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POSITIVE PIECEWISE PSEUDO ALMOST PERIODIC SOLUTIONS OF A GENERALIZED HEMATOPOIESIS MODEL WITH HARVESTING TERMS AND IMPULSES

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Abstract. This paper is mainly concerned with a generalized hematopoiesis model with harvesting terms and impulses. Based on the contraction mapping principle and generalized Gronwall-Bellmain inequality, the new results on the existence, uniqueness and globally exponential stability of the positive piecewise pseudo almost periodic solutions of the addressed model are established. Some corresponding results in the literature can be complemented and extended. In addition, an example is given to illustrate the effectiveness of the new obtained results.

Key Words and Phrases: Positive piecewise pseudo almost periodic solution, hematopoiesis model, harvesting term, impulses, contraction mapping principle, generalized Gronwall-Bellmain inequality.
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1. Introduction

Hematopoiesis model which arisen in blood cell production, was firstly introduced and studied by Mackey and Glass [15]. Due to its applications in our daily lives, in last years, the qualitative properties for hematopoiesis model and its generalized models have been extensively investigated in literature, see for example [8], [9], [7], [18], [25], [26]. In the real world phenomena, the variation of the environment plays an important role. In this case, some researchers, such as Saker in [18], Wang and Li in [24], Yao in [27], studied the dynamic behavior of the following nonautonomous delay differential equation with time-varying coefficients:

$$\dot{x}(t) = -a(t)x(t) + \frac{b(t)}{1 + x^n(t - \tau(t))},\ t \geq 0. \tag{1.1}$$

For example, Yao in [27] studied the existence and exponential stability of the unique positive almost periodic solution of (1.1).

Meanwhile, In 1991, Gyori and Ladas [10] investigate the global attractivity of unique positive equilibrium of the following equation:

$$\dot{x}(t) = -ax(t) + \frac{b}{1 + x^n(t - \tau)},\ t \geq 0,$$

and Gyori and Ladas gave the open problem of extending the results to equations with several delays. In order to solve this problem, the following equation was come up with:

$$\dot{x}(t) = -ax(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \ n > 0, \tag{1.2}$$

and researchers have paid lots of attention on the qualitative properties of (1.2). For example, Zhang et al in [32] discussed the existence and exponential convergence of the positive almost periodic solution for (1.2), and Meng [16] studied the global exponential stability of positive pseudo-almost periodic solutions for (1.2).

Many dynamical systems have impulsive dynamical behaviors due to abrupt changes at certain instants during the evolution process. The mathematical description of these phenomena leads to impulsive differential equations and the theory of impulsive problems is experiencing a rapid development, see [5], [13], [12], [17], [23], [21], [22], [28] and the references therein. Compared with the classical hematopoiesis model or impulsive equations, the study on hematopoiesis model with impulses has been few considered in the literature, see [2], [20]. For example, Alzabut et al in [2] studied the existence and exponential stability of positive almost periodic solutions for the following impulsive hematopoiesis model of the form:

$$\begin{cases} \dot{x}(t) = -a(t)x(t) + \dfrac{b(t)}{1 + x^n(t - \tau)}, \ t \in \mathbb{R}, \ t \neq t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k^-) = \gamma_k x(t_k) + \delta_k, \ k \in \mathbb{N}, \end{cases}$$

where t_k represent the instants at which the density suffers an increment of δ_k units, $x(t_k^+)$, $x(t_k^-)$ denote the limit from right and left, respectively.

On the other hand, the existence of pseudo almost periodicity which was first treated by Zhang [29] around 1990 are the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics and physics. The concept of pseudo almost periodicity is a natural generalization of almost periodicity and the properties of the almost periodic functions do not always hold in the set of pseudo almost periodic functions. For example, the function $f(t) = \sin^2 t + \sin^4 \sqrt{11t} + \exp(-t^6 \sin^4 t)$, $t \in \mathbb{R}$ is pseudo almost periodic but not almost periodic. During the last several years, some criteria ensuring the existence and stability of pseudo almost periodic solutions have been established for the some certain and valuable problems, for more details, we refer the readers to see the references [4], [6], [14], [29], [30] and the books [20], [31].

