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MOMENTS SOLUTION OF FRACTIONAL EVOLUTION EQUATION FOUND BY NEW KRASNOSELSKII TYPE FIXED POINT THEOREMS

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Abstract. In this note, we establish the existence of solutions in the moment mode of a fractional evolution equation, as well as a fractional coupled system, obtained by new Krasnoselskii type fixed point and coupled fixed point theorems. We use new Krasnoselskii type contraction conditions in the sense of measure of noncompactness in Banach spaces. The new outcomes extend some special well known recent results.
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1. Introduction

Most of the physical problems in real life are considered through non-linear models. These problems can be modelled using various mathematical techniques, especially using differential equations and integral equations. Nonlinear problems are of interest to scientists because almost all the mathematical models are inherently nonlinear in nature. Nonlinear equations are tricky to solve but give rise to real life phenomena. The problem of existence of a solution often becomes equivalent to the problems of detecting a fixed point of a certain operator. Hence, results from fixed point theory can then be employed to obtain solutions of an operator equation. There are various notions of solutions of differential equations, and in numerous situations one cannot just switch to one of them and study solutions in that sense. It is worthy to establish solutions with strong differentiability, but it might be difficult to verify their existence. Therefore, one typical request is to deliver solutions in a weaker sense. Fixed point theory (FPT) has two main branches: Constructive fixed point theorems in the line of Banach Contraction Principle (BCP) and nonconstructive fixed point theorems, where results are obtained by using topological properties in the direction of Brouwer's/ Schauder's (SFPT)/ Darbo's FPT (DFPT). Schauder discussed the convexity of domains and the compactness of operators. Darbo relaxed the strong condition of compactness of operators. He used the notion of measure of noncompactness (MNC) and defined appropriate classes of operators [16]. Krasnoselskii combined SFPT and BCP together in one result (see [5, 6, 7, 14, 15]). To discuss more related results, we need to recall the notion of MNC.

Let $(\mathcal{X}, \|\cdot\|)$ be an infinite dimensional Banach space and let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. If $\mathcal{B} \subseteq \mathcal{X}$, we denote by $conv(\mathcal{B})$ the convex hull of $\mathcal{B}$. Let

$$\mathfrak{B}_{\mathcal{X}} = \{\mathcal{D} : \emptyset \neq \mathcal{D} \subseteq \mathcal{X} \text{ and } \mathcal{D} \text{ is bounded}\}$$

and $\mathfrak{C}_{\mathcal{X}} = \{\mathcal{D} : \mathcal{D} \subseteq \mathcal{X} \text{ and } \mathcal{D} \text{ is relatively compact }\}$.

Definition 1.1. [13] A mapping $\mu : \mathfrak{B}_X \to \mathbb{R}^+$ is called an MNC on $\mathcal{X}$ if it attains the subsequent requirements:

(1°) $Ker\mu := \{\mathcal{K} \in \mathfrak{B}_{\mathcal{X}} : \mu(\mathcal{K}) = 0\}$ is nonempty and $Ker\mu \subset \mathfrak{C}_{\mathcal{X}}$;
(2°) $\mu(\mathcal{K}_1) \leq \mu(\mathcal{K}_2)$ for all subsets $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}_{\mathcal{X}}$ with $\mathcal{B}_1 \subset \mathcal{B}_2$;
(3°) $\mu(conv(\mathcal{K})) = \mu(\mathcal{K})$ for any subset $\mathcal{K} \in \mathfrak{B}_{\mathcal{X}}$;
(4°) $\mu(\lambda\mathcal{K}_1 + (1-\lambda)\mathcal{K}_2) \leq \lambda\mu(\mathcal{K}_1) + (1-\lambda)\mu(\mathcal{K}_2)$ for all subsets $\mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{B}_{\mathcal{X}}$ and $\lambda \in [0,1]$;
(5°) If $\{\mathcal{K}_n\}$ is a sequence of closed sets from $\mathfrak{B}_{\mathcal{X}}$ such that $\mathcal{K}_{n+1} \subset \mathcal{K}_n, n \geq 1$ and if $\lim\limits_{n\to\infty} \mu(\mathcal{K}_n) = 0$, then the intersection $\mathcal{K}_\infty = \bigcap\limits_{n=1}^{\infty} \mathcal{K}_n$ is nonempty.

In Definition 1.1 (1°), the family $Ker\mu$ is called the kernel of MNC μ in $\mathcal{X}$. Also, (5°) implies that $\mathcal{K}_\infty$ belongs to $Ker\mu$. As a matter of fact, since $\mu(\mathcal{K}_\infty) \leq \mu(\mathcal{K}_n)$ for any $n \geq 1$, we can conclude that $\mu(\mathcal{K}_\infty) = 0$, consequently $\mathcal{K}_\infty \in Ker\mu$. In the sequel, we denote $\Lambda = \{\mathcal{D} : \mathcal{D} \neq \emptyset$, closed, bounded and convex subset of a Banach space $\mathcal{X}\}$. We denote the set of fixed points of a mapping $\mathcal{T}$ by $Fix(\mathcal{T})$. Also, we denote $\mathfrak{F} := \{\mathcal{T} : \mathcal{T}$ is a self continuous operator on $\mathcal{D} \subseteq \mathcal{X}\}$.

Theorem 1.2. (SFPT [9]). *Let $\mathcal{D} \in \Lambda$ without boundedness, and $\mathcal{T} \in \mathfrak{F}$ with compactness. Then $Fix(\mathcal{T}) \neq \emptyset$ in $\mathcal{D}$.*

Lemma 1.3. (DFPT [13]) *Let $\mathcal{D} \in \Lambda$, $\mathcal{T} \in \mathfrak{F}$ be a μ-set contraction operator, that is, there is a constant $k \in [0,1)$ with $\mu(\mathcal{T}(\mathcal{P})) \leq k\mu(\mathcal{P})$ for all $\emptyset \neq \mathcal{P} \subset \mathcal{D}$, where μ is the Kuratowski MNC on $\mathcal{X}$. Then $Fix(\mathcal{T}) \neq \emptyset$ in $\mathcal{D}$.*

