|_{L^2(\Omega)} = \left(\int_{\Omega}|x(t)-y(t)|^2dt\right)^{\frac{1}{2}},\ \text{for all } x,y\in X.$$

We consider the operator $A: X\to X$, $x\mapsto Ax$, defined by

$$Ax(t) = \int_{\Omega} K(t,s,x(s))ds + g(t),\ \text{for all } t\in\overline{\Omega}.$$

Then the integral equation (2.3.10) is equivalent with the fixed point problem $x = Ax$.

Notice that, since A is a continuous operator on $(X, \xrightarrow{\|\cdot\|_\infty})$, we get that A has closed graph in $(X, \xrightarrow{\|\cdot\|_\infty})$.

On the other hand, A is a contraction in (X,d). Indeed, by the definition of d we have

$$\begin{aligned} d(Ax,Ay) &= \left(\int_{\Omega}|Ax(t)-Ay(t)|^2dt\right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega}\left|\int_{\Omega}\big[K(t,s,x(s))-K(t,s,y(s))\big]ds\right|^2dt\right)^{\frac{1}{2}}. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_\Omega \big[K(t,s,x(s))-K(t,s,y(s))\big]ds\right| \le \int_\Omega \big|K(t,s,x(s))-K(t,s,y(s))\big|ds \\
&\le \int_\Omega L(t,s)|x(s)-y(s)|ds \overset{\text{Hölder}}{\le} \left(\int_\Omega L(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_\Omega |x(s)-y(s)|^2 ds\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega L(t,s)^2 ds\right)^{\frac{1}{2}} \cdot d(x,y).
\end{aligned}$$

Hence, for all $x, y \in X$ we have

$$d(Ax,Ay) \le \left(\int_\Omega \left(\int_\Omega L(t,s)^2 ds\right) d(x,y)^2 dt\right)^{\frac{1}{2}} = \left(\int_\Omega \int_\Omega L(t,s)^2 ds dt\right)^{\frac{1}{2}} d(x,y)$$

and by (iii), we get that A is a contraction in (X,d).

Thus, the triple $(C(\Omega), \xrightarrow{\|\cdot\|_\infty}, d)$ is a Kasahara space with respect to the operator A (see also Example 2.3.2). Applying Theorem 2.3.1 the conclusion follows. □

We consider next the following boundary value problem

$$\begin{cases} y''(t) = f(t,y(t)), \ \textit{for all } t \in [a,b] \\ a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b) = 0 \\ b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b) = 0 \end{cases} \tag{2.3.11}$$

where $a_i, b_i \in \mathbb{R}$, $i = \overline{1,4}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We consider also the following linear mappings:

(1) $L : C^2([a,b]) \to C([a,b])$, $L(y) = y''(t)$;

(2) $l_1 : C^2([a,b]) \to \mathbb{R}$, $l_1(y) = a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b)$

(3) $l_2 : C^2([a,b]) \to \mathbb{R}$, $l_2(y) = b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b)$

Then the boundary value problem (2.3.11) can be written as follows:

$$L(y) = f(\cdot, y), \ l_1(y) = 0, \ l_2(y) = 0. \tag{2.3.12}$$

We recall that the Green's function associated to the boundary value problem (2.3.12) is the mapping

$$G : [a,b] \times [a,b] \to \mathbb{R}; \ (t,s) \mapsto G(t,s)$$

which satisfies the following conditions:

(i) $G \in C([a,b] \times [a,b])$;

(*ii*) For any $s \in [a,b]$, $G(\cdot,s) \in C^2([a,s[\cup]s,b])$ and

$$\frac{\partial}{\partial t}G(s+0,s) - \frac{\partial}{\partial t}G(s-0,s) = -\frac{1}{p(s)},$$

where $p \in C([a,b])$ and $p(s) \neq 0$ for any $s \in [a,b]$;

(*iii*) $G(\cdot,s)$ is a solution for $L(y) = 0$ on $[a,b] \setminus \{s\}$ and satisfies the boundary conditions $l_1(y) = l_2(y) = 0$.

We have the following result:

Theorem 2.3.4 (A.-D. Filip [34]). *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the boundary value problem* (2.3.12). *We assume that:*

(*i*) *there exists $L_f > 0$ such that*

$$|f(s,u) - f(s,v)| \leq L_f|u-v|,$$

for all $s \in [a,b]$ and $u, v \in \mathbb{R}$;

(*ii*) $\displaystyle\int_a^b \int_a^b G(t,s)^2 dsdt < 1$, *where G is the Green's function associated to the boundary value problem* (2.3.12).

If the homogeneous boundary value problem

$$\begin{cases} L(y) = 0 \\ l_1(y) = l_2(y) = 0 \end{cases} \tag{2.3.13}$$

admits only the trivial solution $y \equiv 0$, then the boundary value problem (2.3.12) *has a unique solution in $C([a,b])$.*

Proof. Since the problem (2.3.13) admits only the trivial solution $y \equiv 0$, there exists a unique Green function G, associated to the problem (2.3.12). Moreover, (see for example P. Pavel and I.A. Rus [98], p.160) the boundary value problem (2.3.12) is equivalent with the Fredholm type integral equation

$$y(t) = -\int_a^b G(t,s)f(s,y(s))ds, \text{ for all } t \in [a,b]. \tag{2.3.14}$$

Let $X = C([a,b])$, $\to := \xrightarrow{\|\cdot\|_\infty}$ be the convergence structure on X, where $\|x\|_\infty = \sup\limits_{t\in[a,b]} |x(t)|$, for all $x \in C([a,b])$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = \|x-y\|_{L^2([a,b])} = \left(\int_a^b |x(t)-y(t)|^2 dt\right)^{\frac{1}{2}}, \text{ for all } x, y \in X.$$

We consider the operator $A : X \to X$, $x \mapsto Ax$, defined by

$$Ax(t) = -\int_a^b G(t,s)f(s,x(s))ds, \text{ for all } t \in [a,b].$$

Then the integral equation (2.3.14) is equivalent with the fixed point problem $y = Ay$.

Notice that, since A is a continuous operator on $(X, \xrightarrow{\|\cdot\|_\infty})$, we have that A has closed graph in $(X, \xrightarrow{\|\cdot\|_\infty})$.

On the other hand, A is a contraction in (X,d). Indeed, by the definition of d we have

$$\begin{aligned} d(Ax, Ay) &= \left(\int_a^b |Ax(t) - Ay(t)|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_a^b \left|\int_a^b G(t,s)\big[f(s,x(s)) - f(s,y(s))\big]ds\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} &\left|\int_a^b G(t,s)\big[f(s,x(s)) - f(s,y(s))\big]ds\right| \\ &\quad \leq \int_a^b G(t,s)\big|f(s,x(s)) - f(s,y(s))\big|ds \leq \int_a^b G(t,s)L_f|x(s) - y(s)|ds \\ &\quad \overset{\text{Hölder}}{\leq} L_f\left(\int_a^b G(t,s)^2 ds\right)^{\frac{1}{2}}\left(\int_a^b |x(s) - y(s)|^2 ds\right)^{\frac{1}{2}} \\ &\quad = L_f\left(\int_a^b G(t,s)^2 ds\right)^{\frac{1}{2}} \cdot d(x,y). \end{aligned}$$

Hence, for all $x, y \in X$ we have

$$\begin{aligned} d(Ax, Ay) &\leq \left(\int_a^b L_f^2\left(\int_a^b G(t,s)^2 ds\right)d(x,y)^2 dt\right)^{\frac{1}{2}} \\ &= L_f\left(\int_a^b\int_a^b G(t,s)^2 dsdt\right)^{\frac{1}{2}} d(x,y). \end{aligned} \tag{2.3.15}$$

and by (ii), we get that A is a contraction in (X,d).

The triple $(C([a,b]), \xrightarrow{\|\cdot\|_\infty}, d)$ is a Kasahara space with respect to the operator A (see Example 2.3.2 and take $\overline{\Omega} = [a,b]$). By Theorem 2.3.1 the conclusion follows. □

As a particular case, we have the following result.

Theorem 2.3.5 (A.-D. Filip [34]). *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We assume that:*

(i) there exists $L_f > 0$ such that

$$|f(s,u) - f(s,v)| \leq L_f |u - v|,$$

for all $s \in [a,b]$ and $u, v \in \mathbb{R}$;

(ii) $L_f \frac{(b-a)^2}{4} < 1$.

Then the boundary value problem

$$\begin{cases} y''(t) = f(t, y(t)), \ \textit{for all } t \in [a,b] \\ y(a) = y(b) = 0 \end{cases} \tag{2.3.16}$$

has a unique solution in $C([a,b])$.

Proof. The boundary value problem (2.3.16) is equivalent with the Fredholm type integral equation (2.3.14) where the Green's function $G : [a,b] \times [a,b] \to \mathbb{R}$ is defined by

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & s \leq t \\ \frac{(b-s)(t-a)}{b-a}, & s > t \end{cases}$$

for all $t, s \in [a,b]$.

In this particular case, G is symmetric, continuous, positive on $[a,b]^2$ and

$$G(t,s) \leq \frac{b-a}{4}, \text{ for all } t, s \in [a,b]. \tag{2.3.17}$$

Let $X = C([a,b])$, $\to := \xrightarrow{\|\cdot\|_\infty}$ be the convergence structure on X, where $\|x\|_\infty = \sup\limits_{t \in [a,b]} |x(t)|$, for all $x \in C([a,b])$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = \|x - y\|_{L^2([a,b])} = \left(\int_a^b |x(t) - y(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } x, y \in X.$$

We consider the operator $A : X \to X$, $x \mapsto Ax$, defined by

$$Ax(t) = -\int_a^b G(t,s) f(s, x(s)) ds, \text{ for all } t \in [a,b].$$

Then the integral equation (2.3.14) is equivalent with the fixed point problem $y = Ay$.

Notice that, since A is a continuous operator on $(X, \xrightarrow{\|\cdot\|_\infty})$, we have that A has closed graph in $(X, \xrightarrow{\|\cdot\|_\infty})$.

On the other hand, A is a contraction in (X,d). Indeed, by following the proof of Theorem 2.3.4 we get the inequality (2.3.15).

Taking into account the property (2.3.17) of the Green's function, we have that

$$\int_a^b \int_a^b G(t,s)^2 ds dt \leq \frac{(b-a)^4}{16}. \tag{2.3.18}$$

By (2.3.15) and (2.3.18) we obtain

$$d(Ax, Ay) \leq L_f \frac{(b-a)^2}{4} d(x, y)$$

and by (ii), we get that A is a contraction in (X, d).

The triple $(C([a, b]), \xrightarrow{\|\cdot\|_\infty}, d)$ is a Kasahara space with respect to the operator A (see Example 2.3.2 and take $\overline{\Omega} = [a, b]$). Applying Theorem 2.3.1 the conclusion follows. $\square$

Chapter 3

Multivalued generalized contractions on Kasahara spaces

The aim of this chapter is to present some fixed point results for multivalued generalized contractions in Kasahara spaces, generalized Kasahara spaces and large Kasahara spaces. We give also several Maia type theorems in close connexion with the results given in the first section of this chapter. The case of Kasahara spaces with respect to a multivalued operator is also studied.

The references which were followed in order to obtain the fixed point results presented in this chapter are: M. Berinde and V. Berinde [8]; A.-D. Filip [39], [31], [32], [33], [37]; S. Kasahara [65]; A. Petruşel and I.A. Rus, [102], [103]; I.A. Rus [112], [115]; I.A. Rus, A. Petruşel and G. Petruşel [123].

3.1 Fixed point theorems in Kasahara spaces

In this section we give corresponding results to Nadler's fixed point theorem, multivalued φ-contractions, multivalued Caristi operators, multivalued (θ, L)-weak contractions, multivalued Kannan and Reich operators which were given in complete metric spaces. We shall adapt these results in order to hold in Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying some properties.

We also present some fixed point theorems in generalized Kasahara spaces and large Kasahara spaces, more precisely:

- fixed point theorems for multivalued generalized contractions in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional, satisfying some properties.
- fixed point theorems for multivalued Zamfirescu operators in large Kasahara spaces $(X, \overset{d}{\to}, p)$, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X.

We recall first some auxiliary notions concerning multivalued operators. The notions and notations given in Section 1.7 for multivalued operators are also considered.

Definition 3.1.1. *Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a multivalued operator.*

(i) A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ if and only if $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$.

(ii) We define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by

$$T^\infty(x, y) := \big\{t^\infty(x, y) \in F_T \mid \text{there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ that converges to } t^\infty(x, y)\big\}$$

(iii) $T : X \to P(X)$ is a multivalued weakly Picard operator if and only if the following statements hold:

(iii_1) $F_T \neq \emptyset$;

(iii_2) there exists a sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ which converges to a fixed point of T;

(iv) $T : X \to P(X)$ is a multivalued Picard operator if and only if the following statements hold:

(iv_1) T is a multivalued weakly Picard operator;

(iv_2) $Card(F_T) = 1$.

Definition 3.1.2 (S. Kasahara [65])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $x \in X$. Then a set $A \in P(X)$ is said to be d-closed if and only if*

$$D(x, A) = 0 \Rightarrow x \in A$$

We define the set

$$P_d(X) := \{A \in P(X) \mid A \text{ is } d\text{-closed }\}.$$

Concerning d-closed sets in Kasahara spaces, we have the following result.

Lemma 3.1.1 (Kasahara, [65])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying the property $d(x, x) = 0$ for every $x \in X$. If $A, B \in P_d(X)$ then $H_d(A, B) = 0$ if and only if $A = B$.*

In the following fixed point results, we consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying the properties:

- ⋄ $d(x, x) = 0$, for all $x \in X$;
- ⋄ $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$.

The study of fixed point theorems for multivalued mappings has been initiated by Markin [85] and Nadler [94]. The following result, usually referred as Nadler's fixed point theorem, extends the Banach-Caccioppolli contraction principle from single-valued maps to set-valued contractive maps.