Moreover, biologists have proposed that the process of harvesting population is of great significance in the exploitation of biological resources. However, to the best of our knowledge, little attention has been devoted to the study of positive piecewise pseudo almost periodic solutions of the hematopoiesis model with linear harvesting terms and impulses. In order to overcome this deficiency, motivated by the above works, in this paper, we study the positive piecewise pseudo almost periodic solutions for the following generalized hematopoiesis model with linear harvesting terms and impulses of the form:

$$\begin{cases} \dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{P} \frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))} - H(x(t-\delta(t))),\ t \in \mathbb{R},\ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = \gamma_k x(t_k) + I_k(x(t_k)),\ k \in \mathbb{N}, \end{cases} \tag{E}$$

where $0 \leq m < n$, $a \in C(\mathbb{R}, \mathbb{R}^+)$ is almost periodic function, $b_i \in C(\mathbb{R}, \mathbb{R}^+)$ and $\tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ are continuous and pseudo almost periodic functions for $i = 1, 2, ..., P$, $P \geq 1$ is a positive constant, $\delta \in C(\mathbb{R}, \mathbb{R}^+)$ is pseudo almost periodic function, $H \in C(\mathbb{R}^+, \mathbb{R}^+)$ is continuous and pseudo almost periodic, $\mathbb{R}^+ = (0, +\infty)$. γ_k, $k \in \mathbb{N}$ is pseudo almost periodic sequence, $\Delta(x(t_k)) = x(t_k^+) - x(t_k^-)$ are impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k \to \pm\infty} t_k = \pm\infty$. The sequence of functions $\{I_k(x)\}_{k \in \mathbb{Z}}$ is pseudo almost periodic uniform for $x \in \Omega$, where Ω is a subset of $\mathbb{R}$.

The unknown x in (E) stands for the density of mature cells in blood circulation, a is the rate of lost cells from the circulation at time t, the flux

$$f(x(t-\tau_i(t))) := \frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))}$$

of cells in the circulation depends on $x(t-\tau_i(t))$ at the time $t-\tau_i(t)$, and $\tau_i(t)$ $(i = 1, 2, ..., P)$ are time delays between the production of immature cells in the bone marrow and their maturation.

Compared with some recent results in the literature, such as [2], [16], [20], [32], the chief contributions of our study contain at least the following two:

1. The hematopoiesis model we are concerned with is more generalized, some related ones in the literature are the special cases of it. Moreover, we also extend the hematopoiesis model to the impulsive case. Thus, the generalized hematopoiesis model with linear harvesting terms and impulses are originally discussed in the present paper.
2. An innovative method based on contraction mapping principle and generalized Gronwall-Bellmain inequality is exploited to discuss the existence, uniqueness and globally exponential stability of the piecewise pseudo almost periodic solutions for the generalized hematopoiesis model with harvesting terms and impulses. The results established are essentially new.

The rest of this paper is organized as follows. In Section 2, we will give some definitions and some useful lemmas. Section 3 and 4 is devoted to establishing some criteria for the existence, uniqueness and globally exponentially stable of positive piecewise pseudo almost periodic solution for (E). Finally, in section 5, an numerical example is given to illustrate the effectiveness of the obtained results.

2. Essential definitions and lemmas

Since the solution for the (E) is a piecewise continuous functions with points of discontinuity of the first kind $t = t_k$, $k \in \mathbb{Z}$ and we adopt the following notations, definitions and lemmas for piecewise pseudo almost periodicity.

• $BC(\mathbb{R},\mathbb{R})$ (respectively, $BC(\mathbb{R}\times\Omega,\mathbb{R})$): the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (respectively, from $\mathbb{R}\times\Omega$ to $\mathbb{R}$) with the supremum norm.

• $PC(\mathbb{R},\mathbb{R})$: the space formed by all piecewise continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(\cdot)$ is continuous at t for any $t\notin\{t_k\}_{k\in\mathbb{Z}}$, $f(t_k^+)$, $f(t_k^-)$ exist, and $f(t_k^-)=f(t_k)$ for all $k\in\mathbb{Z}$.