Thereafter, various types of DFPT and their coupled versions were obtained by using different types of contractive conditions in the sense of MNC (see [3]–[24]). Recently, Yang et al. [29] proved a coupled fixed point theorem of Krasnoselskii type based on results in [9]. In our investigation, we focus on findings in the sense of moments in a separable Banach space $\mathcal{X}$ equipped with the weak topology, induced by $\mathcal{X}'$. Also, we deal with the fractional evolution equation of the form

$$D_0^\alpha \nu(t) = F(t, \nu(t)), \tag{1.1}$$
$$\big(\nu \in \mathcal{X},\ \nu(0) = \nu_0,\ \alpha \in (0,1),\ t \in [0,\infty)\big),$$

where D_0^α denotes the Riemann-Liouville fractional differential operator (the classic fractional calculus), given by

$${}_0D_t^\alpha \nu(t) = \frac{d}{dt} I^{1-\alpha}\nu(t),$$

corresponding to the fractional integral operator (see [20, 22])

$$I^\alpha(\nu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\nu(s)\, ds,$$

$F : [0,\infty) \times \mathcal{X} \to \mathcal{X}$ is integrable ($F \in L^1([0,T] \times \mathcal{X}, \mathcal{X})$), Lipschitz function with the Lipschitz constant $\ell > 0$, We say that the equation (1.1) has a solution $\nu : [0,T] \to \mathcal{X}$ in the moment mode if it satisfies the following fractional integral equation:

$$\begin{aligned}\langle \nu(t), \rho\rangle = \langle \nu(\eta), \rho\rangle &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F\big(s, \nu(s)\big), \rho\right\rangle ds \\ &+ \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \left\langle F'\big(s, \nu(s)\big), \rho\right\rangle ds,\end{aligned} \tag{1.2}$$

where $0 \le \eta < t - \varepsilon$, for all $\rho \in \mathcal{X}'$ (weak topology in $\mathcal{X}$) and ν is continuous in $\mathcal{X}'$ (weakly continuous). In this work, we shall assume that ρ is bounded by some finite constant $K > 0$; thus, we obtain $| \left\langle F\big(s, \nu(s)\big), \rho\right\rangle | \le K\|F\|$.
In this work, we discuss new Krasnoselskii type fixed point and coupled fixed point results using the concept of DFPT (see [17] and [26]). We generalize the results from [3, 2, 4, 1, 9, 17, 29, 21, 27, 28].We find the moment solution of (1.1) as well as of the coupled system

$$\begin{aligned} D_0^\alpha \nu(t) &= F_1(t, \upsilon(t)) \\ D_0^\alpha \upsilon(t) &= F_2(t, \nu(t)). \end{aligned}$$

2. Krasnoselskii type fixed point outcomes

In this section, we discuss two main Krasnoselskii type fixed point results.
To achieve them, we use the results of DFPT given in [26].

2.1. **Result 1.** We start with two notions that are required for completing the results [18].

Definition 2.1. Let Δ be the set of functions $\chi : \mathbb{R}^+ \to [0,1)$ satisfying $\chi(t_n) \to 1 \to t_n \to 0$.

Definition 2.2. Let Ψ denote the class of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:
$(i)\, \psi$ is nondecreasing, $\quad (ii)\, \psi$ is continuous, $\quad (iii)\, \psi^{-1}(\{0\}) = \{0\}$.

From now on, we take $\mu(\cdot)$ as an arbitrary MNC in Banach space $\mathcal{X}$.

Lemma 2.3. [26] *Let $\mathcal{D} \in \Lambda$, $\mathcal{T} \in \mathfrak{F}$ satisfy*

$$\psi(\mu(\mathcal{T}(\mathcal{B})) + \varphi(\mu(\mathcal{T}(\mathcal{B})))) \leq \chi(\psi(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))], \tag{2.1}$$

for all $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous mapping, $\chi \in \Delta$ and $\psi \in \Psi$. Then $Fix(\mathcal{T}) \neq \emptyset$ in $\mathcal{D}$.

We call (2.1) the condition of μ-(ψ, φ)-set contraction. Also, we define the family $\Gamma := \{\mathcal{K} : \mathcal{K} : \mathcal{D} \to \mathcal{X}$ is a continuous operator$\}$. Next we prove our first main result.

Theorem 2.4. *Let $\mathcal{D} \in \Lambda$ and $\mathcal{K}_i \in \Gamma$ $(i = 1, 2)$ satisfy*

(F1) *$\mathcal{K}_1 x + \mathcal{K}_2 y \in \mathcal{D}$, $\forall\ x, y \in \mathcal{D}$;*
(F2) *$\mathcal{K}_1 \mathcal{D} \subset \mathcal{R}$, where $\mathcal{R}$ is a compact set,*
(F3) *$\mathcal{K}_2$ is a μ-(ψ, φ)-set contraction.*

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ in $\mathcal{D}$.

Proof. Consider an operator $\mathcal{T} : \mathcal{D} \to \mathcal{X}$ defined, for $x \in \mathcal{D}$, by

$$\mathcal{T}x = \mathcal{K}_1 x + \mathcal{K}_2 x. \tag{2.2}$$

Obviously, if $u \in Fix(\mathcal{T})$ in $\mathcal{D}$, then u is the solution of equation $y = \mathcal{K}_1 y + \mathcal{K}_2 y$. Now, due to continuity of operators $\mathcal{K}_1, \mathcal{K}_2$, $\mathcal{T} : \mathcal{D} \to \mathcal{X}$ is continuous. Also, by the virtue of $(F1)$, we can have $\mathcal{T}x = \mathcal{K}_1 x + \mathcal{K}_2 x \in \mathcal{D}$ for any $x \in \mathcal{D}$. Then $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is continuous. By the virtue of (2.2), for any $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, we can obtain

$$\mathcal{T}\mathcal{B} \subset \mathcal{K}_1\mathcal{B} + \mathcal{K}_2\mathcal{B}. \tag{2.3}$$

By the virtue of $(F2)$ and MNC, we have

$$\mu(\mathcal{K}_1\mathcal{B}) = 0. \tag{2.4}$$

Finally, in view of $(F3)$, (2.3) and (2.4), we have

$$\begin{aligned}
\psi(\mu(\mathcal{T}\mathcal{B}) + \varphi(\mu(\mathcal{T}\mathcal{B}))) &\leq \psi[\mu(\mathcal{A}_1\mathcal{B} + \mathcal{A}_2\mathcal{B}) + \varphi(\mu(\mathcal{A}_1\mathcal{B} + \mathcal{A}_2\mathcal{B}))] \\
&\leq \psi[\mu(\mathcal{A}_1\mathcal{B}) + \mu(\mathcal{A}_2\mathcal{B}) + \varphi(\mu(\mathcal{A}_1\mathcal{B}) + \mu(\mathcal{A}_2\mathcal{B}))] \\
&= \psi[\mu(\mathcal{A}_2\mathcal{B})) + \varphi(\mu(\mathcal{A}_2\mathcal{B}))] \\
&\leq \chi(\psi(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))].
\end{aligned}$$

Consequently, all the requirements of Lemma 2.3 are fulfilled and thus the result. □

Taking $\varphi(t) \equiv \lambda t$ for $t \in \mathbb{R}^+$ with $\lambda \geq 0$ in Theorem 2.4, we have the following FPT:

Corollary 2.5. *Let all the conditions of Theorem 2.4 be satisfied, apart from the hypothesis (F3) which is replaced by*

(F3') *$\psi(\mu(\mathcal{K}_2(\mathcal{B}))) \leq \chi(\psi(\mu(\mathcal{B})))\psi(\mu(\mathcal{B}))$, for any $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, where $\chi \in \Delta$ and $\psi \in \Psi$.*

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ in $\mathcal{D}$.