Theorem 3.1.1 (S.B. Nadler Jr. [94]). *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ a set-valued α-contraction, i.e., a mapping for which there exists a constant $\alpha \in]0, 1[$ such that $H(Tx, Ty) \leq \alpha \cdot d(x, y)$, for all $x, y \in X$. Then T has at least one fixed point.*

In the above result, $P_{b,cl}(X)$ stands for the set of all bounded and closed subsets of X. In addition, H is the Pompeiu-Hausdorff functional (see [8], [15]).

We remark also that the Nadler's fixed point theorem is given in the context of metric spaces. We adapt this result into the context of Kasahara spaces.

First we prove the following lemma:

Lemma 3.1.2 (A.-D. Filip [32]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $A, B \in P_d(X)$ and a real number $q > 1$. Then for every $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq q \cdot H_d(A, B).$$

Proof. If $A = B$ then, by Lemma 3.1.1 we have $H_d(A, B) = 0$. Hence $d(a, b) = 0 \Rightarrow a = b$. So, for every $a \in A$, there exists $b := a \in B$ such that the conclusion holds.

Now let $A, B \in P_d(X)$ such that $A \neq B$. By the same Lemma 3.1.1 we get that $H_d(A, B) > 0$.

Supposing contrary: there exists $q > 1$ and there exists $a \in A$ such that for every $b \in B$,

$$d(a, b) > q \cdot H_d(A, B).$$

By taking the $\inf\limits_{b \in B}$ in the above inequality, we get that

$$H_d(A, B) \geq D(a, B) \geq q \cdot H_d(A, B).$$

Hence $q \leq 1$ which is a contradiction. □

Theorem 3.1.2 (A.-D. Filip [32]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that*

i) $Graph(T)$ is closed in $(X, \to)$;

ii) T is a multivalued α-contraction, i.e.,

$$\text{there exists } \alpha \in [0, 1[\text{ such that } H_d(Tx, Ty) \leq \alpha \cdot d(x, y), \text{ for all } x, y \in X.$$

Then T has at least one fixed point.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$.

If $x_0 = x_1$ then $x_0 \in F_T$ and the proof is complete.

If $x_0 \neq x_1$ then by Lemma 3.1.2, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq q \cdot H_d(Tx_0, Tx_1) \leq q\alpha \cdot d(x_0, x_1).$$

We take $q > 1$ such that $\theta := q\alpha < 1$. Hence

$$d(x_1, x_2) \leq \theta \cdot d(x_0, x_1).$$

For $x_2 \in Tx_1$ we have the following cases:
If $x_1 = x_2$ then $x_1 \in F_T$ and the proof is complete.
If $x_1 \neq x_2$ then by Lemma 3.1.2, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq q \cdot H_d(Tx_1, Tx_2) \leq q\alpha \cdot d(x_1, x_2) \leq \theta^2 \cdot d(x_0, x_1).$$

By induction, there exists the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$ which starts from $(x_0, x_1) \in Graph(T)$ with $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \theta^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We have the following estimations:

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n\in\mathbb{N}} \theta^n \cdot d(x_0, x_1) = \frac{1}{1-\theta} d(x_0, x_1) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. In addition, since $Graph(T)$ is closed, we have that $x^* \in F_T$. □

Remark 3.1.1. *Theorem 3.1.2 extends Nadler's fixed point theorem 3.1.1 in the sense that the context of the complete metric space is replaced by the context of a Kasahara space, where the functional $d : X \times X \to \mathbb{R}_+$ is not necessarily a metric.*

Remark 3.1.2. *By considering multivalued Rakotch operators instead of multivalued α-contractions, the following generalization of Theorem 3.1.2 holds.*

Theorem 3.1.3 (A.-D. Filip [32]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We suppose that:*

i) $Graph(T)$ is closed in $(X, \to)$;

ii) T is a multivalued Rakotch operator, i.e., there exists $\Lambda : \mathbb{R}_+ \to [0, 1[$ with $\limsup\limits_{s\to t^+} \Lambda(s) < 1$, for all $t \in \mathbb{R}_+$ such that

$$H_d(Tx, Ty) \leq \Lambda(d(x, y)) \cdot d(x, y), \text{ for all } x, y \in X.$$

Then T has at least one fixed point.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$.
If $x_0 = x_1$ then $x_0 \in F_T$ and the proof is complete.
If $x_0 \neq x_1$ then by Lemma 3.1.2, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq q \cdot H_d(Tx_0, Tx_1) \leq q \cdot \Lambda(d(x_0, x_1)) \cdot d(x_0, x_1).$$

For $x_2 \in Tx_1$, we have the following cases:
If $x_1 = x_2$ then $x_1 \in F_T$ and the proof is complete.
If $x_1 \neq x_2$ then by Lemma 3.1.2, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq q \cdot H_d(Tx_1, Tx_2) \leq q \cdot \Lambda(d(x_1, x_2)) \cdot d(x_1, x_2) \\ &\leq q^2 \cdot \Lambda(d(x_1, x_2)) \cdot \Lambda(d(x_0, x_1)) \cdot d(x_0, x_1). \end{aligned}$$

By induction, we get that there exists the sequence of successive approximations for T which starts from $(x_0, x_1) \in Graph(T)$ with $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq q^n \cdot \prod_{k=0}^{n-1} \Lambda(d(x_k, x_{k+1})) \cdot d(x_0, x_1).$$

Let $M = \max\limits_{k=\overline{0,n-1}} \{\Lambda(d(x_k, x_{k+1}))\} < 1$.
Hence we have the following estimation

$$d(x_n, x_{n+1}) \leq (qM)^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We take $q > 1$ such that $\theta = qM < 1$ and hence

$$d(x_n, x_{n+1}) \leq \theta^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

By following the proof of Theorem 3.1.2, the conclusion follows. □

Remark 3.1.3. *Theorem 3.1.3 extends a similar result given by N. Mizoguchi and W. Takahashi in [89] for multivalued Rakotch operators defined on complete metric spaces.*

A. Petruşel and I.A. Rus introduced in [103] the concept of *theory of a metric fixed point theorem* and uses this theory for the case of multivalued contractions. By following [103], we present next a fixed point theory for Theorem 3.1.2.

Theorem 3.1.4. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that*

(i) $Graph(T)$ is closed in $(X, \to)$;

(ii) T is a multivalued α-contraction, i.e.,

$$\textit{there exists } \alpha \in [0, 1[\textit{ such that } H_d(Tx, Ty) \leq \alpha \cdot d(x, y), \textit{ for all } x, y \in X;$$

(iii) *d satisfies the triangle inequality and it is continuous with respect to the second argument.*

Then

(1) *T is a multivalued weakly Picard operator and for every $x^* \in F_T$, $x_0 \in X$ and $x_1 \in Tx_0$ we have*

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, x_1) \tag{3.1.1}$$

(2) *Let $S : X \to P_d(X)$ be a multivalued α-contraction and $\eta > 0$ such that for each $x \in X$, $H_d(Sx, Tx) \leq \eta$. Then $H_d(F_S, F_T) \leq \frac{\eta}{1-\alpha}$.*

(3) *Let $T_n : X \to P_d(X)$, $n \in \mathbb{N}$ be a sequence of multivalued α-contractions such that $T_n x \xrightarrow{H_d} Tx$ as $n \to \infty$, uniformly with respect to $x \in X$. Then $F_{T_n} \xrightarrow{H_d} F_T$ as $n \to \infty$.*

(4) *If in addition, Tx is a compact set in X for each $x \in X$, then we have*

- ⋄ *(Ulam-Hyers stability of the inclusion $x \in Tx$)*
  Let $\varepsilon > 0$ and $x \in X$ be such that $D(x, Tx) \leq \varepsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \frac{\varepsilon}{1-\alpha}$.

Proof. (1). By following the proof of Theorem 3.1.2 we construct the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$. This sequence satisfies

(*j*) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

(*jj*) $d(x_n, x_{n+1}) \leq (q\alpha)^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

By Theorem 3.1.2, T is a multivalued weakly Picard operator.

On the other hand, let $n, p \in \mathbb{N}$. Since d satisfies the triangle inequality, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (q\alpha)^n [1 + (q\alpha) + (q\alpha)^2 + \ldots + (q\alpha)^{p-1}] d(x_0, x_1) \\ &= (q\alpha)^n \cdot \frac{1-(q\alpha)^p}{1-q\alpha} \cdot d(x_0, x_1), \text{ for each } n, p \in \mathbb{N}. \end{aligned} \tag{3.1.2}$$

By (3.1.2), letting $p \to \infty$, we get that

$$d(x_n, x^*) \leq (q\alpha)^n \frac{1}{1-q\alpha} d(x_0, x_1), \text{ for each } n \in \mathbb{N}. \tag{3.1.3}$$

For $n = 1$ we get

$$d(x_1, x^*) \leq \frac{q\alpha}{1-q\alpha} d(x_0, x_1).$$

Then

$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \frac{1}{1-q\alpha} d(x_0, x_1). \tag{3.1.4}$$

By letting $q \searrow 1$ in (3.1.4) we obtain the relation (3.1.1).

(2). Let $x_0 \in F_S$ be arbitrary chosen. Then, for a fixed point $t^\infty(x_0, x_1) \in F_T$, by (3.1.1), we have

$$d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1-\alpha} d(x_0, x_1), \text{ for each } x_1 \in Tx_0.$$

Let $q > 1$ be arbitrary. Then there exists $x_1 \in Tx_0$ such that

$$d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1-\alpha} q H_d(Sx_0, Tx_0) \leq \frac{q\eta}{1-\alpha}. \tag{3.1.5}$$

By a similar procedure we can prove that for each $y_0 \in F_T$, there exists $y_1 \in Sx_0$ such that

$$d(y_0, s^\infty(x_0, y_1)) \leq \frac{q\eta}{1-\alpha}. \tag{3.1.6}$$

By (3.1.5) and (3.1.6), we obtain $H_d(F_S, F_T) \leq \frac{q\eta}{1-\alpha}$, for each $q > 1$. Letting $q \searrow 1$, we get the conclusion.

(3). Follows immediately from (2).

(4). Let $\varepsilon > 0$ and $x \in X$ be such that $D(x, Tx) \leq \varepsilon$. Since Tx is compact, there exists $y \in Tx$ such that $d(x, y) \leq \varepsilon$.

By the proof of (1), we have that

$$d(x, t^\infty(x, y)) \leq \frac{1}{1-\alpha} d(x, y)$$

and since $x^* = t^\infty(x, y) \in F_T$, we get that $d(x, x^*) \leq \frac{\varepsilon}{1-\alpha}$. □

In addition, we have the following result:

Theorem 3.1.5. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that*

(i) $Graph(T)$ is closed in $(X, \to)$;

(ii) T is a multivalued α-contraction, i.e.,

$$\textit{there exists } \alpha \in [0, 1[\textit{ such that } H_d(Tx, Ty) \leq \alpha \cdot d(x, y), \textit{ for all } x, y \in X;$$

(iii) $(SF)_T \neq \emptyset$.

Then, the following assertions hold:

(1) $F_T = (SF)_T = \{x^*\}$;

(2) $F_{T^n} = (SF)_{T^n} = \{x^*\}$ for each $n \in \mathbb{N}^*$;

(3) $H_d(T^n x, x^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$, for each $x \in X$;

(4) *If d satisfies the triangle inequality, then*

(4_a) *Let $S : X \to P_d(X)$ be a multivalued operator and $\eta > 0$ such that $F_S \neq \emptyset$ and $H_d(Sx, Tx) \leq \eta$, for each $x \in X$. Then $H_d(F_S, F_T) \leq \frac{\eta}{1-\alpha}$;*

(4_b) *Let $T_n : X \to P_d(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $F_{T_n} \neq \emptyset$ for esch $n \in \mathbb{N}$ and $H_d(T_n x, Tx) \to 0$ as $n \to \infty$, uniformly with respect to $x \in X$. Then $H_d(F_{T_n}, F_T) \to 0$ as $n \to \infty$;*

(5) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$;*

(6) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H_d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$;*

(7) *Assuming that d satisfies the triangle inequality, the limit shadowing property for T holds, i.e. if $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(Ty_n, y_{n+1}) \to 0$ as $n \to \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T, such that $d(x_n, y_{n+1}) \to 0$ as $n \to \infty$.*

Proof. (1). Let $x^* \in (SF)_T$. Then by definition, $Tx^* = \{x^*\}$. Hence $x^* \in F_T$ and so $(SF)_T \subset F_T$.

On the other hand, let us suppose that $y \in F_T$. Then

$$d(x^*, y) = D(Tx^*, y) \leq H_d(Tx^*, Ty) \leq \alpha d(x^*, y) \Rightarrow d(x^*, y) = 0$$

which implies further that $x^* = y$. Hence $y \in (SF)_T$, so $F_T \subset (SF)_T$.

By the proved inclusions, we get that $F_T = (SF)_T$ and since $(SF)_T$ contains a unique element, we get the conclusion.

(2). Notice first that $x^* \in F_{T^n} \subset (SF)_{T^n}$ for each $n \in \mathbb{N}^*$.

Let $y \in (SF)_{T^n}$ for an arbitrary $n \in \mathbb{N}^*$. Then

$$d(x^*, y) = H_d(T^n x^*, T^n y) \leq \alpha H_d(T^{n-1}x^*, T^{n-1}y) \leq \ldots \leq \alpha^n d(x^*, y) \Rightarrow d(x^*, y) = 0.$$

We get further that $x^* = y$. Thus $(SF)_{T^n} = \{x^*\}$.

Let $y \in F_{T^n}$. Then

$$d(x^*, y) = D(T^n x^*, y) \leq H_d(T^n x^*, T^n y) \leq \alpha H_d(T^{n-1}x^*, T^{n-1}y) \leq \ldots \leq \alpha^n d(x^*, y).$$

Thus, $d(x^*, y) = 0$ which implies that $x^* = y$. Hence $F_{T^n} = \{x^*\}$ and the conclusion follows.