• $PC(\mathbb{R}\times\Omega,\mathbb{R})$: the space formed by all piecewise continuous functions $f:\mathbb{R}\times\Omega\to\mathbb{R}$ such that for any $x\in\Omega$, $f(\cdot,x)\in PC(\mathbb{R},\mathbb{R})$ and for any $t\in\mathbb{R}$, $f(t,\cdot)$ is continuous at $x\in\Omega$.

Let

$$T=\left\{\{t_k\}_{k=-\infty}^{\infty}:t_k\in\mathbb{R},t_k<t_{k+1},k\in\mathbb{Z},\lim_{k\to\pm\infty}t_k=\pm\infty\right\}$$

denote the set of all sequence unbounded and strictly increasing.

Definition 2.1. [19] A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon>0$, there exists a relatively dense set of its ε-periods, *i.e.*, there exists a natural number $l=l(\varepsilon)$, such that for $k\in\mathbb{Z}$, there is at least one number p in $[k,k+l]$, for which inequality $|x_{n+p}-x_n|<\varepsilon$ holds for all $n\in\mathbb{Z}$. Denote by $AP(\mathbb{Z},\mathbb{R})$ the set of such sequences.

Definition 2.2. [20] Let $\{t_k\}\in T$, $k\in\mathbb{Z}$. We say $\{t_k^j\}$ is a derivative sequence of $\{t_k\}$ and $t_k^j=t_{j+k}-t_k,\ k,j\in\mathbb{Z}$.

Define

$$PAP_0(\mathbb{Z},\mathbb{R})=\left\{x_n\in l^\infty(\mathbb{Z},\mathbb{R}):\lim_{n\to+\infty}\frac{1}{2n}\sum_{k=-n}^{n}|x_k|=0\right\}.$$

Definition 2.3. [1] A sequence $\{x_n\}_{n\in\mathbb{Z}}\in l^\infty(\mathbb{Z},\mathbb{R})$ is called pseudo-almost periodic if $x_n=\overline{x}_n+\widehat{x}_n$, where $\overline{x}_n\in AP(\mathbb{Z},\mathbb{R})$, $\widehat{x}_n\in PAP_0(\mathbb{Z},\mathbb{R})$. Denote by $PAP(\mathbb{Z},\mathbb{R})$ the set of such sequences.

Definition 2.4. [20] A piecewise continuous function $f:\mathbb{R}\to\mathbb{R}$ with discontinuity of first kind at the points t_k is said to be almost periodic, if

(a) the set of sequence $\{t_k^j\}$, $t_k^j=t_{k+j}-t_k$, $k,\ j\in\mathbb{Z}$, $\{t_k\}\in T$ is equipotentially almost periodic.

(b) For any $\varepsilon>0$, there exists a real number $\delta>0$ such that if the points t' and t'' belong to one and the same interval of continuity of $f(t)$ and satisfy the inequality $|t'-t''|<\delta$, then $\|f(t')-f(t'')\|<\varepsilon$.

(c) for any $\varepsilon>0$, there exists a relatively dense set Ω_ε such that if $\tau\in\Omega_\varepsilon$, then $\|f(t+\tau)-f(t)\|<\varepsilon$ for all $t\in\mathbb{R}$ satisfying the condition $|t-t_k|>\varepsilon$, $k\in\mathbb{Z}$.

We denote by $AP_T(\mathbb{R},\mathbb{R})$ the space of all piecewise almost periodic functions.
Obviously, the space $AP_T(\mathbb{R},\mathbb{R})$ endowed with the norm defined by

$$\|\varphi\|=\sup_{t\in\mathbb{R}}|\varphi(t)|$$

is a Banach space. Let $UPC(\mathbb{R},\mathbb{R})$ be the space of all functions $f\in PC(\mathbb{R},\mathbb{R})$ such that f satisfies the condition (b) in Definition 2.4 .

Definition 2.5. [14] $f \in PC(\mathbb{R} \times \Omega, \mathbb{R})$ is said to be piecewise almost periodic in t uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that

$$|f(t+\tau, x) - f(t, x)| \leq \varepsilon$$

for all $x \in K$, $\tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - t_k| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times \Omega, \mathbb{R})$ the set of all such functions.