Taking $\chi(t) \equiv \lambda \in [0, 1)$ for $t \in \mathbb{R}^+$ in Theorem 2.4, we obtain the following FPT.

Corollary 2.6. *Let all the conditions of Theorem 2.4 be satisfied, apart from the hypothesis (F3) which is replaced by*

(F3") $\psi(\mu(\mathcal{K}_2(\mathcal{B})) + \varphi(\mu(\mathcal{K}_2(\mathcal{B})))) \leq \lambda(\psi(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))]$ *for any* $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous mapping and* $\psi \in \Psi$.

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ *in* $\mathcal{D}$.

2.2. **Result 2.** To start our second main result, we need the following preliminaries:

Definition 2.7. [23] Let Υ denote the class of all MT-functions $\zeta : [0, \infty) \to [0, 1)$ which satisfies Mizoguchi-Takahashi's condition

$$\limsup_{s \to t^+} \zeta(s) < 1 \text{ for all } t \in [0, \infty).$$

It is noted that if $\zeta : [0, \infty) \to [0, 1)$ is a non-decreasing function or a non-increasing function, then ζ is an MT-function. So, the set of MT-functions is a rich class, but it is worth to mention that there exist functions which are not MT-function.

Definition 2.8. Let Ω denote the set of all functions $\omega : [0, +\infty) \to [0, +\infty)$ satisfying:

(i) ω is non-decreasing, (ii) $\omega(t) = 0 \Leftrightarrow t = 0$.

Lemma 2.9. [26] *Let* $\mathcal{D} \in \Lambda$, $\mathcal{T} \in \mathfrak{F}$ *with*

$$\omega(\mu(\mathcal{T}(\mathcal{B})) + \varphi(\mu(\mathcal{T}(\mathcal{B})))) \leq \zeta(\omega(\mu(\mathcal{B})))\omega[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))], \tag{2.5}$$

for all $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous mapping,* $\zeta \in \Upsilon$ *and* $\omega \in \Omega$. *Then* $Fix(\mathcal{T}) \neq \emptyset$ *in* $\mathcal{D}$.

We call (2.5) a μ-(ω, φ)-set contractive condition. We have the following result:

Theorem 2.10. *Let* $\mathcal{D} \in \Lambda$ *and* $\mathcal{K}_i \in \Gamma$ $(i = 1, 2)$ *satisfy*

(F4) $\mathcal{K}_1 x + \mathcal{K}_2 y \in \mathcal{D}$, $\forall\, x, y \in \mathcal{D}$;
(F5) $\mathcal{K}_1 \mathcal{D} \subset \mathcal{R}$, *where* $\mathcal{R}$ *is a compact set;*
(F6) $\mathcal{K}_2$ *is a* μ-(ω, φ)*-set contraction.*

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ *in* $\mathcal{D}$.

Proof. Consider an operator $\mathcal{W} : \mathcal{D} \to \mathcal{X}$ defined by

$$\mathcal{W}x = \mathcal{K}_1 x + \mathcal{K}_2 x, \quad \forall x \in \mathcal{D}.$$

Following the proof of Theorem 2.4, we have only to prove that $\mathcal{W}$ satisfy μ-(ω, φ)-set contractive condition. From $(F6)$, (2.3) and (2.4), we have

$$\begin{aligned}
\omega[\mu(\mathcal{W}\mathcal{B}) + \varphi(\mu(\mathcal{W}\mathcal{B}))] &\leq \omega[\mu(\mathcal{K}_1\mathcal{B} + \mathcal{K}_2\mathcal{B}) + \varphi(\mu(\mathcal{K}_1\mathcal{B} + \mathcal{K}_2\mathcal{B}))] \\
&\leq \omega[\mu(\mathcal{K}_1\mathcal{B}) + \mu(\mathcal{K}_2\mathcal{B}) + \varphi(\mu(\mathcal{K}_1\mathcal{B}) + \mu(\mathcal{K}_2\mathcal{B}))] \\
&= \omega[\mu(\mathcal{K}_2\mathcal{B})) + \varphi(\mu(\mathcal{K}_2\mathcal{B}))] \\
&\leq \zeta(\omega(\mu(\mathcal{B})))\omega[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))].
\end{aligned}$$

Thus $\mathcal{W}$ satisfy all the conditions of Lemma 2.9 and hence the result. □

Putting $\varphi(t) \equiv 0$ for $t \in \mathbb{R}^+$ in the condition $(F6)$ of Theorem 2.10, we have the following outcome:

Corollary 2.11. *Let all the conditions of Theorem 2.10 be satisfied, apart from the hypothesis (F6) which is replaced by*

(F6') $\omega(\mu(\mathcal{K}_1(\mathcal{B}))) \leq \zeta(\omega(\mu(\mathcal{B})))\omega(\mu(\mathcal{B}))$, *for any* $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, *where* $\zeta \in \Upsilon$ *and* $\omega \in \Omega$.

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ *in* $\mathcal{D}$.

Taking $\chi(t) \equiv \lambda \in [0, 1)$ for $t \in \mathbb{R}^+$ in Theorem 2.10, we have the following fixed point result.

Corollary 2.12. *Let all the conditions of Theorem 2.10 be satisfied, apart from the hypothesis (F6) which is replaced by*

(F6") $\omega(\mu(\mathcal{K}_2(\mathcal{B})) + \varphi(\mu(\mathcal{K}_2(\mathcal{B})))) \leq \lambda(\omega(\mu(\mathcal{B})))\omega[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))]$ *for any* $\emptyset \neq \mathcal{B} \subset \mathcal{D}$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous mapping and* $\omega \in \Omega$.

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ *in* $\mathcal{D}$.