(3). Let $x \in X$ be arbitrary chosen. Then

$$H_d(T^n x, x^*) = H_d(T^n x, T^n x^*) \leq \alpha H_d(T^{n-1}x, T^{n-1}x^*) \leq \ldots \leq \alpha^n d(x, x^*) \to 0 \text{ as } n \to \infty.$$

(4_a). Let $y \in F_S$. Then

$$d(y, x^*) \leq H_d(Sy, x^*) \leq H_d(Sy, Ty) + H_d(Ty, x^*) \leq \eta + \alpha d(y, x^*).$$

Thus $d(y,x^*) \le \frac{\eta}{1-\alpha}$. The conclusion follows by the relations

$$H_d(F_S,F_T) = \sup_{y\in F_S} d(y,x^*) \le \frac{\eta}{1-\alpha}.$$

(4_b) It follows from (4_a).

(5). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(x_n,Tx_n)\to 0$ as $n\to\infty$. Then

$$d(x_n,x^*) \le D(x_n,Tx_n) + H_d(Tx_n,Tx^*) \le D(x_n,Tx_n) + \alpha d(x_n,x^*)$$

which implies further that

$$d(x_n,x^*) \le \frac{1}{1-\alpha} D(x_n,Tx_n) \to 0 \text{ as } n\to\infty.$$

(6). Since $D(x_n,Tx_n)\le H_d(x_n,Tx_n)$, the conclusion follows from (5).

(7). Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(Ty_n,y_{n+1})\to 0$ as $n\to\infty$. Then there exists $u_n\in Ty_n$, $n\in\mathbb{N}$ such that $d(u_n,y_{n+1})\to 0$ as $n\to\infty$.

We successively have:

$$\begin{aligned}
d(x^*,y_{n+1}) &\le H_d(x^*,Ty_n) + D(Ty_n,y_{n+1})\\
&\le \alpha d(x^*,y_n) + D(Ty_n,y_{n+1})\\
&\le \alpha[\alpha d(x^*,y_{n-1}) + D(Ty_{n-1},y_n)] + D(Ty_n,y_{n+1})\\
&\le \ldots \le \alpha^{n+1} d(x^*,y_0) + \alpha^n D(Ty_0,y_1) + \ldots + \alpha D(Ty_{n-1},y_n) + D(Ty_n,y_{n+1}).
\end{aligned}$$

By a Cauchy's Lemma [114], we get that $d(x^*,y_{n+1})\to 0$ as $n\to\infty$.

On the other hand, by following the proof Theorem 3.1.2 and choosing $q\in]1,\frac{1}{\alpha}[$ we have

$$d(x_n,x^*) \le qH_d(Tx_{n-1},Tx^*) \le (q\alpha)d(x_{n-1},x^*) \text{ for each } n\in\mathbb{N}^*.$$

By an inductive procedure, we get

$$d(x_n,x^*) \le (q\alpha)^n d(x_0,x^*) \to 0 \text{ as } n\to\infty.$$

Hence

$$d(x_n,y_{n+1}) \le d(x_n,x^*) + d(x^*,y_{n+1}) \to 0 \text{ as } n\to\infty.$$

$\square$

Remark 3.1.4. *Theorems 3.1.4 and 3.1.5 extend Theorems 3.1 and 3.2 given by A. Petruşel and I.A. Rus in [103] in the sense that Kasahara spaces are considered instead of complete metric spaces.*

We study next the case of multivalued φ-contractions.

Lemma 3.1.3 (A.-D. Filip [32])**.** *Let $(X,\to,d)$ be a Kasahara space, where $d: X\times X\to\mathbb{R}_+$ is a functional satisfying $d(x,x)=0$ and $d(x,y)=0 \Rightarrow x=y$, for all $x,y\in X$. Let $A,B\in P_d(X)$ and $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a function that satisfies the following conditions:*

i) $\psi(0) = 0$;

ii) $\psi(t) > t$, *for all* $t > 0$.

Then for all $a \in A$, *there exists* $b \in B$ *such that*

$$d(a, b) \leq \psi(H_d(A, B)). \tag{3.1.7}$$

Proof. If $A = B$, then by Lemma 3.1.1, $H_d(A, B) = 0$. Hence

$$0 \leq d(a, b) \leq \psi(0) = 0,$$

so $d(a, b) = 0$ which implies that $a = b$. Thus, for all $a \in A$ there exists $b := a \in B$ such that (3.1.7) holds.

If $A \neq B$ we suppose contrary: there exists $a \in A$ such that for all $b \in B$, $d(a, b) > \psi(H_d(A, B))$. We take the $\inf\limits_{b \in B}$ and we get

$$H_d(A, B) \geq D(a, B) \geq \psi(H_d(A, B)) > H_d(A, B),$$

which is a contradiction. □

Theorem 3.1.6 (A.-D. Filip [32]). *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a functional satisfying* $d(x, x) = 0$ *and* $d(x, y) = 0 \Rightarrow x = y$, *for all* $x, y \in X$. *Let* $T : X \to P_d(X)$ *be a multivalued* φ*-contraction, i.e., there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *(*φ *is a comparison function if* φ *is increasing and* $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, *for all* $t \in \mathbb{R}_+$*) such that*

$$H_d(Tx, Ty) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

We suppose that T *has closed graph in* $(X, \to)$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that:*

i) $\psi(0) = 0$;

ii) $\psi(t) > t$, *for all* $t > 0$;

iii) $\psi \circ \varphi$ *is a comparison function;*

iv) ψ *is increasing;*

v) $\sum\limits_{n \in \mathbb{N}} (\psi \circ \varphi)^n(t) < \infty$, *for all* $t > 0$.

Then T *is a multivalued weakly Picard operator.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$.

If $x_1 = x_0$ then $x_0 \in F_T$ and the proof is complete.

If $x_1 \neq x_0$ then by Lemma 3.1.3, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \psi(H_d(Tx_0, Tx_1)) \leq (\psi \circ \varphi)(d(x_0, x_1)).$$

Since $x_2 \in Tx_1$, we take into account the following two cases.
If $x_2 = x_1$ then $x_1 \in F_T$ and the proof is complete.
If $x_2 \neq x_1$ then there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \psi(H_d(Tx_1, Tx_2)) \leq (\psi \circ \varphi)(d(x_1, x_2)) \leq (\psi \circ \varphi)^2(d(x_0, x_1)).$$

By induction, there exists the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that:

j) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

jj) $d(x_n, x_{n+1}) \leq (\psi \circ \varphi)^n(d(x_0, x_1))$, for all $n \in \mathbb{N}$.

By v), the following estimations hold

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n\in\mathbb{N}} (\psi \circ \varphi)^n(d(x_0, x_1)) < \infty$$

and since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ is convergent in $(X, \to)$. Hence, there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

Since $Graph(T)$ is closed in $(X, \to)$, we have $x^* \in F_T$. □

Remark 3.1.5. *By considering Kasahara spaces, Theorem 3.1.6 extends a similar result given by R. Wegrzyk in [148] for multivalued φ-contractions defined on complete metric spaces.*

We analyze next the case of multivalued Caristi operators. For more considerations on multivalued Caristi operators see N. Mizoguchi and W. Takahashi [89], A. Petruşel [100] and J.-P. Aubin and J. Siegel [5].

Theorem 3.1.7 (A.-D. Filip [32])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $T : X \to P(X)$ be a multivalued Caristi operator, i.e., for all $x \in X$, there exists $y \in Tx$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y),$$

having closed graph. Then T has at least one fixed point.

Proof. Let $x_0 \in X$. Then there exists $x_1 \in Tx_0$. If $x_1 = x_0$ then $x_0 \in F_T$ and the proof is complete. If $x_1 \neq x_0$ then

$$d(x_0, x_1) \leq \varphi(x_0) - \varphi(x_1).$$

Since $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$. If $x_2 = x_1$ then $x_1 \in F_T$ and the proof is complete. If $x_2 \neq x_1$ then

$$d(x_1, x_2) \leq \varphi(x_1) - \varphi(x_2).$$

By induction, there exists $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

We have the following estimations

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \le \varphi(x_0) - \varphi(x_{n+1}) \le \varphi(x_0) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

Since $Graph(T)$ is closed, $x^* \in F_T$. □

The case of multivalued (θ, L)-weak contractions is studied bellow. For more considerations on multivalued (θ, L)-weak contractions see M. Berinde and V. Berinde [8] and the references therein.

Let $\tilde{P}_d(X) := \{A \subset X \mid D(x, A) = 0 \Leftrightarrow x \in A\}$. Clearly $\tilde{P}_d(X) \subset P_d(X)$.

Theorem 3.1.8 (A.-D. Filip [32])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to \tilde{P}_d(X)$ be a multivalued (θ, L)-weak contraction, i.e., there exist two constants $\theta \in [0, 1[$ and $L \ge 0$ such that*

$$H_d(Tx, Ty) \le \theta \cdot d(x, y) + L \cdot D(y, Tx), \text{ for all } x, y \in X.$$

We assume that $Graph(T)$ is closed. Then T has at least one fixed point in X.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$ then $x_0 \in F_T$ and the proof is complete. If $x_1 \ne x_0$ then by Lemma 3.1.2, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \le q \cdot H_d(Tx_0, Tx_1) \le q\theta \cdot d(x_0, x_1) + qL \cdot D(x_1, Tx_0) = q\theta \cdot d(x_0, x_1).$$

We take $q > 1$ such that $\lambda := q\theta < 1$.

Since $x_2 \in Tx_1$, if $x_2 = x_1$ then $x_1 \in F_T$ and the proof is complete. If $x_2 \ne x_1$ then by Lemma 3.1.2, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\le q \cdot H_d(Tx_1, Tx_2) \le q\theta \cdot d(x_1, x_2) + qL \cdot D(x_2, Tx_1) \\ &= \lambda \cdot d(x_1, x_2) \le \lambda^2 \cdot d(x_0, x_1). \end{aligned}$$

By induction, there exists the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$ which starts from $(x_0, x_1) \in Graph(T)$ with $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \le \lambda^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We have the following estimations:

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \le \sum_{n\in\mathbb{N}} \lambda^n \cdot d(x_0, x_1) = \frac{1}{1-\lambda} d(x_0, x_1) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. In addition, since $Graph(T)$ is closed, we have that $x^* \in F_T$. □

We present next the case of multivalued Kannan and Reich operators.

Theorem 3.1.9 (A.-D. Filip [32]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued Kannan operator, i.e.,*

$$\exists \alpha \in [0, \tfrac{1}{2}[\text{ such that } H_d(Tx, Ty) \le \alpha[D(x, Tx) + D(y, Ty)], \text{ for all } x, y \in X.$$

We assume that T has closed graph. Then T has at least one fixed point.

Proof. Let $q > 1$, $x_0 \in X$ and $x_1 \in Tx_0$. Then by Lemma 3.1.2 there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \le q \cdot H_d(Tx_0, Tx_1) \le q\alpha[D(x_0, Tx_0) + D(x_1, Tx_1)].$$

If $x_2 = x_1$ then $x_1 \in F_T$ and the proof is complete. We assume that $x_2 \neq x_1$ and we take $q > 1$ such that $\theta := q\alpha < \frac{1}{2}$. Hence we have

$$d(x_1, x_2) \le \theta \cdot D(x_0, Tx_0) + \theta \cdot D(x_1, Tx_1) \le \theta \cdot d(x_0, x_1) + \theta \cdot d(x_1, x_2).$$

So $d(x_1, x_2) \le \frac{\theta}{1-\theta} d(x_0, x_1)$.

Since $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \le q \cdot H_d(Tx_1, Tx_2)$. If $x_3 = x_2$ then $x_2 \in F_T$ and the proof is complete. We assume that $x_3 \neq x_2$. Then we have

$$d(x_2, x_3) \le \theta[D(x_1, Tx_1) + D(x_2, Tx_2)] \le \theta d(x_1, x_2) + \theta d(x_2, x_3).$$

So $d(x_2, x_3) \le \frac{\theta}{1-\theta} d(x_1, x_2) \le \left(\frac{\theta}{1-\theta}\right)^2 d(x_0, x_1)$.

By induction, there exists $(x_n)_{n\in\mathbb{N}}$ in X, a sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ with the properties:

1) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

2) $d(x_n, x_{n+1}) \le \left(\frac{\theta}{1-\theta}\right)^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

Next, we have the estimation

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \le \sum_{n\in\mathbb{N}} \left(\frac{\theta}{1-\theta}\right)^n d(x_0, x_1) = \frac{1-\theta}{1-2\theta} d(x_0, x_1) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$. Since $Graph(T)$ is closed, the conclusion follows. □

Theorem 3.1.10 (A.-D. Filip [32]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued Reich operator, i.e., there exists $\alpha, \beta, \gamma > 0$ with $\alpha+\beta+\gamma < 1$ such that*

$$H_d(Tx, Ty) \le \alpha d(x, y) + \beta D(x, Tx) + \gamma D(y, Ty), \text{ for all } x, y \in X.$$

We assume that T has closed graph. Then T has at least one fixed point.

Proof. Let $1 < q < \frac{1}{\alpha+\beta+\gamma}$, $x_0 \in X$ and $x_1 \in Tx_0$. Then by Lemma 3.1.2, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \le q \cdot H_d(Tx_0, Tx_1)$. If $x_2 = x_1$ then $x_1 \in F_T$ and the proof is complete. We assume that $x_2 \neq x_1$. In this case we have

$$\begin{aligned} d(x_1, x_2) &\le q \cdot H_d(Tx_0, Tx_1) \le q\alpha d(x_0, x_1) + q\beta D(x_0, Tx_0) + q\gamma D(x_1, Tx_1) \\ &\le q\alpha d(x_0, x_1) + q\beta d(x_0, x_1) + q\gamma d(x_1, x_2). \end{aligned}$$

Hence, we get that

$$d(x_1, x_2) \le \frac{q(\alpha+\beta)}{1-q\gamma} d(x_0, x_1).$$

Let $\theta := \frac{q(\alpha+\beta)}{1-q\gamma}$. Then $\theta < 1$.