Define

$$PC_T^0(\mathbb{R}, \mathbb{R}) = \left\{ f \in PC(\mathbb{R}, \mathbb{R}) : \lim_{r \to +\infty} |f(t)| dt = 0 \right\},$$

$$PAP_T^0(\mathbb{R}, \mathbb{R}) = \left\{ f \in PC(\mathbb{R}, \mathbb{R}) : \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} |f(t)| dt = 0 \right\},$$

$$PAP_T^0(\mathbb{R} \times \Omega, \mathbb{R}) = \left\{ f \in PC(\mathbb{R} \times \Omega, \mathbb{R}) : \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} |f(t, x)| dt = 0, \text{ uniformly} \right.$$

$$\left. \text{with respect to } x \in K, \text{ where } K \text{ is an arbitrary compact subset of } \Omega \right\},$$

Definition 2.6. [14] A function $f \in PC(\mathbb{R}, \mathbb{R})$ is said to be piecewise pseudo almost periodic if it can be decomposed as $f = g + \phi$, where $g \in AP_T(\mathbb{R}, \mathbb{R})$ and $\phi \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Denote by $PAP_T(\mathbb{R}, \mathbb{R})$ the set of all such functions. $PAP_T(\mathbb{R}, \mathbb{R})$ is a Banach space when endowed with the supremum norm.

Remark 2.7. $PAP_T^0(\mathbb{R}, \mathbb{R})$ is a translation invariant set of $PC(\mathbb{R}, \mathbb{R})$ and it is easy to see that $PC_T^0(\mathbb{R}, \mathbb{R}) \subset PAP_T^0(\mathbb{R}, \mathbb{R})$.

Lemma 2.8. *Let $\{f_n\}_{n \in \mathbb{N}} \subset PAP_T^0(\mathbb{R}, \mathbb{R})$ be a sequence of functions. If $\{f_n\}$ converges uniformly to f, then $f \in PAP_T^0(\mathbb{R}, \mathbb{R})$.*

Proof. First of all, note that f is necessarily a bounded continuous function from $\mathbb{R}$ into $\mathbb{R}$. For each $n \in \mathbb{N}$, let

$$f_n = h_n + \phi_n,$$

where $\{h_n\}_{n \in \mathbb{N}} \subset AP_T(\mathbb{R}, \mathbb{R})$ and $\{\phi_n\}_{n \in \mathbb{N}} \subset PAP_T^0(\mathbb{R}, \mathbb{R})$. From [29, Lemma 1.3], we can have that $\|h_n\| \leq \|f_n\|$. Then, there exists $h \in AP_T(\mathbb{R}, \mathbb{R})$ such that

$$\|h_n - h\| \to 0 \text{ as } n \to +\infty.$$

Similarly, it easily follows that there exists a function $\phi \in BC(\mathbb{R}, \mathbb{R})$ such that

$$\|\phi_n - \phi\| \to 0 \text{ as } n \to +\infty.$$

Furthermore, for $r > 0$, we have

$$\begin{aligned} \frac{1}{2r} \int_{-r}^{r} |\phi(t)| dt \leq & \frac{1}{2r} \int_{-r}^{r} |\phi_n(t) - \phi(t)| dt + \frac{1}{2r} \int_{-r}^{r} |\phi_n(t)| dt \\ \leq & \|\phi_n - \phi\| + \frac{1}{2r} \int_{-r}^{r} |\phi_n(t)| dt, \end{aligned}$$

hence, we can obtain that $\phi \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Therefore, $f = h + \phi \in PAP_T(\mathbb{R}, \mathbb{R})$. This completes the proof. $\square$

Definition 2.9. [14] Let $PAP_T(\mathbb{R}\times\Omega,\mathbb{R})$ consist of all functions $f\in PC(\mathbb{R}\times\Omega,\mathbb{R})$ such that $f=g+\phi$, where $g\in AP_T(\mathbb{R}\times\Omega,\mathbb{R})$ and $\phi\in PAP_T^0(\mathbb{R}\times\Omega,\mathbb{R})$.