2.3. **Result 3.** To complete the result of this section, we recall the following preliminaries:

Definition 2.13. Let Ξ denote the set of continuous functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\beta(t_n) \to 0 \Rightarrow t_n \to 0.$$

Definition 2.14. Let Φ denote the class of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:

(i) ϕ is nondecreasing;
(ii) ϕ is lower semicontinuous;
(iii) $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$.

Lemma 2.15. [17] *Let* $\mathcal{D} \in \Lambda$ *and* $\mathcal{T} \in \mathfrak{F}$ *such that*

$$\mu(\mathcal{T}(\mathcal{P})) \leq \mu(\mathcal{P}) - \phi(\beta(\mu(\mathcal{P}))) \tag{2.6}$$

$\forall\ \emptyset \neq \mathcal{B} \subset \mathcal{D}$, *where* $\beta \in \Xi$ *and* $\phi \in \Phi$. *Then* $Fix(\mathcal{T}) \neq \emptyset$ *in* $\mathcal{D}$.

We call (2.6) the μ-(ϕ, β)-set condition. We have the main result as follows:

Theorem 2.16. *Let* $\mathcal{D} \in \Lambda$ *and* $\mathcal{K}_i \in \Gamma$ $(i = 1, 2)$ *satisfy*

(F7) $\mathcal{K}_1 x + \mathcal{K}_2 y \in \mathcal{D}$, $\forall\ x, y \in \mathcal{D}$;
(F8) $\mathcal{K}_1\mathcal{D} \subset \mathcal{R}$, *where* $\mathcal{R}$ *is a compact set;*
(F9) $\mathcal{K}_2$ *is a* μ-(ϕ, β)*-set contraction.*

Then $Fix(\mathcal{K}_1 + \mathcal{K}_2) \neq \emptyset$ *in* $\mathcal{D}$.

Proof. Consider the operator $\mathcal{Z} : \mathcal{D} \to \mathcal{X}$ defined by

$$\mathcal{Z}x = \mathcal{K}_1 x + \mathcal{K}_2 x, \quad \forall x \in \mathcal{D}.$$

Following the proof of Theorem 2.4, we have only to prove that $\mathcal{Z}$ satisfy μ-(ϕ, β)-set contraction condition. From $(F9)$, (2.3) and (2.4), we have

$$\begin{aligned} \mu(\mathcal{Z}(\mathcal{B})) &\leq \mu(\mathcal{K}_1\mathcal{B} + \mathcal{K}_2\mathcal{B}) \\ &\leq \mu(\mathcal{K}_1\mathcal{B}) + \mu(\mathcal{K}_2\mathcal{B}) \\ &\leq \mu(\mathcal{B}) - \phi(\beta(\mu(\mathcal{B}))). \end{aligned}$$

Thus, $\mathcal{Z}$ satisfy all the constraints of Lemma 2.15 and hence the result. $\square$

3. Krasnoselskii type coupled fixed point theorem

Before we start our discussion, we set the following terminology:

Let $\mathcal{X}$ be a real Banach space with the norm $\|\cdot\|$ and let $\widehat{\mathcal{X}} = \mathcal{X} \times \mathcal{X}$. For any $u = (u_1, u_2), v = (v_1, v_2)$ and $\zeta \in \mathbb{R}$, we define

$$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2),$$

$$\zeta u = \zeta(u_1, u_2) = (\zeta u_1, \zeta u_2)$$

and

$$\|u\|_{\widehat{\mathcal{X}}} = \|(u_1, u_2)\|_{\mathcal{X}} = \|u_1\| + \|u_2\|.$$

Then $\widehat{\mathcal{X}}$ is a Banach space with the norm $\|\cdot\|_{\widehat{\mathcal{X}}}$.

Definition 3.1. An element $(u, v) \in \widehat{\mathcal{X}}$ is called a coupled fixed point (CFP) of the mapping $\mathcal{G} : \widehat{\mathcal{X}} \to \mathcal{X}$ if $\mathcal{G}(u, v) = u$ and $\mathcal{G}(v, u) = v$. The set of all coupled fixed points of $\mathcal{G}$ is denoted by $CFix(\mathcal{G})$.

3.1. **Result 4.** In this section, we discuss two new Krasnoselskii-type coupled fixed point theorems and some consequences by applying the Krasnoselskii fixed point theorem established in Section 2. We consider the following setup of new MNC to obtain the result. Let $\emptyset \neq \mathcal{H}_1, \emptyset \neq \mathcal{H}_2 \subseteq \mathcal{X}$ be bounded. Then $\mathcal{H}_1 \times \mathcal{H}_2 \subseteq \widehat{\mathcal{X}}$ is bounded. We construct new MNC in $\widehat{\mathcal{X}}$ as

$$\widehat{\beta}(\mathcal{H}_1 \times \mathcal{H}_2) = \frac{\beta(\mathcal{H}_1) + \beta(\mathcal{H}_2)}{2}. \tag{3.1}$$

Theorem 3.2. *Let $\mathcal{D} \in \Lambda$ and $\widehat{\mathcal{D}} = \mathcal{D} \times \mathcal{D}$. Assume that $\mathcal{J}_i \in \Gamma$ $(i = 1, 2)$ satisfy*

- (H1) *$\mathcal{J}_1 x + \mathcal{J}_2 y \in \mathcal{D}$, $\forall\, x, y \in \mathcal{D}$,*
- (H2) *$\mathcal{J}_1 \mathcal{D} \subset \mathcal{R}$, where $\mathcal{R}$ is a compact set,*
- (H3) *$\mathcal{J}_2$ is μ-(ϕ, φ)-set contraction*

Then $CFix(\mathcal{G}) \neq \emptyset$ in $\widehat{\mathcal{D}}$, where $\mathcal{G}(x, y) = \mathcal{J}_1 x + \mathcal{J}_2 y$.