Since $x_2 \in Tx_1$ there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \le q \cdot H_d(Tx_1, Tx_2)$. If $x_3 = x_2$ then $x_2 \in F_T$ so the proof is complete. We assume now that $x_3 \neq x_2$. Then we have

$$\begin{aligned} d(x_2, x_3) &\le q \cdot H_d(Tx_1, Tx_2) \le q\alpha d(x_1, x_2) + q\beta D(x_1, Tx_1) + q\gamma D(x_2, Tx_2) \\ &\le q\alpha d(x_1, x_2) + q\beta d(x_1, x_2) + q\gamma d(x_2, x_3). \end{aligned}$$

Hence, we get that $d(x_2, x_3) \le \theta d(x_1, x_2) \le \theta^2 d(x_0, x_1)$.

By induction, there exists $(x_n)_{n\in\mathbb{N}}$ in X, a sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ with the properties:

1) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

2) $d(x_n, x_{n+1}) \le \theta^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

Next, we have the estimation

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \le \sum_{n\in\mathbb{N}} \theta^n d(x_0, x_1) = \frac{1}{1-\theta} d(x_0, x_1) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$. Since $Graph(T)$ is closed, the conclusion follows. □

- We present next local and global fixed point results for multivalued Zamfirescu operators in Kasahara spaces, by extending the results given for single-valued Zamfirescu operators in A.-D. Filip [36].

Let us recall first the notion of multivalued Zamfirescu operator.

Definition 3.1.3 (A.-D. Filip, [37]). *Let $(X, \to, d)$ be a Kasahara space. The mapping $T : X \to P(X)$ is called multivalued Zamfirescu operator if there exist α, β, $\gamma \in \mathbb{R}_+$ with $\alpha < 1$, $\beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}$ such that for each $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that at least one of the following conditions is true:*

(1_m) *$d(u, v) \le \alpha d(x, y)$;*

(2_m) $d(u,v) \leq \beta[d(x,u) + d(y,v)]$;

(3_m) $d(u,v) \leq \gamma[d(x,v) + d(y,u)]$.

In our following results, we consider the Kasahara space $(X, \to, d)$ and assume that $d : X \times X \to \mathbb{R}_+$ is a premetric, i.e. the functional d satisfies the following conditions:

(d_1) $d(x,x) = 0$, for all $x \in X$;

(d_2) $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in X$.

We assume in addition that

(d_3) d is continuous with respect to the second argument.

Remark 3.1.6. *Under the above assumptions on $(X, \to, d)$, the right closed ball*

$$\tilde{B}_d(x_0, r) := \{x \in X \mid d(x_0, x) \leq r\}$$

where $x_0 \in X$ and $r \in \mathbb{R}_+$, is a closed set with respect to $\to$, in the sense that for any sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$, with $z_n \to z \in X$ as $n \to \infty$, we get that $z \in \tilde{B}_d(x_0, r)$.

We give next our fixed point results in Kasahara spaces.

Theorem 3.1.11 (A.-D. Filip, [37])**.** *Let $(X, \to, d)$ be a Kasahara space and $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued Zamfirescu operator. We assume that:*

(i) T has closed graph with respect to $\to$;

(ii)
$$d(x_0, z) \leq (1 - \delta)r; \tag{3.1.8}$$
where $z \in Tx_0$ and $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(iii) $d : X \times X \to \mathbb{R}_+$ is a premetric, which is continuous with respect to the second argument.

Then the following statements hold:

(1) T has at least one fixed point in $\tilde{B}_d(x_0, r)$.

(2) there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$ such that

(2.a) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

(2.b) $x_n \to x^* \in F_T$ as $n \to +\infty$;

(2.c) we have
$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{3.1.9}$$
where $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. Let $x_0 \in X$. By (ii), there exists an element $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \leq (1-\delta)r \leq r \Rightarrow x_1 \in \tilde{B}_d(x_0, r).$$

If $x_1 = x_0$ then $x_0 \in F_T$. We suppose that $x_1 \neq x_0$. Since T is a multivalued Zamfirescu operator, there exists $x_2 \in Tx_1$ such that one of the following conditions holds:

- ◇ $d(x_1, x_2) \leq \alpha d(x_0, x_1)$ or
- ◇ $d(x_1, x_2) \leq \beta[d(x_0, x_1) + d(x_1, x_2)] \Leftrightarrow d(x_1, x_2) \leq \frac{\beta}{1-\beta} d(x_0, x_1)$ or
- ◇ $d(x_1, x_2) \leq \gamma[d(x_0, x_2) + d(x_1, x_1)] \leq \gamma[d(x_0, x_1) + d(x_1, x_2)]$
  $\Leftrightarrow d(x_1, x_2) \leq \frac{\gamma}{1-\gamma} d(x_0, x_1)$.

Since $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, we have $\delta < 1$. Hence, we get

$$d(x_1, x_2) \leq \delta d(x_0, x_1).$$

Notice that if $x_2 = x_1$ then we already have a fixed point for T ($x_1 \in F_T$).
On the other hand,

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \leq d(x_0, x_1) + \delta d(x_0, x_1) \\ &\leq (1-\delta)r + \delta(1-\delta)r = (1-\delta^2)r \leq r \Rightarrow x_2 \in \tilde{B}_d(x_0, r). \end{aligned}$$

For $x_2 \in Tx_1$ there exists $x_3 \in Tx_2$ (if $x_3 = x_2$ then $x_2 \in F_T$, so we assume that $x_3 \neq x_2$) such that one of the following conditions holds:

- ◇ $d(x_2, x_3) \leq \alpha d(x_1, x_2) \leq \delta d(x_1, x_2)$ or
- ◇ $d(x_2, x_3) \leq \beta[d(x_1, x_2) + d(x_2, x_3)] \Leftrightarrow d(x_2, x_3) \leq \frac{\beta}{1-\beta} d(x_1, x_2) \leq \delta d(x_1, x_2)$ or
- ◇ $d(x_2, x_3) \leq \gamma[d(x_1, x_3) + d(x_2, x_2)] \leq \gamma[d(x_1, x_2) + d(x_2, x_3)]$
  $\Leftrightarrow d(x_2, x_3) \leq \frac{\gamma}{1-\gamma} d(x_1, x_2) \leq \delta d(x_1, x_2)$.

Thus, in all three cases we have

$$d(x_2, x_3) \leq \delta d(x_1, x_2) \leq \delta^2 d(x_0, x_1).$$

On the other hand,

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \leq (1 + \delta + \delta^2) d(x_0, x_1) \\ &\leq (1 + \delta + \delta^2)(1-\delta)r = (1-\delta^3)r \leq r \Rightarrow x_3 \in \tilde{B}_d(x_0, r). \end{aligned}$$

By induction with respect to $n \in \mathbb{N}$, we get that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\tilde{B}_d(x_0, r)$ such that

(1°) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

(2°) $d(x_n, x_{n+1}) \leq \delta^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

Next, we have the following estimations

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n\in\mathbb{N}} \delta^n d(x_0, x_1) = \frac{1}{1-\delta} d(x_0, x_1) \leq r < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, by (iii) we get that $(\tilde{B}_d(x_0, r), \to, d)$ is also a Kasahara space. Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $\tilde{B}_d(x_0, r)$, so there exists an element $x^* \in \tilde{B}_d(x_0, r)$ such that $x_n \to x^*$ as $n \to +\infty$.

Knowing that $Graph(T)$ is closed in $X \times X$ with respect to $\to$, we get that $x^* \in F_T$.

Let $p \in \mathbb{N}$, $p \geq 1$. Then, by (2°) we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \ldots + \delta^{n+p-1} d(x_0, x_1) \\ &\leq \delta^n \big(1 + \delta + \ldots + \delta^{p-1} + \ldots\big) d(x_0, x_1) = \frac{\delta^n}{1-\delta} d(x_0, x_1) \leq \delta^n r. \end{aligned}$$

By letting $p \to +\infty$, we get the estimation (3.1.9). □

In the sequel, we present a global version of Theorem 3.1.11.

Corollary 3.1.1 (A.-D. Filip, [37]). *Let $(X, \to, d)$ be a Kasahara space and $T : X \to P(X)$ be a multivalued Zamfirescu operator, having closed graph with respect to $\to$. We assume that $d : X \times X \to \mathbb{R}_+$ is a premetric, which is continuous with respect to the second argument. Then the following statements are true:*

(1) T has at least one fixed point in X.

(2) the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ converges to an element $x^* \in F_T$ as $n \to +\infty$.

(3) we have

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \text{ for all } n \in \mathbb{N}, \tag{3.1.10}$$

where $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. Fix $x_0 \in X$ and choose $r > 0$ such that the relation (3.1.8) holds. Then the conclusions follow by taking into account the proof of Theorem 3.1.11. □

Remark 3.1.7. *Regarding the Corollary 3.1.1, notice that the functional d need not to be necessarily a premetric in order to prove the existence of fixed points for an operator $T : X \to P(X)$ satisfying one of the conditions (1_m) or (2_m) from the Definition 3.1.3. However, the functional d must be at least a premetric in case T satisfies condition (3_m).*

- The following fixed point results are given for multivalued operators in the context of generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional.

We consider the following set

$$\mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+) := \left\{ Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ 0 & q_{22} & \cdots & q_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & q_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} q_{ii} < \frac{1}{2} \right\}.$$

Then the following lemma holds.

Lemma 3.1.4 (A.-D. Filip, [37]). *If $Q \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ then*

(1) the matrix Q is convergent to zero;

(2) the matrix $(I_m - Q)^{-1}Q$ is convergent to zero.

Proof. The conclusions follow from Theorem 2.2.3, since the eigenvalues of the matrices Q and $(I_m - Q)^{-1}Q$ are in the open unit disk. $\square$

Remark 3.1.8. *For more considerations on matrices convergent to zero, see Section 2.2.*

We give next our local and global fixed point results for multivalued operators in generalized Kasahara spaces.

Theorem 3.1.12 (A.-D. Filip, [37]). *Let $(X, \to, d)$ be a generalized Kasahara space and $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued operator. We assume that:*

(i) T has closed graph with respect to $\to$;

(ii) one of the following conditions holds:

(ii_1) there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ convergent to zero such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \leq Ad(x, y);$$

(ii_2) there exists a matrix $B \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \leq B[d(x, u) + d(y, v)];$$

(ii_3) there exists a matrix $C \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \leq C[d(x, v) + d(y, u)];$$

(iii) if $u \in \mathbb{R}^m_+$ is such that $u(I_m - M)^{-1} \leq (I_m - M)^{-1}r$ then $u \leq r$, for all $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;

(iv)
$$d(x_0, z)(I_m - W)^{-1} \leq r \tag{3.1.11}$$
where $z \in Tx_0$ *and* $W := \max\left\{A, (I_m - B)^{-1}B, (I_m - C)^{-1}C\right\} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$*;*

(v) $d : X \times X \to \mathbb{R}^m_+$ *is a premetric, which is continuous with respect to the second argument on* X*.*

Then the following statements hold:

(1) T *has at least one fixed point in* $\tilde{B}_d(x_0, r)$*.*

(2) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$ *such that*

(2.a) $x_{n+1} \in Tx_n$*, for all* $n \in \mathbb{N}$*;*

(2.b) $x_n \to x^* \in F_T$ *as* $n \to +\infty$*;*

(2.c) *we have*
$$d(x_n, x^*) \leq W^n (I_m - W)^{-1} d(x_0, x_1), \text{ for all } n \in \mathbb{N}, \tag{3.1.12}$$
where $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$*.*

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. By (iv) we get
$$d(x_0, x_1)(I_m - W)^{-1} \leq r \leq (I_m - W)^{-1} r \Leftrightarrow d(x_0, x_1) \leq r \Rightarrow x_1 \in \tilde{B}_d(x_0, r).$$
We assume that $x_1 \neq x_0$, otherwise we have $x_0 \in F_T$.

Then there exists $x_2 \in Tx_1$ such that one of the following conditions holds:

- $\diamond$ $d(x_1, x_2) \leq Ad(x_0, x_1)$;
- $\diamond$ $d(x_1, x_2) \leq B[d(x_0, x_1) + d(x_1, x_2)]$, or equivalent with
$$d(x_1, x_2) \leq (I_m - B)^{-1} Bd(x_0, x_1);$$
- $\diamond$ $d(x_1, x_2) \leq C[d(x_0, x_2) + d(x_1, x_1)] \leq C[d(x_0, x_1) + d(x_1, x_2)]$, i.e.
$$d(x_1, x_2) \leq (I_m - C)^{-1} Cd(x_0, x_1);$$

Since A is convergent to zero and $B, C \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$, by Lemma 3.1.4 we get that the matrix W is also convergent to zero. In addition, we have
$$d(x_1, x_2) \leq Wd(x_0, x_1).$$
On the other hand,
$$\begin{aligned} d(x_0, x_2)(I_m - W)^{-1} &\leq d(x_0, x_1)(I_m - W)^{-1} + d(x_1, x_2)(I_m - W)^{-1} \\ &\leq I_m r + Wr \leq (I_m + W + W^2 + \ldots) r \\ &= (I_m - W)^{-1} r \Rightarrow d(x_0, x_2) \leq r, \text{ so } x_2 \in \tilde{B}_d(x_0, r). \end{aligned}$$
Notice that if $x_2 = x_1$ then $x_1 \in F_T$, so we assume that $x_2 \neq x_1$.