Lemma 2.10. (Generalized Gronwall-Bellmain inequality) [19] *Let a non-negative function $u(t)\in PC(\mathbb{R},\mathbb{R})$ satisfy for $t\geq t_0$,*

$$u(t)\leq C+\int_{t_0}^{t}v(\tau)u(\tau)d\tau+\sum_{t_0<t_i<t}\beta_i u(\tau_i),$$

with $C\geq 0$, $\beta_i\geq 0$, $u(\tau)\geq 0$ and τ_i are the first kind discontinuity points of the functions $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t)\leq C\prod_{t_0<t_i<t}(1+\beta_i)e^{\int_{t_0}^{t}v(\tau)d\tau}.$$

In order to establish the main results, the following assumptions are needed:

(H1) $a\in C(\mathbb{R},\mathbb{R}^+)$ is almost periodic and there exists a constant $a>0$ such that $a(t)\geq a$;

(H2) The sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ is almost periodic and $-1\leq\gamma_k\leq 0$, $k\in\mathbb{N}$;

(H3) The set of sequences $\{t_k^j\}$ are equipotentially almost periodic and there exists $\sigma>0$ such that $\sigma=\inf_{k\in\mathbb{Z}}t_k^1=\inf_{k\in\mathbb{Z}}(t_{k+1}-t_k)>0$;

(H4) $H:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ is uniformly pseudo almost periodic and there exists a constant $L^H>0$ such that

$$|H(t,x_1)-H(t,x_2)|\leq L^H|x_1-x_2|,\ \forall x_1,x_2\in\Omega,\ t\in\mathbb{R}.$$

(H5) The sequence of functions $\{I_k(x)\}_{k\in\mathbb{N}}$, $k\in\mathbb{N}$ is pseudo almost periodic uniform with respect to $x\in\Omega$, and there exist two positive constants μ and L^I such that

$$|I_k(x)|\leq\mu,\ \text{for all } x\in\Omega,\ k\in\mathbb{N};$$

and

$$|I_k(x_1)-I_k(x_2)|\leq L^I|x_1-x_2|,\ \forall x_1,x_2\in\Omega,\ k\in\mathbb{N}.$$

3. Existence and uniqueness of positive piecewise pseudo almost periodic solution

Define

$$a^+=\sup_{t\in\mathbb{R}}|a(t)|,\ b_i^+=\max_{1\leq i\leq P}\sup_{t\in\mathbb{R}}b_i(t),\ H^+=\sup_{x\in\mathbb{R}^+}|H(x)|,$$
$$a^-=\inf_{t\in\mathbb{R}}|a(t)|,\ b_i^-=\min_{1\leq i\leq P}\inf_{t\in\mathbb{R}}b_i(t),\ H^-=\sup_{x\in\mathbb{R}^+}|H(x)|.$$

Consider the following auxiliary linear equation:

$$\begin{cases}\dot{x}(t)=-a(t)x(t),\ t\in\mathbb{R},\ t\neq t_k,\\ \Delta x(t_k)=\gamma_k x(t_k),\ k\in\mathbb{N},\end{cases}\tag{3.1}$$

By [19], it is well known that (3.1) with an initial condition $x(t_0) = x_0$ has a unique solution represented by the following form:

$$x(t; t_0, x_0) = W(t, t_0)x_0, \ t_0, x_0 \in \mathbb{R},$$

where W is the Cauchy matrix of (3.1) defined as follows:

$$W(t,s) = \begin{cases} e^{-\int_s^t a(r)dr}, & t_{k-1} < s \le t \le t_k; \\ \prod_{i=m}^{k+1} (1+\gamma_i) \cdot e^{-\int_s^t a(r)dr}, & t_{m-1} < s \le t_m \le t_k < t \le t_{k+1}. \end{cases} \tag{3.2}$$