Proof. It is easy to see that $\widehat{\mathcal{D}} \in \Lambda$. We define two operators $\widehat{\mathcal{J}}_1, \widehat{\mathcal{J}}_2$ by

$$\widehat{\mathcal{J}}_1(x, y) = (\mathcal{J}_1 x, \mathcal{J}_1 y), \qquad \widehat{\mathcal{J}}_2(x, y) = (\mathcal{J}_2 y, \mathcal{J}_2 x). \tag{3.2}$$

Since $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{D} \to \mathcal{X}$, it follows that $\widehat{\mathcal{J}}_1, \widehat{\mathcal{J}}_2 : \widehat{\mathcal{D}} \to \widehat{\mathcal{X}}$. If $\exists\ \hat{u} = (\hat{x}, \hat{y}) \in \widehat{\mathcal{D}}$ such that $\hat{u} = \widehat{\mathcal{J}}_1 \hat{u} + \widehat{\mathcal{J}}_2 \hat{u}$, by (3.2), we have

$$\begin{aligned}
(\hat{x}, \hat{y}) &= \widehat{\mathcal{J}}_1(\hat{x}, \hat{y}) + \widehat{\mathcal{J}}_2(\hat{x}, \hat{y}) \\
&= (\mathcal{J}_1 \hat{x}, \mathcal{J}_1 \hat{y}) + (\mathcal{J}_2 \hat{y}, \mathcal{J}_2 \hat{x}) \\
&= (\mathcal{J}_1 \hat{x} + \mathcal{J}_2 \hat{y}, \mathcal{J}_1 \hat{y} + \mathcal{J}_2 \hat{x}) \\
&= (\mathcal{G}(\hat{x}, \hat{y}), \mathcal{G}(\hat{y}, \hat{x})).
\end{aligned}$$

By virtue of Definition 3.1, $(\hat{x}, \hat{y}) \in \widehat{\mathcal{D}}$ is a CFP of $\mathcal{G}$. To come up to this conclusion, we make use of Theorem 2.4.

Part (I): For any $x, y \in \mathcal{D}$, by $(H1)$, we have $\mathcal{J}_1 x + \mathcal{J}_2 y \in \mathcal{D}$. Let $u = (x, y) \in \widehat{\mathcal{D}}$ be an arbitrary element. We assert that $\widehat{\mathcal{J}}_1 u + \widehat{\mathcal{J}}_2 u \in \widehat{\mathcal{D}}$. In fact, by the virtue of (3.2), we have

$$\widehat{\mathcal{J}}_1 u + \widehat{\mathcal{J}}_2 u = (\mathcal{J}_1 x, \mathcal{J}_1 y) + (\mathcal{J}_2 y, \mathcal{J}_2 x) = (\mathcal{J}_1 x + \mathcal{J}_2 y, \mathcal{J}_1 y + \mathcal{J}_2 x) \in \mathcal{D} \times \mathcal{D}.$$

This shows that $\widehat{\mathcal{J}}_1 u + \widehat{\mathcal{J}}_2 u \in \widehat{\mathcal{D}}$.

Part (II): In view of the assumption $(H2)$, $\mathcal{J}_1 \mathcal{D}$ is involved in a compact set. For all $\emptyset \neq \mathcal{D}_i \subset \mathcal{D}(i = 1, 2)$, it follows from $\widehat{\mathcal{J}}_1(\mathcal{D}_1 \times \mathcal{D}_2) = (\mathcal{J}_1 \mathcal{D}_1, \mathcal{J}_1 \mathcal{D}_2)$ that $\widehat{\mathcal{J}}_1 \widehat{\mathcal{D}}$ is contained in a compact set. Let $\emptyset \neq \mathcal{B} \subseteq \mathcal{D}$ be arbitrary. Then $\widehat{\mathcal{B}} = \mathcal{B} \times \mathcal{B}$ is arbitrary in $\widehat{\mathcal{D}}$.

Part (III): Last of all, we argue that $\widehat{\mathcal{J}}_2$ is $\hat{\mu}$-(ϕ, φ)-set contractive. To attain this, we use $(H3)$ and (3.2) and we have

$$\begin{aligned}
&\psi[\widehat{\mu}(\widehat{\mathcal{J}}_2(\widehat{\mathcal{B}})) + \varphi(\widehat{\mu}(\widehat{\mathcal{J}}_2(\widehat{\mathcal{B}})))] \\
&= \psi[\widehat{\mu}(\widehat{\mathcal{J}}_2(\mathcal{B} \times \mathcal{B})) + \varphi(\widehat{\mu}(\widehat{\mathcal{J}}_2(\mathcal{B} \times \mathcal{B})))] \\
&= \psi[\widehat{\mu}(\mathcal{J}_2(\mathcal{B}) \times \mathcal{J}_2(\mathcal{B})) + \varphi(\widehat{\mu}(\mathcal{J}_2(\mathcal{B}) \times \mathcal{J}_2(\mathcal{B})))] \\
&= \psi\left[\frac{\mu(\mathcal{J}_2(\mathcal{B})) + \mu(\mathcal{J}_2(\mathcal{B}))}{2} + \varphi(\frac{\mu(\mathcal{J}_2(\mathcal{B})) + \mu(\mathcal{J}_2(\mathcal{B}))}{2})\right] \\
&= \psi[\mu(\mathcal{J}_2(\mathcal{B})) + \varphi(\mu(\mathcal{J}_2(\mathcal{B})))] \\
&\leq \chi(\psi(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))] \\
&= \chi\left(\psi\left(\frac{\mu(\mathcal{B}) + \mu(\mathcal{B})}{2}\right)\right)\psi\left[\frac{\mu(\mathcal{B}) + \mu(\mathcal{B})}{2} + \varphi\left(\frac{\mu(\mathcal{B}) + \mu(\mathcal{B})}{2}\right)\right] \\
&= \chi(\psi(\widehat{\mu}(\widehat{\mathcal{B}})))\psi[\widehat{\mu}(\widehat{\mathcal{B}}) + \varphi(\widehat{\mu}(\widehat{\mathcal{B}}))].
\end{aligned}$$

Thus, in view of Theorem 2.4, we have the conclusion. □

Putting $\varphi(t) \equiv \lambda t$ for $t \in \mathbb{R}^+$ with $\lambda \geq 0$ in Theorem 3.2, we have the following CFPT.

Corollary 3.3. *Let all the conditions of Theorem 3.2 be satisfied, apart from the hypothesis (H3) which is replaced by*

(H3') $\psi(\mu(\mathcal{J}_2(\mathcal{B}))) \leq \chi(\psi(\mu(\mathcal{B})))\psi(\mu(\mathcal{B}))$, $\forall\ \mathcal{B} \neq \emptyset \subset \mathcal{D}$, *where* $\chi \in \Delta$ *and* $\psi \in \Psi$.

Then $CFix(\mathcal{G}(x, y) = \mathcal{J}_1 x + \mathcal{J}_2 y) \neq \emptyset$ *in* $\widehat{\mathcal{D}}$.