Then, there exists $x_3 \in Tx_2$ (and we assume that $x_3 \neq x_2$, otherwise $x_2 \in F_T$) such that one of the following conditions holds:

- $\diamond$ $d(x_2, x_3) \leq Ad(x_1, x_2)$;
- $\diamond$ $d(x_2, x_3) \leq B[d(x_1, x_2) + d(x_2, x_3)]$, or equivalent with

$$d(x_2, x_3) \leq (I_m - B)^{-1} B d(x_1, x_2);$$

- $\diamond$ $d(x_2, x_3) \leq C[d(x_1, x_3) + d(x_2, x_2)] \leq C[d(x_1, x_2) + d(x_2, x_3)]$, i.e.

$$d(x_2, x_3) \leq (I_m - C)^{-1} C d(x_1, x_2),$$

so, in both three cases we get

$$d(x_2, x_3) \leq W d(x_1, x_2) \leq W^2 d(x_0, x_1).$$

On the other hand,

$$\begin{aligned} d(x_0, x_3)(I_m - W)^{-1} &\leq [d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3)](I_m - W)^{-1} \\ &\leq I_m r + W r + W^2 r \leq (I_m + W + W^2 + \ldots) r \\ &= (I_m - W)^{-1} r \Rightarrow d(x_0, x_3) \leq r, \text{ so } x_3 \in \tilde{B}_d(x_0, r). \end{aligned}$$

By induction after $n \in \mathbb{N}$, we deduce the existence of a sequence $(x_n)_{n \in \mathbb{N}}$ in $\tilde{B}_d(x_0, r)$ which satisfies the following conditions:

(1°) $x_{n+1} \in T x_n$, for all $n \in \mathbb{N}$;

(2°) $d(x_n, x_{n+1}) \leq W^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

We have next the following estimations

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n \in \mathbb{N}} W^n d(x_0, x_1) = (I_m - W)^{-1} d(x_0, x_1) < +\infty.$$

Since $(X, \to, d)$ is a generalized Kasahara space, by (v) we get that $(\tilde{B}_d(x_0, r), \to, d)$ is also a generalized Kasahara space. Hence, the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent in $\tilde{B}_d(x_0, r)$, so there exists an element $x^* \in \tilde{B}_d(x_0, r)$ such that $x_n \to x^*$, as $n \to +\infty$.

Knowing that $Graph(T)$ is closed in $X \times X$ with respect to $\to$, we get that $x^* \in F_T$.

Now let $p \in \mathbb{N}$, $p \geq 1$. Then, by (2°) we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq W^n d(x_0, x_1) + W^{n+1} d(x_0, x_1) + \ldots + W^{n+p-1} d(x_0, x_1) \\ &\leq W^n \big(I_m + W + \ldots + W^{p-1} + \ldots \big) d(x_0, x_1) \\ &= W^n (I_m - W)^{-1} d(x_0, x_1). \end{aligned}$$

By letting $p \to +\infty$, we get the estimation (3.1.12). $\square$

Remark 3.1.9. *Any matrix* $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, *with* $a, b \in \mathbb{R}_+$ *and* $\max\{a, b\} < 1$, *is convergent towards zero and satisfies the assumption* (iii) *of Theorem 3.1.12.*

Remark 3.1.10. *Theorem 3.1.12 holds even if the assumption* (ii_1) *is replaced by the following one:*

(ii_1') *there exists a matrix* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *convergent to zero and a matrix* $B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that for all* $x, y \in X$ *and* $u \in Tx$ *there exists* $v \in Ty$ *such that*

$$d(u, v) \le Ad(x, y) + Bd(y, u).$$

The corresponding global result for Theorem 3.1.12 is the following:

Corollary 3.1.2 (A.-D. Filip, [37]). *Let* $(X, \to, d)$ *be a generalized Kasahara space and* $T : X \to P(X)$ *be a multivalued operator. We assume that:*

(i) T *has closed graph with respect to* $\to$*;*

(ii) *one of the conditions* (ii_1), (ii_2), (ii_3) *of Theorem 3.1.12 holds.*

(iii) $d : X \times X \to \mathbb{R}^m_+$ *is a premetric, which is continuous with respect to the second argument.*

Then the following statements hold:

(1) T *has at least one fixed point in* X.

(2) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that* $(2.a)$, $(2.b)$ *and* $(2.c)$ *of Theorem 3.1.12 hold.*

Proof. Fix $x_0 \in X$ and choose $r > 0$ such that the relation (3.1.11) holds. Then the conclusions follow by taking into account the proof of Theorem 3.1.12. □

As an application of the previous results, we present a fixed point theorem concerning the existence of solutions for semi-linear inclusion systems.

Theorem 3.1.13 (A.-D. Filip, [37]). *Let* $\varphi, \psi : [0,1]^2 \to]0, \frac{1}{2}]$ *be two functions and* $T_1, T_2 : [0,1]^2 \to P([0,1])$ *be two multivalued operators defined as follows:*

$$T_1(x_1, x_2) = [\varphi(x_1, x_2), \tfrac{1}{2} + \varphi(x_1, x_2)] \text{ and}$$
$$T_2(x_1, x_2) = [\psi(x_1, x_2), \tfrac{1}{2} + \psi(x_1, x_2)].$$

We assume that for each (x_1, x_2), $(y_1, y_2) \in [0,1]^2$ *and each* $u_1 \in T_1(x_1, x_2)$ *and* $u_2 \in T_2(x_1, x_2)$, *there exist* $v_1 \in T_1(y_1, y_2)$ *and* $v_2 \in T_2(y_1, y_2)$ *such that one of the following couples of conditions holds:*

(I) *for all* $a, b, c, d \in \mathbb{R}_+$ *with* $|a + d \pm \sqrt{(a-d)^2 + 4bc}| < 2$,

$$|u_1 - v_1| \le a|x_1 - y_1| + b|x_2 - y_2|,$$
$$|u_2 - v_2| \le c|x_1 - y_1| + d|x_2 - y_2|,$$

(*II*) *for all* $a, b, c \in \mathbb{R}_+$ *with* $a, c < \frac{1}{2}$,

$$\begin{aligned} |u_1 - v_1| &\leq a\big(|x_1 - u_1| + |y_1 - v_1|\big) + b\big(|x_2 - u_2| + |y_2 - v_2|\big), \\ |u_2 - v_2| &\leq c\big(|x_2 - u_2| + |y_2 - v_2|\big), \end{aligned}$$

(*III*) *for all* $a, b, c \in \mathbb{R}_+$ *with* $a, c < \frac{1}{2}$,

$$\begin{aligned} |u_1 - v_1| &\leq a\big(|x_1 - v_1| + |y_1 - u_1|\big) + b\big(|x_2 - v_2| + |y_2 - u_2|\big), \\ |u_2 - v_2| &\leq c\big(|x_2 - v_2| + |y_2 - u_2|\big). \end{aligned}$$

Then the system

$$\begin{cases} x_1 \in T_1(x_1, x_2) \\ x_2 \in T_2(x_1, x_2), \end{cases}$$

has at least one solution in $[0,1]^2$.

Proof. Let $T := (T_1, T_2) : [0,1]^2 \to P([0,1]^2)$. Then the above system can be represented as a fixed point problem of the form

$$x \in Tx, \text{ where } x = (x_1, x_2) \in [0,1]^2.$$

We consider the generalized Kasahara space $([0,1]^2, \xrightarrow{\rho_e}, d)$ where:

i) $\rho_e : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$\rho_e(x, y) = (|x_1 - y_1|, |x_2 - y_2|),$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$;

ii) $d : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$d(x,y) = \begin{cases} \rho_e(x,y) & , \ x \neq \theta \text{ and } y \neq \theta \\ (1,1) & , \ x = \theta \text{ or } y = \theta \end{cases},$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$, where $\theta = (0,0)$.

For each $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$ and $u = (u_1, u_2) \in Tx$, there exists $v = (v_1, v_2) \in Ty$ such that one of the following conditions holds:

(*I*) $d(u, v) \leq Ad(x, y)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}_+)$;

(*II*) $d(u, v) \leq A[d(x, u) + d(y, v)]$, where $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$;

(*III*) $d(u, v) \leq A[d(x, v) + d(y, u)]$, where $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{M}^{\Delta}_{2,2}(\mathbb{R}_+)$.

In all three cases, the matrix A is convergent to zero, having its eigenvalues in the open unit disc.

Since $Graph(T)$ is closed in $[0,1]^2 \times [0,1]^2$ with respect to $\xrightarrow{\rho_e}$, the conclusion follows from the Corollary 3.1.2. $\square$

Some other fixed point results in generalized Kasahara spaces are presented in the sequel.

Remark 3.1.11. *Kasahara's Lemma 2.1.1 holds also in the case when $(X, \to, d)$ is a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a functional. The lemma is proved in the work of S. Kasahara [66].*

Theorem 3.1.14 (A.-D. Filip, [33])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a functional. Let $T : X \to P(X)$ be a multivalued operator. We assume that:*

i) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq Ad(x, y);$$

ii) T has closed graph with respect to $\to$.

If A converges to zero, then $F_T \neq \emptyset$. If, in addition, $(I_m - A)$ is non-singular, $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ and

$$\max\{d(u, v) \mid u \in Fx,\ v \in Ty\} \leq Ad(x, y),\ \textit{for all } x, y \in X$$

then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$ then $x_0 \in F_T$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1).$$

Since $x_2 \in Tx_1$, if $x_2 = x_1$ then $x_1 \in F_T$. If we consider $x_2 \neq x_1$ then there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq Ad(x_1, x_2) \leq A^2 d(x_0, x_1).$$

By induction, we construct the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$. This sequence has the following properties:

1°) $x_{n+1} \in Tx_n$, $\forall n \in \mathbb{N}$;

2°) $d(x_n, x_{n+1}) \leq A^n d(x_0, x_1)$, $\forall n \in \mathbb{N}$.

Next, we have the following estimation:

$$\sum_{n\in\mathbb{N}} d(x_n . x_{n+1}) \leq \sum_{n\in\mathbb{N}} A^n d(x_0, x_1) = (I - A)^{-1} d(x_0, x_1) < \infty.$$

Since $(X, \to, d)$ is a generalized Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X, so there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, T has closed graph, so $x^* \in F_T$.

We prove now the uniqueness of the fixed point x^*.

Let $x^*, y^* \in F_T$ such that $x^* \neq y^*$. Since $x^* \in Tx^*$ and $y^* \in Ty^*$, we get that

$$d(x^*, y^*) \leq \max_{\substack{u\in Tx^*\\ v\in Ty^*}} d(u, v) \leq Ad(x^*, y^*) \Leftrightarrow (I_m - A)d(x^*, y^*) \leq 0_m.$$

Since $I_m - A$ is a non-singular matrix and $(I_m - A)^{-1}$ has non-negative elements, it follows that $d(x^*, y^*) = 0_m$. By the same way of proof, we get that $d(y^*, x^*) = 0_m$.

Next, by Lemma 2.1.1 and Remark 3.1.11 we get $x^* = y^*$. □

Corollary 3.1.3 (A.-D. Filip, [33]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a functional satisfying $d(x, x) = 0_m$, for all $x \in X$. Let $T : X \to P(X)$ be a multivalued operator. We assume that:*

i) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $B \in \mathcal{M}_{m,m}(\mathbb{R})$ and for all $x, y \in X$ and $u \in Tx$, there exists $y \in Tv$ such that

$$d(u, v) \leq Ad(x, y) + Bd(y, u);$$

ii) T has closed graph with respect to $\to$.

If A converges to zero, then T has at least one fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$ then $x_0 \in F_T$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_1, x_1) = Ad(x_0, x_1).$$

By following the proof of Theorem 3.1.14, the conclusion follows. □

A fixed point result for multivalued Kannan operators is presented bellow.

Theorem 3.1.15 (A.-D. Filip, [33]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a functional. Let $T : X \to P(X)$ be a multivalued operator. We assume that:*

i) there exists $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq A[d(x, u) + d(y, v)];$$

ii) T has closed graph with respect to $\to$.

Then T has at least one fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$, then we already have a fixed point for T ($x_0 \in F_T$). Assuming that $x_1 \neq x_0$, then by i), there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq A[d(x_0, x_1) + d(x_1, x_2)] \Leftrightarrow d(x_1, x_2) \leq (I_m - A)^{-1} Ad(x_0, x_1).$$

We denote $\Lambda = (I_m - A)^{-1}A$ and we have

$$d(x_1, x_2) \leq \Lambda d(x_0, x_1).$$

By taking into account Lemma 3.1.4, item (2) and by following the proof of Theorem 3.1.14, replacing A with Λ, the conclusion follows. □

Next we present a result regarding the fixed points for the multivalued operators of Reich type.

Theorem 3.1.16 (A.-D. Filip, [33]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^m_+$ is a functional. Let $T : X \to P(X)$ be a multivalued operator. We assume that:*

i) there exist $A = (a_{ij})_{i,j=\overline{1,m}}$, $B = (b_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $C = (c_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$, with $A + B \leq C$, i.e., $a_{ij} + b_{ij} \leq c_{ij}$, for all $i, j = \overline{1,m}$ and for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq Ad(x, y) + Bd(x, u) + Cd(y, v);$$

ii) T has closed graph with respect to $\to$.

Then T has at least one fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$, then we already have a fixed point for T ($x_0 \in F_T$). Assuming that $x_1 \neq x_0$, then by i), there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} &d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_0, x_1) + Cd(x_1, x_2) \\ \Leftrightarrow\ &d(x_1, x_2) \leq (I_m - C)^{-1}(A + B)d(x_0, x_1) \leq (I_m - C)^{-1}Cd(x_0, x_1). \end{aligned}$$

We denote $\Lambda = (I_m - C)^{-1}C$. By taking into account Lemma 3.1.4, item (2) and by following the proof of Theorem 3.1.14, replacing A with Λ, the conclusion follows. □

- We give next some fixed point results for multivalued Zamfirescu operators in large Kasahara spaces in the sense of Definition 2.1.9.

Theorem 3.1.17 (A.-D. Filip, [37]). *Let $(X, \xrightarrow{d}, p)$ be a large Kasahara space in the sense of Definition 2.1.9, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X. Let $x_0 \in X$, $r > 0$ and $T : \tilde{B}_p(x_0, r) \to P(X)$ be a multivalued Zamfirescu operator w.r.t. p. We assume that*

(i) *T has closed graph with respect to* $\xrightarrow{d}$;

(ii) $p(x_0, z) < (1-\delta)r$, *where* $z \in Tx_0$ *and* $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(iii) $p(x, x) = 0$, *for all* $x \in X$.

Then the following statements hold:

(1) *T has at least one fixed point in* $\tilde{B}_p(x_0, r)$.

(2) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_p(x_0, r)$ *such that*

(2.a) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(2.b) $x_n \to x^* \in F_T$ *as* $n \to +\infty$;

(2.c) *the following estimation holds*

$$p(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{3.1.13}$$

where $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$.

Proof. By following the proof of Theorem 3.1.11 we get that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\tilde{B}_p(x_0, r)$ which has the properties

(1°) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

(2°) $p(x_n, x_{n+1}) \leq \delta^n p(x_0, x_1)$, for all $n \in \mathbb{N}$.