Assuming $\chi(t) \equiv \lambda \in [0, 1)$ for $t \in \mathbb{R}^+$ in Theorem 3.2, we have the following CFPT:

Corollary 3.4. *Let all the conditions of Theorem 3.2 be satisfied, apart from the hypothesis (H3) which is replaced by*

(H3”) $\psi(\mu(\mathcal{J}_2(\mathcal{B})) + \varphi(\mu(\mathcal{J}_2(\mathcal{B})))) \leq \lambda(\psi(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))]\ \forall\ \mathcal{B} \neq \emptyset \subset \mathcal{D}$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous mapping and* $\psi \in \Psi$.

Then $CFix(\mathcal{G}(x, y) = \mathcal{J}_1 x + \mathcal{J}_2 y) \neq \emptyset$ *in* $\widehat{\mathcal{D}}$.

3.2. **Result 5.** Now, we state and prove the second Krasnoselskii type CFP result using Theorem 2.10. For this, we select different MNC as follows:
Let $\emptyset \neq \mathcal{H}_1, \emptyset \neq \mathcal{H}_2 \subseteq \mathcal{X}$ be bounded. Then $\mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{X}$ is bounded. We construct a new MNC in $\mathcal{X}$ as follows:

$$\widehat{\beta}(\mathcal{H}_1 \times \mathcal{H}_2) = \max\{\beta(\mathcal{H}_1), \beta(\mathcal{H}_2)\} \tag{3.3}$$

Theorem 3.5. *Let $\mathcal{D} \in \Lambda$ and $\widehat{\mathcal{D}} = \mathcal{D} \times \mathcal{D}$. Assume that $\mathcal{J}_i \in \Gamma$ $(i = 1, 2)$ satisfy*

(H4) $\mathcal{J}_1 x + \mathcal{J}_2 y \in \mathcal{D}$, *$\forall\ x, y \in \mathcal{D}$,*
(H5) $\mathcal{J}_1 \mathcal{D} \subset \mathcal{R}$, *where $\mathcal{R}$ is a compact set,*
(H6) $\mathcal{J}_2$ *is μ-(ω, φ)-set contraction, where $\zeta \in \Upsilon$ and $\omega \in \Omega$.*

Then $CFix(\mathcal{G}) \neq \emptyset$ in $\widehat{\mathcal{D}}$, where $\mathcal{G}(x, y) = \mathcal{J}_1 x + \mathcal{J}_2 y$.

Proof. Following the proof of Theorem 3.2, we define the operators $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{D} \to \mathcal{X}$ and prove the Parts (I) and (II). We only have to prove Part (III), i.e., that $\widehat{\mathcal{J}}_2$ is μ-(ω, φ)-set contractive. For this, we use $(H6)$ and (3.2) and we have

$$\begin{aligned}
&\omega[\widehat{\mu}(\widehat{\mathcal{J}}_2(\widehat{\mathcal{B}})) + \varphi(\widehat{\mu}(\widehat{\mathcal{J}}_2(\widehat{\mathcal{B}})))] \\
&= \omega[\widehat{\mu}(\widehat{\mathcal{J}}_2(\mathcal{B} \times \mathcal{B})) + \varphi(\widehat{\mu}(\widehat{\mathcal{J}}_2(\mathcal{B} \times \mathcal{B})))] \\
&= \omega[\widehat{\mu}(\mathcal{J}_2(\mathcal{B}) \times \mathcal{J}_2(\mathcal{B})) + \varphi(\widehat{\mu}(\mathcal{J}_2(\mathcal{B}) \times \mathcal{J}_2(\mathcal{B})))] \\
&= \omega[\max\{\mu(\mathcal{J}_2(\mathcal{B})), \mu(\mathcal{J}_2(\mathcal{B}))\} + \varphi(\max\{\mu(\mathcal{J}_2(\mathcal{B})), \mu(\mathcal{J}_2(\mathcal{B}))\})] \\
&= \omega[\mu(\mathcal{J}_2(\mathcal{B})) + \varphi(\mu(\mathcal{J}_2(\mathcal{B})))] \\
&\leq \zeta(\omega(\mu(\mathcal{B})))\psi[\mu(\mathcal{B}) + \varphi(\mu(\mathcal{B}))] \\
&= \zeta(\omega(\max\{\mu(\mathcal{B}), \mu(\mathcal{B})\}))\omega[\max\{\mu(\mathcal{B}), \mu(\mathcal{B})\} + \varphi(\max\{\mu(\mathcal{B}), \mu(\mathcal{B})\})] \\
&= \zeta(\omega(\widehat{\mu}(\widehat{\mathcal{B}})))\omega[\widehat{\mu}(\widehat{\mathcal{B}}) + \varphi(\widehat{\mu}(\widehat{\mathcal{B}}))].
\end{aligned}$$

Hence, we conclude the result from Theorem 2.10. □

3.3. **Result 6.** The third type of Krasnoselskii CFP result using Theorem 2.16 is the following. We need to use (3.1) or (3.3). Similarly as in Theorems 3.2 and 3.5, we have the following result:

Theorem 3.6. *Let $\mathcal{D} \in \Lambda$ and $\widehat{\mathcal{D}} = \mathcal{D} \times \mathcal{D}$. Assume that $\mathcal{J}_i \in \Gamma$ $(i = 1, 2)$ satisfy*

(H4) $\mathcal{J}_1 x + \mathcal{J}_2 y \in \mathcal{D}$, *$\forall\ x, y \in \mathcal{D}$,*
(H5) $\mathcal{J}_1 \mathcal{D} \subset \mathcal{R}$, *where $\mathcal{R}$ is a compact set,*
(H6) $\mathcal{J}_2$ *is μ-(ϕ, β)-set contraction, where $\beta \in \Xi$ and $\phi \in \Phi$.*

Then $CFix(\mathcal{G}(x, y) = \mathcal{J}_1 x + \mathcal{J}_2 y) \neq \emptyset \in \widehat{\mathcal{D}}$.

4. Applications

We consider the fractional differential equation (1.1). Our aim is to show that (1.1) has a solution in the sense of moments taking the form (1.2), by employing Theorem 2.4.

Theorem 4.1. *Define the operator $Q : \mathcal{X} \to \mathcal{X}$ as follows:*

$$(Q\nu)(t) = (Q_1\nu)(t) + (Q_2\nu)(t), \tag{4.1}$$

where

$$(Q_1\nu)(t) = \langle \nu(\eta), \rho\rangle + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F\big(s,\nu(s)\big), \rho\right\rangle ds, \tag{4.2}$$

and

$$(Q_2\nu)(t) = \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \left\langle F'\big(s,\nu(s)\big), \rho\right\rangle ds. \tag{4.3}$$

If $K \in (0,1)$ *(the upper bound of* ρ*) satisfies*

$$\ell < \frac{\Gamma(\alpha+1)}{3KT^\alpha}$$

then the operator (4.1) *admits at least one fixed point corresponding to the moment solution of* (1.1).