Further we get for any $m, n \in \mathbb{N}$ with $m > n$ that

$$\begin{aligned} p(x_n, x_m) &\leq \sum_{i=0}^{m-1} p(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} \delta^{n+i} p(x_0, x_1) \\ &\leq \delta^n (1 + \delta + \ldots + \delta^{m-1} + \ldots) p(x_0, x_1) = \frac{\delta^n}{1-\delta} p(x_0, x_1). \end{aligned}$$

Hence

$$\lim_{n,m\to+\infty} p(x_n, x_m) = 0,$$

so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{B}_p(x_0, r)$ with respect to p. Since $(\tilde{B}_p(x_0, r), \xrightarrow{d}, p)$ is a large Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $\tilde{B}_p(x_0, r)$, so there exists $x^* \in \tilde{B}_p(x_0, r)$ such that $x_n \to x^*$ as $n \to +\infty$.

Knowing that $Graph(T)$ is closed in $X \times X$ with respect to $\xrightarrow{d}$, we get that $x^* \in F_T$.

Let $p \in \mathbb{N}$, $p \geq 1$. Then, by (2°) we get

$$\begin{aligned} p(x_n, x_{n+p}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{n+p-1}, x_{n+p}) \\ &\leq \delta^n p(x_0, x_1) + \delta^{n+1} p(x_0, x_1) + \ldots + \delta^{n+p-1} p(x_0, x_1) \\ &\leq \delta^n \big(1 + \delta + \ldots + \delta^{p-1} + \ldots\big) p(x_0, x_1) = \frac{\delta^n}{1-\delta} p(x_0, x_1) \leq \delta^n r. \end{aligned}$$

We have next

$$p(x_n, x^*) \leq \liminf_{p\to+\infty} p(x_n, x_{n+p}) \leq \delta^n r.$$

so the estimation (3.1.13) holds. $\square$

The global version of Theorem 3.1.17 is the following

Corollary 3.1.4 (A.-D. Filip, [37]). *Let $(X, \xrightarrow{d}, p)$ be a large Kasahara space in the sense of Definition 2.1.9, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X. Let $T : X \to P(X)$ be a multivalued Zamfirescu operator w.r.t. p. We assume that T has closed graph with respect to $\xrightarrow{d}$ and $p(x, x) = 0$, for all $x \in X$. Then the following statements hold:*

(1) T has at least one fixed point in X;

(2) the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ converges to an element $x^* \in F_T$ as $n \to \infty$;

(3) the following estimation holds

$$p(x_n, x^*) \leq \frac{\delta^n}{1-\delta} p(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. Choose $r > 0$ and $x_0 \in X$ such that $p(x_0, z) < (1-\delta)r$, where $z \in Tx_0$. The conclusions follow from Theorem 3.1.17. $\square$

We give next a data dependence result for multivalued Zamfirescu operators.

Theorem 3.1.18 (A.-D. Filip, [37]). *Let $(X, \xrightarrow{d}, p)$ be a large Kasahara space in the sense of Definition 2.1.9, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X with $p(x, x) = 0$, for all $x \in X$. Let $T_1, T_2 : X \to P(X)$ be two multivalued Zamfirescu operators w.r.t. p, having closed graph w.r.t $\xrightarrow{d}$. Then*

(i) T_1 and T_2 have at least one fixed point in X;

(ii) If we assume that there exists $\eta > 0$ such that for all $x \in X$ and $u \in T_1x$, there exists $v \in T_2x$ such that $p(u, v) \leq \eta$, then for all $u^* \in F_{T_1}$, there exists $v^* \in F_{T_2}$ such that

$$p(u^*, v^*) \leq \frac{\eta}{1-\delta_2}, \text{ where } \delta_2 = \max\left\{\alpha_2, \frac{\beta_2}{1-\beta_2}, \frac{\gamma_2}{1-\gamma_2}\right\} \tag{3.1.14}$$

respectively, if we assume that there exists $\eta > 0$ such that for all $x \in X$ and $v \in T_2x$, there exists $u \in T_1x$ such that $p(v, u) \leq \eta$, then for all $v^* \in F_{T_2}$, there exists $u^* \in F_{T_1}$ such that

$$p(v^*, u^*) \leq \frac{\eta}{1-\delta_1}, \text{ where } \delta_1 = \max\left\{\alpha_1, \frac{\beta_1}{1-\beta_1}, \frac{\gamma_1}{1-\gamma_1}\right\}. \tag{3.1.15}$$

Proof. (i) follows from Corollary 3.1.4.

(ii) Let $u_0 \in F_{T_1}$. Then for $u_0 \in T_1u_0$, there exists $u_1 \in T_2u_0$ such that $p(u_0, u_1) \leq \eta$.

For every $u_0, u_1 \in X$, with $u_1 \in T_2u_0$, since T_2 is a multivalued Zamfirescu operator, there exists $u_2 \in T_2u_1$ such that at least one of the following conditions holds

- ◇ $p(u_1, u_2) \leq \alpha_2 p(u_0, u_1)$ or
- ◇ $p(u_1, u_2) \leq \frac{\beta_2}{1-\beta_2} p(u_0, u_1)$ or
- ◇ $p(u_1, u_2) \leq \frac{\gamma_2}{1-\gamma_2} p(u_0, u_1)$

Since $\delta_2 = \max\left\{\alpha_2, \frac{\beta_2}{1-\beta_2}, \frac{\gamma_2}{1-\gamma_2}\right\}$, we have $p(u_1, u_2) \leq \delta_2 p(u_0, u_1)$.

For $u_1 \in X$ and $u_2 \in T_2u_1$, there exists $u_3 \in T_2u_2$ such that

$$p(u_2, u_3) \leq \delta_2 p(u_1, u_2) \leq \delta_2^2 p(u_0, u_1).$$

By induction, we obtain a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that for all $n \in \mathbb{N}$ we have

- ◇ $u_{n+1} \in T_2u_n$;
- ◇ $p(u_n, u_{n+1}) \leq \delta_2^n p(u_0, u_1)$.

For $n, p \in \mathbb{N}$, we have

$$p(u_n, u_{n+p}) \leq \sum_{i=n}^{n+p-1} p(u_i, u_{i+1}) \leq \sum_{i=n}^{n+p-1} \delta_2^i p(u_0, u_1) \leq \frac{\delta_2^n}{1-\delta_2} p(u_0, u_1)$$

which implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy with respect to p. Since $(X, \xrightarrow{d}, p)$ is a large Kasahara space, we get that $(u_n)_{n\in\mathbb{N}}$ is convergent in $(X, \xrightarrow{d})$, i.e., there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$, as $n \to +\infty$.

Since p is lower semicontinuous, for all $n \in \mathbb{N}$ we have

$$p(u_n, v^*) \leq \liminf_{p\to+\infty} p(u_n, u_{n+p}) \leq \frac{\delta_2^n}{1-\delta_2} p(u_0, u_1). \tag{3.1.16}$$

For $u_{n-1}, v^* \in X$ and $u_n \in T_2u_{n-1}$, there exists $z_n \in T_2v^*$ such that

$$p(u_n, z_n) \leq \delta_2 p(u_{n-1}, v^*) \leq \frac{\delta_2^n}{1-\delta_2} p(u_0, u_1). \tag{3.1.17}$$

By (3.1.16), (3.1.17) and Lemma 1.4.1, item (ii), we have $z_n \xrightarrow{d} v^*$, as $n \to +\infty$.

Since $z_n \in T_2v^*$, $z_n \xrightarrow{d} v^*$, as $n \to +\infty$ and $Graph(T_2)$ is closed in $X \times X$ w.r.t. d, we get that $v^* \in T_2v^*$.

On the other hand, for $n = 0$ in (3.1.16), we obtain

$$p(u_0, v^*) \leq \frac{1}{1-\delta_2} p(u_0, u_1) \leq \frac{\eta}{1-\delta_2}.$$

Hence (3.1.14) holds. By a similar way of proof, we get (3.1.15). □

3.2 Maia type fixed point theorems

The aim of this section is to present several Maia type theorems for multivalued generalized contractions in close connexion with the results given in Kasahara spaces.

First, we recall the multivalued version of Maia's fixed point theorem 2.2.1.

Theorem 3.2.1 (A. Petruşel and I.A. Rus, [102]). *Let X be a nonempty set, d and ρ be two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X, ρ) is a complete metric space;*

(ii) *there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$, for each $x, y \in X$;*

(iii) *$T : (X, \rho) \to (P(X), H_\rho)$ has closed graph (here H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]));*

(iv) *there exists $\alpha \in [0, 1[$ such that $H_d(Tx, Ty) \leq \alpha d(x, y)$, for each $x, y \in X$.*

Then we have:

(a) *$F_T \neq \emptyset$;*

(b) *for each $x \in X$ and each $y \in Tx$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:*

(1) *$x_0 = x$, $x_1 = y$;*

(2) *$x_{n+1} \in Tx_n$, for each $n \in \mathbb{N}$;*

(3) *$x_n \xrightarrow{\rho} x^* \in Tx^*$, as $n \to \infty$.*

We mention here another two local fixed point results of Maia type.

Theorem 3.2.2 (A.-D. Filip, [31]). *Let X be a nonempty set, ρ and d be two metrics on X, $x_0 \in X$, $r > 0$ and $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued operator. We suppose that*

(i) *(X, ρ) is a complete metric space;*

(ii) *there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$, for each $x, y \in \tilde{B}_d(x_0, r)$;*

(iii) *$T : (\tilde{B}_d(x_0, r), \rho) \to (P(X), H_\rho)$ has closed graph (here H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]));*

(iv) *there exists $L \geq 0$ such that for all $x \in \tilde{B}_d(x_0, r)$, there exists $y \in I^x_{b,d}$ such that*

$$H_d(Tx, Ty) \leq \Lambda(d(x, y)) \cdot d(x, y) + L \cdot D_d(y, Tx)$$

where

◇ $I^x_{b,d} := \{y \in Tx \mid b \cdot d(x, y) \leq D_d(x, Tx)\}$, *where* $b \in]0, 1[$ *and* $D_d(x, Tx) = \inf_{z\in Tx} d(x, z)$.

- ◇ $\Lambda : \mathbb{R}_+ \to [0,1[$ *is a function defined by* $\Lambda(t) = b \cdot \alpha(t)$, *for all* $t \in \mathbb{R}_+$, *where* $b \in]0,1[$ *is the same number used in the definition of the set* $I^x_{b,d}$ *and* $\alpha : \mathbb{R}_+ \to [0,1[$ *is a function with the property* $\limsup_{s\to t^+} \alpha(s) < 1$, *for all* $t \in \mathbb{R}_+$.

(v) $D_d(x_0, Tx_0) < b(1-\theta)r$, *where* $\theta \in [0,1[$ *satisfies* $\Lambda(t) < b\theta$, *for all* $t \in \mathbb{R}_+$.

Then we have:

(a) $F_T \neq \emptyset$;

(b) *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\tilde{B}_d(x_0, r)$ *such that:*

(b1) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(b2) $x_n \xrightarrow{\rho} x^* \in F_T$, *as* $n \to \infty$;

(b3) $\rho(x_n, x^*) \leq c \cdot \theta^n \cdot r$, *for each* $n \in \mathbb{N}$.

Proof. By (iv), since $x_0 \in \tilde{B}_d(x_0, r)$, there exists $x_1 \in I^{x_0}_{b,d}$ such that

$$H_d(Tx_0, Tx_1) \leq \Lambda(d(x_0, x_1)) \cdot d(x_0, x_1) + L \cdot D_d(x_1, Tx_0).$$

By $x_1 \in I^{x_0}_{b,d}$ we have that $x_1 \in Tx_0 \ \Rightarrow D_d(x_1, Tx_0) = 0$ and

$$b \cdot d(x_0, x_1) \leq D_d(x_0, Tx_0) < b(1-\theta)r \Rightarrow d(x_0, x_1) < (1-\theta)r \Rightarrow x_1 \in \tilde{B}_d(x_0, r).$$

We have also that

$$H_d(Tx_0, Tx_1) \leq b\theta \cdot d(x_0, x_1) < b\theta(1-\theta)r.$$

Since $x_1 \in \tilde{B}_d(x_0, r)$, there exists $x_2 \in I^{x_1}_{b,d}$ such that

$$H_d(Tx_1, Tx_2) \leq \Lambda(d(x_1, x_2)) \cdot d(x_1, x_2) + L \cdot D_d(x_2, Tx_1).$$

By $x_2 \in I^{x_1}_{b,d}$ we have that $x_2 \in Tx_1 \ \Rightarrow D_d(x_2, Tx_1) = 0$ and

$$b \cdot d(x_1, x_2) \leq D_d(x_1, Tx_1) \leq H_d(Tx_0, Tx_1) < b\theta(1-\theta)r \Rightarrow d(x_1, x_2) < \theta(1-\theta)r.$$

We estimate

$$\begin{array}{rcl} d(x_0, x_2) & \leq & d(x_0, x_1) + d(x_1, x_2) \\ & < & (1-\theta)r + \theta(1-\theta)r = (1-\theta^2)r \Rightarrow x_2 \in \tilde{B}_d(x_0, r). \end{array}$$

In addition, we get

$$H_d(Tx_1, Tx_2) \leq b\theta \cdot d(x_1, x_2) < b\theta^2(1-\theta)r.$$

Proceeding inductively, we construct a sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$ having the following properties

$$x_{n+1} \in Tx_n, \ \text{for all } n \in \mathbb{N}, \tag{3.2.1}$$

$$d(x_n, x_{n+1}) < \theta^n \cdot (1-\theta) \cdot r. \tag{3.2.2}$$

We want to prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Let $p \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &< \theta^n \cdot (1-\theta) \cdot r \cdot (1 + \theta + ... + \theta^{p-1}) = \theta^n \cdot r \cdot (1-\theta^p). \end{aligned} \tag{3.2.3}$$

Letting $n \to \infty$, since $\theta \in [0, 1[$, we have that $d(x_n, x_{n+p}) \to 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric d. By (ii) we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric ρ. Since (X, ρ) is a complete metric space, there exists $x^* \in X$ such that $(x_n) \xrightarrow{\rho} x^*$ as $n \to \infty$. It remains to show that $x^* \in F_T$. Since $Graph(T)$ is closed with respect to ρ, we get that $x^* \in F_T$.

By (ii) and (3.2.3), we have that there exists $c > 0$ such that $\rho(x_n, x_{n+p}) \leq c \cdot d(x_n, x_{n+p}) < c \cdot \theta^n \cdot r \cdot (1-\theta^p)$. Letting $p \to \infty$ we obtain that $\rho(x_n, x^*) \leq c \cdot \theta^n \cdot r$, for each $n \in \mathbb{N}$. □

Remark 3.2.1. *In Theorem 3.2.2, by taking $n = 0$ in the conclusion $(b3)$, it follows that $x^* \in \tilde{B}_\rho(x_0, cr)$.*

We consider now the case of generalized metric spaces (X, d), where $d : X \times X \to \mathbb{R}^m_+$. The following Maia type theorem holds.

Theorem 3.2.3 (A.-D. Filip and A. Petruşel [39]). *Let X be a nonempty set and $d, \rho : X \times X \to \mathbb{R}^m_+$ be two generalized metrics on X. Let $x_0 \in X$, $r := (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m_+$ and let $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued operator. Suppose that*

(i) (X, ρ) is a complete generalized metric space;

(ii) there exists $C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that $\rho(x, y) \leq C \cdot d(x, y)$, for all $x, y \in X$;

(iii) $T : (\tilde{B}_d(x_0, r), \rho) \to (P(X), H_\rho)$ has closed graph (here H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]));

(iv) there exist $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that A is a matrix that converges to zero and for all $x, y \in \tilde{B}_d(x_0, r)$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \leq Ad(x, y) + Bd(y, u);$$

(v) if $u \in \mathbb{R}^m_+$ is such that $u(I_m - A)^{-1} \leq (I_m - A)^{-1}r$, then $u \leq r$;

(vi) $d(x_0, x_1)(I_m - A)^{-1} \leq r$.

Then $F_T \neq \emptyset$.

Proof. Let $x_0 \in X$ such that $x_1 \in Tx_0$. By (v) and (vi) we have

$$d(x_0, x_1)(I_m - A)^{-1} \leq r \leq (I_m - A)^{-1}r$$

which implies $x_1 \in \tilde{B}_d(x_0, r)$.
Since $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2)(I_m - A)^{-1} \leq Ad(x_0, x_1)(I_m - A)^{-1} + Bd(x_1, x_1)(I_m - A)^{-1} \leq Ar.$$

Hence,

$$\begin{aligned} d(x_0,x_2)(I_m-A)^{-1} &\leq d(x_0,x_1)(I_m-A)^{-1}+d(x_1,x_2)(I_m-A)^{-1} \\ &\leq I_m r + Ar \leq (I_m+A+\ldots+A^n+\ldots)r \leq (I_m-A)^{-1}r \end{aligned}$$

which implies that $d(x_0,x_2)\leq r$ i.e. $x_2\in \tilde{B}_d(x_0,r)$.
For $x_2\in Tx_1$, there exists $x_3\in Tx_2$ such that

$$d(x_2,x_3)(I_m-A)^{-1}\leq Ad(x_1,x_2)(I_m-A)^{-1}+Bd(x_2,x_2)(I_m-A)^{-1}\leq A^2r.$$

Then the following estimation holds

$$\begin{aligned} d(x_0,x_3)(I_m-A)^{-1} &\leq d(x_0,x_1)(I_m-A)^{-1}+d(x_1,x_2)(I_m-A)^{-1}+d(x_2,x_3)(I_m-A)^{-1} \\ &\leq I_m r+Ar+A^2r\leq (I_m-A)^{-1}r \end{aligned}$$

and thus $d(x_0,x_3)\leq r$, i.e., $x_3\in\tilde{B}_d(x_0,r)$.
Inductively, we can construct the sequence $(x_n)_{n\in\mathbb{N}}$ which has its elements in the closed ball $\tilde{B}_d(x_0,r)$ and satisfies the conditions:

(1) $x_{n+1}\in Tx_n$, for all $n\in\mathbb{N}$;

(2) $d(x_n,x_{n+1})(I_m-A)^{-1}\leq A^n r$, for all $n\in\mathbb{N}$.

By (2), for all $n\in\mathbb{N}$ we have

$$d(x_n,x_{n+1})(I_m-A)^{-1}\leq I_m\cdot A^n r\leq (I_m+A+A^2+\ldots)A^n r\leq (I_m-A)^{-1}A^n r.$$

By (v) we obtain

$$d(x_n,x_{n+1})\leq A^n r,\ \text{for all } n\in\mathbb{N}.$$

We show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X with respect to d. In order to do that, let $p\in\mathbb{N}$, $p>0$. The following estimations hold

$$\begin{aligned} d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{n+p-1},x_{n+p}) \\ &\leq A^n r+A^{n+1}r+\ldots+A^{n+p-1}r \\ &\leq A^n(I_m+A+\ldots+A^{p-1}+\ldots)r=A^n(I_m-A)^{-1}r. \end{aligned}$$

Since the matrix A converges towards zero, one has $A^n\to\Theta_m$ as $n\to\infty$. By letting $n\to\infty$ we get that $d(x_n,x_{n+p})\to 0$, which implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d.
By (ii), $\rho(x_n,x_{n+p})\leq C\cdot d(x_n,x_{n+p})\to 0$ as $n\to\infty$. Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ.

Since (X,ρ) is a complete metric space, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X. Thus there exists $x^*\in X$ such that $x_n\xrightarrow{\rho}x^*$ as $n\to\infty$. By (iii), we get $x^*\in Tx^*$. $\square$

Remark 3.2.2. *Notice that in Theorem 3.2.3, the fixed point* $x^* \in \tilde{B}_\rho(x_0, Cr)$.

Indeed, we have proved that the sequence of successive approximations for T *starting from* $x_0 \in X$ *is* $(x_n)_{n\in\mathbb{N}}$ *with* $x_n \in \tilde{B}_d(x_0, r)$, *for all* $n \in \mathbb{N}$ *and there exists* $x^* \in X$ *such that* $x_n \xrightarrow{\rho} x^*$ *as* $n \to \infty$.

By (ii), *there exists* $C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that*

$$\rho(x_0, x_n) \leq C \cdot d(x_0, x_n) \leq Cr, \text{ for all } n \in \mathbb{N}. \tag{3.2.4}$$

Hence $x_n \in \tilde{B}_\rho(x_0, Cr)$, *for all* $n \in \mathbb{N}$.

By letting $n \to \infty$ *in* (3.2.4), *we get that* $x^* \in \tilde{B}_\rho(x_0, Cr)$.

Remark 3.2.3. *Some other Maia type fixed point results can be obtained in the case when* d *is not necessarily a metric.*

Let X *be a nonempty set and* $\rho : X \times X \to \mathbb{R}_+$ *be a complete metric on* X. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *and let* $x \in X$. *We consider the convergence structure* $\xrightarrow{\rho}$ *induced by* ρ *on* X *and defined by*

$$x_n \xrightarrow{\rho} x \;\Leftrightarrow\; \rho(x_n, x) \to 0, \text{ as } n \to \infty.$$

We have the following Maia type result:

Corollary 3.2.1 (A.-D. Filip [32])**.** *Let* X *be a nonempty set and* $\rho : X \times X \to \mathbb{R}_+$ *be a complete metric on* X. *Let* $d : X \times X \to \mathbb{R}_+$ *be a functional with the property that for all* $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$. *Let* $T : X \to P_d(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $\alpha \in [0, 1[$ *such that* $H_d(Tx, Ty) \leq \alpha \cdot d(x, y)$, *for all* $x, y \in X$;

ii) $Graph(T)$ *is closed in* $(X, \xrightarrow{\rho})$;

iii) *there exists* $c > 0$ *such that* $\rho(x, y) \leq c \cdot d(x, y)$.

Then the following statements hold:

1) $F_T \neq \emptyset$;

2) *there exists* $\theta \in [0, 1[$ *such that*

$$\rho(x_n, x^*) \leq c\frac{\theta^n}{1-\theta} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$.

Proof. By $i)$ and by following the proof of Theorem 3.1.2, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ such that:

j) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

jj) $d(x_n, x_{n+1}) \leq \theta^n \cdot d(x_0, x_1)$, for all $n \in \mathbb{N}$

By $iii)$, there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \le c \cdot d(x_n, x_{n+1}) \le c \cdot \theta^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric, we have that

$$\begin{aligned}\rho(x_n, x_{n+p}) &\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\le c \cdot \theta^n \cdot d(x_0, x_1) + c \cdot \theta^{n+1} \cdot d(x_0, x_1) + \ldots + c \cdot \theta^{n+p-1} \cdot d(x_0, x_1) \\ &= c \cdot \theta^n (1 + \theta + \ldots + \theta^{p-1}) \cdot d(x_0, x_1).\end{aligned}$$

So, the following estimation hold

$$\rho(x_n, x_{n+p}) \le c \cdot \theta^n \cdot \frac{1 - \theta^p}{1 - \theta} \cdot d(x_0, x_1),\ \forall n \in \mathbb{N},\ \forall p \in \mathbb{N},\ p > 0. \tag{3.2.5}$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0$, so $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, ρ). Therefore $(x_n)_{n \in \mathbb{N}}$ is convergent in (X, ρ), so there exists $x^* \in X$ such that $x_n \overset{\rho}{\to} x^*$.

By $ii)$, it follows that $x^* \in F_T$.

By letting $p \to \infty$ in (3.2.5), we get the estimation mentioned in the conclusion 2) of the corollary. □

Corollary 3.2.2 (A.-D. Filip [32])**.** *Let X be a nonempty set, $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X and $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$.*

Let $T : X \to P_d(X)$ be a multivalued operator such that:

i) $Graph(T)$ is closed in $(X, \overset{\rho}{\to})$;

ii) T is a multivalued φ-contraction.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that:

j) $\psi(0) = 0$;

jj) $\psi(t) > t$, for all $t > 0$;

jjj) $\psi \circ \varphi$ is a comparison function;

jv) ψ is increasing;

v) $\sum_{n \in \mathbb{N}} (\psi \circ \varphi)^n(t) < \infty$, for all $t > 0$.

We assume that there exists $c > 0$ such that

$$\rho(x, y) \le c \cdot d(x, y), \text{ for all } x, y \in X.$$

Then $F_T \neq \emptyset$.

Proof. By $ii)$ and the proof of the Theorem 3.1.6, we get that there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

1) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

2) $d(x_n, x_{n+1}) \leq (\psi \circ \varphi)^n(d(x_0, x_1))$, for all $n \in \mathbb{N}$.

Hence, we have

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot (\psi \circ \varphi)^n(d(x_0, x_1)), \text{ for all } n \in \mathbb{N}.$$

Let $p \in \mathbb{N}$, $p > 0$. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by $\xi(t) = (\psi \circ \varphi)(t)$, for all $t \in \mathbb{R}_+$. Since ξ is a comparison function, each iterate ξ^k, $k \in \mathbb{N}$, $k \neq 0$ is a comparison function. Since ρ is a metric, we have

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\leq c \cdot \xi^n(d(x_0, x_1)) + c \cdot \xi^{n+1}(d(x_0, x_1)) + \ldots + c \cdot \xi^{n+p-1}(d(x_0, x_1))\end{aligned}$$

We consider now

$$\xi^{N(n)}(d(x_0, x_1)) := \max\{\xi^n(d(x_0, x_1)), \xi^{n+1}(d(x_0, x_1)), \ldots, \xi^{n+p-1}(d(x_0, x_1))\}.$$

Hence $\xi^{N(n)}$ is also a comparison function, where

$$N(n) \in \{n, n+1, \ldots, n+p-1\}.$$

So we get

$$\rho(x_n, x_{n+p}) \leq c \cdot p \cdot \xi^{N(n)}(d(x_0, x_1)) \to 0, \text{ as } n \to \infty.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, ρ). Hence there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

By $i)$ we have $x^* \in F_T$. □

Corollary 3.2.3 (A.-D. Filip [32]). *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional. Let $\varphi : X \to \mathbb{R}_+$ be a functional.*

Let $T : X \to P(X)$ be a multivalued operator such that

i) $Graph(T)$ is closed;

ii) for all $x \in X$, there exists $y \in Tx$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$;

iii) there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$.

Then T has at least one fixed point in X.

Proof. By $ii)$ and the proof of the Theorem 3.1.7, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

1) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

2) $d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1})$, for all $n \in \mathbb{N}$.

By *iii*) there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot (\varphi(x_n) - \varphi(x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

We will prove that the series $\sum\limits_{n\in\mathbb{N}} \rho(x_n, x_{n+1})$ is convergent. For this purpose, we need to show that the sequence of its partial sums is convergent in $\mathbb{R}_+$.

Denote by $s_n = \sum\limits_{k=0}^{n} \rho(x_k, x_{k+1})$. Then $s_{n+1} - s_n = \rho(x_{n+1}, x_{n+2}) \geq 0$, for each $n \in \mathbb{N}$. Moreover $s_n \leq \sum\limits_{k=0}^{n} \left[c\varphi(x_k) - c\varphi(x_{k+1})\right] \leq c\varphi(x_0)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}_+$. So the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}_+$.

It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and, from the completeness of the metric space (X, ρ), convergent to a certain element $x^* \in X$.

The conclusion follows from i). □

Corollary 3.2.4 (A.-D. Filip [32]). *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$. Let $T : X \to \tilde{P}_d(X)$ be a multivalued operator. We assume that:*

i) T is a multivalued (θ, L)-weak contraction;

ii) Graph(T) is closed in $(X, \xrightarrow{\rho})$;

iii) there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$.