Proof. Our aim is to achieve all the conditions of Theorem 2.4.
Boundedness. By the definition of operator Q, a computation implies

$$\begin{aligned}(Q\nu)(t) &= \langle \nu(\eta), \rho\rangle + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F\big(s,\nu(s)\big), \rho\right\rangle ds \\ &+ \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \left\langle F'\big(s,\nu(s)\big), \rho\right\rangle ds \\ &= \langle \nu(\eta), \rho\rangle + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F\big(s,\nu(s)\big), \rho\right\rangle ds \\ &+ \alpha \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} \left\langle F\big(s,\nu(s)\big), \rho\right\rangle ds \\ &+ \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \langle F(t-\varepsilon), \rho\rangle - \frac{t^\alpha}{\Gamma(\alpha+1)} \langle F(\eta), \rho\rangle .\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|(Q\nu)(t)| &\le K\|\nu\| + K\|F\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \alpha K\|F\| \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} ds \\ &+ \frac{\varepsilon^\alpha K\|F\|}{\Gamma(\alpha+1)} + \frac{(t-\eta)^\alpha K\|F\|}{\Gamma(\alpha+1)}. \\ &\le K\|\nu\| + K\|F\| \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\eta)^\alpha}{\Gamma(\alpha+1)} \Big).\end{aligned}$$

Taking the maximum norm on $t \in [0,T]$, we have

$$\|Q\nu\| \le \frac{4K\|F\|T^\alpha}{(1-K)\Gamma(\alpha+1)} := r, \quad K \in (0,1).$$

Hence, Q is bounded in B_r.

Continuity. Let $\delta > 0$ and $\nu, \upsilon \in B_r$ such that $\|\nu - \upsilon\| \le \delta$. Then a computation implies

$$\begin{aligned}
|(Q\nu)(t) - (Q\upsilon)(t)| \le & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, | \big\langle [F\big(s, \nu(s)\big) - F\big(s, \upsilon(s)\big)], \rho \big\rangle \, | ds \\
& + \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \, | \big\langle [F'\big(s, \nu(s) - F'\big(s, \nu(s)\big)], \rho \big\rangle \, | ds \\
& \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, K \| F\big(s, \nu(s)\big) - F\big(s, \upsilon(s)\big) \| ds \\
& + \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} \, K \| F\big(s, \nu(s) - F\big(s, \nu(s)\big) \| ds \\
& + \frac{\varepsilon^\alpha K}{\Gamma(\alpha+1)} \| F\big(s, \nu(s) - F\big(s, \nu(s)\big) \|, \quad \nu_0 = \upsilon_0 \\
& \le \|\nu - \upsilon\| \ell K \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \Big) \\
& \le \frac{3\delta \ell T^\alpha K}{\Gamma(\alpha+1)} = \epsilon,
\end{aligned}$$

where

$$\delta := \frac{\Gamma(\alpha+1)\epsilon}{3T^\alpha \ell K}, \quad K \in (0,1).$$

Contractivity. Let ν and $\upsilon \in B_r$. Then we have

$$\begin{aligned}
|(Q_2\nu)(t) - (Q_2\upsilon)(t)| &\le \|\nu - \upsilon\| \ell K \Big(\frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \Big) \\
&\le \|\nu - \upsilon\| \frac{2\ell K T^\alpha}{\Gamma(\alpha+1)} < \|\nu - \upsilon\| \frac{3\ell K T^\alpha}{\Gamma(\alpha+1)} < \|\nu - \upsilon\|.
\end{aligned}$$

Hence, Q_2 is a contraction mapping.

Measurement. Here, we aim to prove $\mu(Q)(B_r) \le \mu(B_r)$. For ν and $\upsilon \in B_r$, we have

$$\begin{aligned}
|(Q\nu)(t) - (Q\upsilon)(t)| &\le \|\nu - \upsilon\| \ell K \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \Big) \\
&\le \|\nu - \upsilon\| \frac{3\ell K T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

This yields that $diam(Q(B_r)) \le K_\alpha \, diam(B_r)$, where for sufficient value of $0 < \ell < \frac{\|F\|}{1-K}$, we have $K_\alpha := \frac{3\ell T^\alpha K}{\Gamma(\alpha+1)} < r$. Consequently, $diam(Q(B_r)) \le diam(B_r)$. Now, we define the function $\psi : (0, \infty) \to (0, \infty)$ as follows: $\psi(\varsigma) = \varsigma + \frac{1}{2}$. Obviously,

$$\begin{aligned}
\psi(\mu(Q(B_r))) &\le \psi(K_\alpha \, \mu(B_r)) = K_\alpha \, \mu(B_r) + \frac{1}{2} \\
&< \mu(B_r) + \frac{1}{2} = \psi(\mu(B_r)), \quad K_\alpha < 1.
\end{aligned}$$

Hence, Q admits a fixed point analogous to the solution of (4.1). $\square$

Next, we consider the fractional coupled system

$$\begin{aligned} D_0^\alpha \nu(t) &= F_1(t, \upsilon(t)) \\ D_0^\alpha \upsilon(t) &= F_2(t, \nu(t)), \end{aligned} \tag{4.4}$$

where F_1 and F_2 are integrable ($F_1, F_2 \in L^1([0,T] \times \mathcal{X}, \mathcal{X})$), Lipschitz function with the Lipschitz constants $\ell_1, \ell_2 > 0$. We aim to show that the system (4.4) has a couple moment solution, taking the form (1.2).

Theorem 4.2. *Define the operator* $\Theta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *as follows:*

$$\Theta(\nu, \upsilon) = (\Theta_1 \nu)(t) + (\Theta_2 \upsilon)(t), \tag{4.5}$$

where Θ_1 *and* Θ_2 *are given by*

$$\begin{aligned} (\Theta_1 \nu)(t) = \langle \nu(\eta), \rho \rangle &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F_1\big(s, \upsilon(s)\big), \rho \right\rangle ds \\ &+ \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \left\langle F_1'\big(s, \upsilon(s)\big), \rho \right\rangle ds, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} (\Theta_2 \upsilon)(t) = \langle \upsilon(\eta), \rho \rangle &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle F_2\big(s, \nu(s)\big), \rho \right\rangle ds \\ &+ \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \left\langle F_2'\big(s, \nu(s)\big), \rho \right\rangle ds. \end{aligned} \tag{4.7}$$

If $K \in (0,1)$ *(the upper bound of* ρ*) satisfies* $\ell_1 + \ell_2 := \ell < \frac{\Gamma(\alpha+1)}{3KT^\alpha}$*, then the operator* (4.5) *admits at least a couple fixed point corresponding to the moment solution of* (4.4)*.*

Proof. Our aim is to apply Theorem 3.2.
Boundedness. For $\nu, \upsilon \in \mathcal{X}$, we have

$$\begin{aligned} |\Theta(\nu, \upsilon)| \le\; & K(\|\nu\| + \|\upsilon\|) + K(\|F_1\| + \|F_2\|) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\ & + \alpha K(\|F_1\| + \|F_2\|) \int_\eta^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)}\, ds \\ & + \frac{\varepsilon^\alpha K(\|F_1\| + \|F_2\|)}{\Gamma(\alpha+1)} + \frac{(t-\eta)^\alpha K(\|F_1\| + \|F_2\|)}{\Gamma(\alpha+1)}. \\ \le\; & K(\|\nu\| + \|\upsilon\|) + K(\|F_1\| + \|F_2\|) \\ & \times \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\eta)^\alpha}{\Gamma(\alpha+1)} \Big). \end{aligned}$$

Taking the maximum norm on $t \in [0,T]$, we have

$$\|\Theta(\nu, \upsilon)\| \le \frac{4K(\|F_1\| + \|F_2\|)T^\alpha}{(1-K)\Gamma(\alpha+1)} := r, \quad K \in (0,1).$$

Hence, Θ is bounded in $\mathbb{B}_r$.

Continuity. Let $\delta > 0$ and $\nu_i, \upsilon_i \in \mathbb{B}_r$ such that $\|\nu_1 - \nu_2\| \le \delta/2$, $\|\upsilon_1 - \upsilon_2\| \le \delta/2$. Then a computation implies

$$\begin{aligned}
|\Theta(\nu_1, \upsilon_1) - \Theta(\nu_2, \upsilon_2)| &= |(\Theta_1\nu_1)(t) + (\Theta_2\upsilon_1)(t) - (\Theta_1\nu_2)(t) - (\Theta_2\upsilon_2)(t)| \\
&= |(\Theta_1\nu_1)(t) - (\Theta_1\nu_2)(t) + (\Theta_2\upsilon_1)(t) - (\Theta_2\upsilon_2)(t)| \\
&\le \|\upsilon_1 - \upsilon_2\|\ell_1 K\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\Big) \\
&+ \|\nu_1 - \nu_2\|\ell_2 K\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\Big) \\
&\le \frac{3\delta\ell_1 T^\alpha K}{2\Gamma(\alpha+1)} + \frac{3\delta\ell_2 T^\alpha K}{2\Gamma(\alpha+1)} = \frac{3\delta\ell T^\alpha K}{\Gamma(\alpha+1)} = \epsilon, \quad \ell = \ell_1 + \ell_2,
\end{aligned}$$

where

$$\delta := \frac{\Gamma(\alpha+1)\epsilon}{3T^\alpha \ell K}, \quad K \in (0,1).$$

Contractivity. Let ν and $\upsilon \in B_r$, we have

$$\begin{aligned}
&|(\Theta_2\nu)(t) - (\Theta_2\upsilon)(t)| \le \|\nu - \upsilon\|\ell_2 K\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\Big) \\
&\le \|\nu - \upsilon\|\frac{3\ell_2 K T^\alpha}{\Gamma(\alpha+1)} < \|\nu - \upsilon\|\frac{3\ell K T^\alpha}{\Gamma(\alpha+1)} < \|\nu - \upsilon\|.
\end{aligned}$$

Hence, Q_2 is a contraction mapping.

Measurement. Here, we aim to prove $\mu(\Theta(\mathbb{B}_r) \le 2\mu(\mathbb{B}_r)$. For ν and $\upsilon \in \mathbb{B}_r$, we have

$$\begin{aligned}
|\Theta(\nu_1, \upsilon_1) - \Theta(\nu_2, \upsilon_2)| &\le \|\upsilon_1 - \upsilon_2\|\ell_1 K\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\Big) \\
&+ \|\nu_1 - \nu_2\|\ell_2 K\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)}\Big) \\
&\le \Big(\|\nu_1 - \nu_2\| + \|\upsilon_1 - \upsilon_2\|\Big)\frac{3\ell T^\alpha K}{\Gamma(\alpha+1)}.
\end{aligned}$$

Then we conclude that $diam(\Theta(\mathbb{B}_r)) \le 2K_\alpha\, diam(\mathbb{B}_r)$, where for sufficient value of $0 < \ell < \frac{\|F_1\|+\|F_2\|}{1-K}$ we have $K_\alpha := \frac{3\ell T^\alpha K}{\Gamma(\alpha+1)} < r$. Consequently, $diam(\Theta(\mathbb{B}_r)) \le 2diam(\mathbb{B}_r)$. Now we consider the function $\psi : (0,\infty) \to (0,\infty)$ given as follows: $\psi(\varsigma) = \varsigma + 1$. Obviously,

$$\begin{aligned}
\psi(\mu(\Theta(\mathbb{B}_r))) &\le \psi(2K_\alpha\, \mu(\mathbb{B}_r)) = 2K_\alpha\, \mu(\mathbb{B}_r) + 1 \\
&< 2[\mu(\mathbb{B}_r) + 1], \quad K_\alpha < 1 \\
&< 2[\mu(\mathbb{B}_r) + 1](\mu(\mathbb{B}_r) + 1) = \chi[\psi(\mu(\mathbb{B}_r))]\psi[\mu(\mathbb{B}_r)],
\end{aligned}$$

where $\chi(\varsigma) = 2\varsigma$. Hence, Θ admits a coupled fixed point corresponding to the couple moment solution of (4.1). $\square$

4.1. **Numerical example.** Consider the following fractional differential equation:

$$D_t^{\alpha}\nu(t) = \frac{1}{3}\nu(t), \quad \nu(0) = \nu_0; \tag{4.8}$$

where $\nu \in \mathcal{X} = C[0,1]$. Let $Q : C[0,1] \to C[0,1]$ be given by $(Q\nu)(t) = \langle \nu(t), \rho \rangle$, where $\rho(y) = \frac{\sin(y)}{2}$. It is clear that $K = \frac{1}{2} < 1$ and $\|F\| = \frac{1}{3}$; thus for $\alpha = 0.5$, we have $\ell = \frac{1}{3} < 0.586$. In view of Theorem 4.1, we conclude that (4.8) has a solution in the mode of moments.

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