Then the following statements hold:

1) T has at least one fixed point in X;

2) there exists $\lambda \in [0, 1[$ such that

$$\rho(x_n, x^*) \leq c\frac{\lambda^n}{1-\lambda} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. By i) and the proof of the Theorem 3.1.8, there exists the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$ which starts from $(x_0, x_1) \in Graph(T)$ with $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$ and $\lambda < 1$ such that

$$d(x_n, x_{n+1}) \leq \lambda^n \cdot d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We follow the proof of the Corollary 3.2.1 by taking $\theta = \lambda$ and the conclusions follow. □

Corollary 3.2.5 (A.-D. Filip [32])**.** *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that:*

i) there exists $\alpha \in [0, \frac{1}{2}[$ such that $H_d(Tx, Ty) \leq \alpha[D(x, Tx) + D(y, Ty)]$, for all $x, y \in X$;

ii) Graph(T) is closed in $(X, \stackrel{\rho}{\to})$;

iii) there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$.

Then the following statements hold:

1) T has at least one fixed point;

2) there exists $\theta \in [0, \frac{1}{2}[$ such that

$$\rho(x_n, x^*) \leq c\frac{\theta}{1-2\theta}\left(\frac{\theta}{1-\theta}\right)^{n-1} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. By $i)$ and following the proof of Theorem 3.1.9, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ such that:

j) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

jj) $d(x_n, x_{n+1}) \leq \left(\frac{\theta}{1-\theta}\right)^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

By $iii)$, there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c\left(\frac{\theta}{1-\theta}\right)^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Now, let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric, we have that

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\leq c\left(\frac{\theta}{1-\theta}\right)^n \left[1 + \frac{\theta}{1-\theta} + \ldots + \left(\frac{\theta}{1-\theta}\right)^{p-1}\right] d(x_0, x_1).\end{aligned}$$

So, for all $n, p \in \mathbb{N}$, $p > 0$, the following estimation hold

$$\rho(x_n, x_{n+p}) \leq c\left(\frac{\theta}{1-\theta}\right)^n \frac{1-\theta}{1-2\theta}\left[1 - \left(\frac{\theta}{1-\theta}\right)^p\right] d(x_0, x_1). \tag{3.2.6}$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0$, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, ρ). Therefore $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ), so there exists $x^* \in X$ such that $x_n \stackrel{\rho}{\to} x^*$.

By $ii)$, it follows that $x^* \in F_T$.

By letting $p \to \infty$ in (3.2.6), we get the estimation mentioned in the conclusion 2) of the corollary. □

Corollary 3.2.6 (A.-D. Filip [32]). *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}_+$ be a complete metric on X. Let $d : X \times X \to \mathbb{R}_+$ be a functional with the property that for all $x, y \in X$, $d(x,y) = 0 \Rightarrow x = y$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that:*

i) there exists $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$ such that

$$H_d(Tx, Ty) \leq \alpha d(x,y) + \beta D(x, Tx) + \gamma D(y, Ty), \text{ for all } x, y \in X.$$

ii) $Graph(T)$ is closed in $(X, \xrightarrow{\rho})$;

iii) there exists $c > 0$ such that $\rho(x,y) \leq c \cdot d(x,y)$.

Then the following statements hold:

1) T has at least one fixed point;

2) there exists $\theta \in [0,1[$ such that

$$\rho(x_n, x^*) \leq c \frac{\theta^n}{1-\theta} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $x^* \in F_T$ and $(x_n)_{n \in \mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

Proof. We follow the proof of the Theorem 3.1.10 and the Corollary 3.2.1, where $\theta := \frac{q(\alpha+\beta)}{1-q\gamma}$, $q > 1$. □

Corollary 3.2.7 (A.-D. Filip, [33]). *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on X. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional and $T : X \to P(X)$ be a multivalued operator. We assume that*

i) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and for all $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$d(u,v) \leq A d(x,y);$$

ii) $Graph(T)$ is closed in $X \times X$.

iii) there exists $c > 0$ such that $\rho(x,y) \leq c \cdot d(x,y)$.

Then the following statements hold:

1) if A converges to zero, then $F_T \neq \emptyset$. If, in addition, $(I_m - A)$ is non-singular, $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ and

$$\max\{d(u,v) \mid u \in Tx,\ v \in Ty\} \leq A d(x,y), \text{ for all } x, y \in X$$

then T has a unique fixed point in X.

2) $\rho(x_n, x^*) \le c \cdot A^n (I_m - A)^{-1} d(x_0, x_1)$, *for all* $n \in \mathbb{N}$, *where* $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$.

Proof. By $i)$ and by following the proof of theorem 3.1.14, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) \le A^n d(x_0, x_1)$, $\forall n \in \mathbb{N}$.

By $iii)$ there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \le c \cdot d(x_n, x_{n+1}) \le c \cdot A^n d(x_0, x_1), \ \forall n \in \mathbb{N}.$$

Now let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric, we have that

$$\begin{aligned}\rho(x_n, x_{n+p}) &\le \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\le c \cdot A^n d(x_0, x_1) + c \cdot A^{n+1} d(x_0, x_1) + \ldots + c \cdot A^{n+p-1} d(x_0, x_1).\end{aligned}$$

So the following estimation hold

$$\rho(x_n, x_{n+p}) \le c \cdot A^n (I_m + A + \ldots + A^{p-1}) d(x_0, x_1), \ \forall n, p \in \mathbb{N}, \ p > 0. \tag{3.2.7}$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0_m$, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete generalized metric space (X, ρ). Therefore $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ), so there exists $x^* \in X$ such that $x_n \stackrel{\rho}{\to} x$.

By $ii)$ it follows that $x^* \in F_T$. The uniqueness of the fixed point x^* follows from Theorem 3.1.14.

By letting $p \to \infty$ in (3.2.7), we get the estimation mentioned in the conclusion 2) of the corollary. □

3.3 Fixed point theorems in Kasahara spaces with respect to an operator

We introduce in this section a new notion: Kasahara spaces with respect to a multivalued operator. Two fixed point results for multivalued α-contractions defined on Kasahara spaces with respect to a multivalued operator are presented.

Definition 3.3.1. *Let* $(X, \to)$ *be an L-space,* $d : X \times X \to \mathbb{R}_+$ *be a functional and* $T : X \to P(X)$ *be a multivalued operator. The triple* $(X, \to, d)$ *is called Kasahara space with respect to the operator* T *if and only if for any sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *satisfying:*

(i) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(ii) $\sum_{n\in\mathbb{N}} H_d(Tx_n, Tx_{n+1}) < \infty$

we have that $(x_n)_{n\in\mathbb{N}}$ *is convergent in* $(X, \to)$.

Example 3.3.1. *Let X be a nonempty set, $T : X \to P_d(X)$ be a multivalued operator and $d, \rho : X \times X \to \mathbb{R}_+$ be two functionals. We suppose that:*

(i) (X, ρ) is a complete metric space;

(ii) for all $x \in X$ and $y \in Tx$, there exist $z \in Ty$ and $c > 0$ such that $H_\rho(Tx, Ty) \leq c \cdot d(y, z)$;

(iii) $d(x, x) = 0$, for all $x \in X$;

(iv) $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$.

Then $(X, \to, d)$ is a Kasahara space with respect to the operator T.

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$ and

$$\sum_{n\in\mathbb{N}} H_d(Tx_n, Tx_{n+1}) < \infty.$$

Then, for $n, p \in \mathbb{N}$ with $p > 1$, there exists $q > 1$ such that

$$\rho(x_{n+1}, x_{n+p+1}) \leq \sum_{k=n}^{n+p-1} \rho(x_{k+1}, x_{k+2}) \leq q \sum_{k=n}^{n+p-1} H_\rho(Tx_k, Tx_{k+1}). \tag{3.3.1}$$

By (ii), we get for all $k \in \mathbb{N}$ that

$$H_\rho(Tx_k, Tx_{k+1}) \leq c \cdot d(x_{k+1}, x_{k+2}). \tag{3.3.2}$$

By (iii) and (iv) together with Lemma 3.1.2, for all $k \in \mathbb{N}$, there exists $\xi > 1$ such that

$$d(x_{k+1}, x_{k+2}) \leq \xi H_d(Tx_k, Tx_{k+1}). \tag{3.3.3}$$

By (3.3.1), (3.3.2) and (3.3.3), it follows that

$$\rho(x_{n+1}, x_{n+p+1}) \leq q \cdot c \cdot \xi \sum_{k=n}^{n+p-1} H_d(Tx_k, Tx_{k+1}) < \infty.$$

Hence, $\rho(x_{n+1}, x_{n+p+1}) \to 0$ as $n \to \infty$, i.e., $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ. By (i), we get further that $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ).

Theorem 3.3.1. *Let $(X, \to, d)$ be a Kasahara space with respect to a multivalued operator $T : X \to P_d(X)$, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. We assume that*

(i) $Graph(T)$ is closed with respect to $\to$;

(ii) T is a multivalued α-contraction with respect to d.

Then we have:

(1) *$F_T \neq \emptyset$;*

(2) *for each $x \in X$ and each $y \in Tx$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that*

(2_a) *$x_0 = x$, $x_1 = y$;*

(2_b) *$x_{n+1} \in Tx_n$, for each $n \in \mathbb{N}$;*

(2_c) *$x_n \to x^* \in F_T$ as $n \to \infty$.*

Proof. Let $x_0 = x \in X$ and $x_1 = y \in Tx_0$.

We assume that $x_1 \neq x_0$, otherwise $x_0 \in F_T$. Let $1 < q < \frac{1}{\alpha}$. Then there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq qH_d(Tx_0, Tx_1) \leq q\alpha d(x_0, x_1) \Rightarrow H_d(Tx_0, Tx_1) \leq \alpha d(x_0, x_1) < \infty.$$

Since $x_2 \in Tx_1$, by assuming that $x_2 \neq x_1$, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq qH_d(Tx_1, Tx_2) \leq q\alpha d(x_1, x_2) \leq q^2\alpha H_d(Tx_0, Tx_1)$$

which implies further that

$$H_d(Tx_1, Tx_2) \leq (q\alpha)H_d(Tx_0, Tx_1).$$

By an inductive procedure we obtain the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 = x \in X$, $x_1 = y \in Tx_0$ and $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$, also known as the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$, such that

$$H_d(Tx_n, Tx_{n+1}) \leq (q\alpha)^n H_d(Tx_0, Tx_1), \text{ for each } n \in \mathbb{N}.$$

We have next

$$\sum_{n\in\mathbb{N}} H_d(Tx_n, Tx_{n+1}) \leq \sum_{n\in\mathbb{N}} (q\alpha)^n H_d(Tx_0, Tx_1) = \frac{1}{1 - q\alpha} H_d(Tx_0, Tx_1) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space with respect to T, we get that $(x_n)_{n\in\mathbb{N}}$ converges in $(X, \to)$ i.e., there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By (i), we get $x^* \in F_T$. □

Theorem 3.3.2. *Let $(X, \to, d)$ be a Kasahara space with respect to a multivalued operator $T : X \to P_d(X)$, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$, for all $x \in X$. We assume that:*

(i) *$Graph(T)$ is closed with respect to $\to$;*

(ii) *T is a multivalued α-contraction with respect to d;*

(iii) *$(SF)_T \neq \emptyset$;*

(iv) *$d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$.*

Then we have:

(1) $F_T = (SF)_T = \{x^*\}$;

(2) $F_{T^n} = (SF)_{T^n} = \{x^*\}$;

(3) $H_d(T^n x, x^*) \leq \alpha^n d(x, x^*)$, *for each* $n \in \mathbb{N}$ *and each* $x \in X$;

(4) *if* d *satisfy the triangle inequality, then*

   (4_a) $d(x, x^*) \leq \frac{1}{1-\alpha} H_d(x, Tx)$ *for each* $x \in X$;

   (4_b) *the fixed point problem for* T *is wll-posed with respect to* D.

Proof. (1). Let $x^* \in (SF)_T$. Then $\{x^*\} = Fx^*$ and hence $x^* \in F_T$.

Suppose that $y \in F_T$. Then

$$d(x^*, y) = D(Tx^*, y) \leq H_d(Tx^*, Ty) \leq \alpha d(x^*, y).$$

We get that $d(x^*, y) = 0$ and by (iv), $x^* = y$. So any fixed point of T is a strict fixed point of T, equal with x^*.

(2). Let $x^* \in F_{T^n}$. Then $x^* \in (SF)_{T^n}$ for each $n \in \mathbb{N}^*$.

Let $y \in (SF)_{T^n}$ for an arbitrary $n \in \mathbb{N}^*$. Then

$$d(x^*, y) = H_d(T^n x^*, T^n y) \leq \alpha H_d(T^{n-1} x^*, T^{n-1} y) \leq \ldots \leq \alpha^n d(x^*, y).$$

We have further that $d(x^*, y) = 0$ and by (iv), $x^* = y$. Hence $(SF)_{T^n} = \{x^*\}$.

Let $y \in F_{T^n}$. Then

$$d(x^*, y) = D(T^n x^*, y) \leq H_d(T^n x^*, T^n y) \leq \ldots \leq \alpha^n d(x^*, y)$$

implying further that $x^* = y$. The conclusion is proved.

(3). Let $x \in X$ and $n \in \mathbb{N}$. Then

$$H_d(T^n x, x^*) = H_d(T^n x, T^n x^*) \leq \alpha H_d(T^{n-1} x, T^{n-1} x^*) \leq \ldots \leq \alpha^n d(x, x^*) \to 0 \text{ as } n \to \infty.$$

(4). We successively have

$$d(x, x^*) \leq H_d(x, Tx) + H_d(Tx, x^*) \leq H_d(x, Tx) + \alpha d(x, x^*)$$

which implies further that

$$d(x, x^*) \leq \frac{1}{1-\alpha} H_d(x, Tx), \text{ for all } x \in X$$

and hence, (4_a) holds.

On the other hand, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $D(x_n, Tx_n) \to 0$ as $n \to \infty$. We have to prove that $d(x_n, x^*) \to 0$ as $n \to \infty$.

We have the following estimations

$$d(x_n, x^*) \leq D(x_n, Tx_n) + H_d(Tx_n, Tx^*) \leq D(x_n, Tx_n) + \alpha d(x_n, x^*).$$

Thus

$$d(x_n, x^*) \leq \frac{1}{1-\alpha} D(x_n, Tx_n) \to 0 \text{ as } n \to \infty$$

so (4_b) holds. □

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