Lemma 3.1. [19] *If (H1)-(H4) are satisfied, then there exists $0 < \varepsilon_1 < \varepsilon$, relatively dense sets Λ of $\mathbb{R}$ and Q of $\mathbb{Z}$ such that the following relations are fulfilled:*
(s_1) $|a(t+\tau) - a(t)| < \varepsilon$ *for all* $t \in \mathbb{R}$, $\tau \in \Lambda$;
(s_2) $|b(t+\tau) - b(t)| < \varepsilon$ *for all* $t \in \mathbb{R}$, $\tau \in \Lambda$;
(s_3) $|e(t+\tau) - e(t)| < \varepsilon$ *for all* $t \in \mathbb{R}$, $\tau \in \Lambda$;
(s_4) $|\gamma_{k+q} - \gamma_k| < \varepsilon$ *for all* $q \in Q$ *and* $k \in \mathbb{N}$;
(s_5) $|t_k^q - \tau| < \varepsilon_1$ *for all* $\tau \in \Lambda$, $q \in Q$ *and* $k \in \mathbb{N}$.

Lemma 3.2. [3] *Let (H1)-(H3) be satisfied, then for the Cauchy matrix $W(t,s)$ of* (3.1)*, there exists a positive constant a such that*

$$|W(t,s)| \le e^{-a(t-s)}, \ t > s, \ t, s \in \mathbb{R}.$$

Lemma 3.3. [3] *Let (H1)-(H3) be satisfied, then for $\varepsilon > 0$, $t \in \mathbb{R}$, $s \in \mathbb{R}$, $t \ge s$, $|t - t_k| > \varepsilon$, $|s - t_k| > \varepsilon$, $k \in \mathbb{N}$, there exists a relatively dense set Λ of ε-almost periods of the function $a(t)$ and a positive constant M such that for $\varepsilon \in \Lambda$ it follows*

$$\Big| W(t+\tau, s+\tau) - W(t,s) \Big| \le \varepsilon M e^{-\frac{a}{2}(t-s)}.$$

Let $D \in PC(\mathbb{R}, \mathbb{R})$ denote the set of all piecewise pseudo almost periodic functions. For any $\varphi \in D$, define $\|\varphi\| = \sup_{t \in \mathbb{R}} |\varphi(t)|$. Then, D is a Banach space.

Due to the biological interpretation of model (E), only positive solutions are meaningful and therefore admissible. In order to obtain the positive solutions of (E), the initial conditions $x_{t_0} = \varphi$, $\varphi \in PC([-r, 0], \mathbb{R}^+)$ and $\varphi(0) > 0$ are needed. We write $x(t; t_0, \varphi)$ for a solution of (E) with the above initial conditions. Let $[t_0, \tau(\varphi))$ be the maximal right interval of existence of $x(t; t_0, \varphi)$. If $x(t)$ is continuous and defined on $[-r + t_0, \delta)$ with t_0, $\delta \in \mathbb{R}$, then for all $t \in [t_0, \delta)$, we define $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

Define

$$\eta_1 = \frac{\sum_{i=1}^{P} \frac{b_i^-}{1+\eta_2^n} - H^+}{a^+}, \quad \eta_2 = \frac{\sum_{i=1}^{P} b_i^+ - H^-}{a^-}. \tag{3.3}$$

Define

$$PC^0 := \big\{ \varphi | \varphi \in PC([-r, 0], \mathbb{R}), \ \eta_1 \le \varphi(t) \le \eta_2, \ t \in [-r, 0] \big\}.$$

Theorem 3.4. *Assume that* $1 < \eta_1 \le \eta_2$. *Then for* $\varphi \in PC^0$, *the solution* $x(t; t_0, \varphi)$ *of* (E) *satisfies*

$$\eta_1 \le x(t; t_0, \varphi) \le \eta_2,$$

for all $t \in [t_0, \tau(\varphi))$ *and* $\tau(\varphi) = +\infty$.

Proof. Set $x(t) = x(t; t_0, \varphi)$. Let $[0, \gamma) \subseteq [t_0, \tau(\varphi))$ be an interval such that $x(t) > 0$ for all $t \in [0, \gamma)$. Firstly, we prove

$$0 < x(t) \le \eta_2, \ t \in [t_0, \gamma). \tag{3.4}$$

By way of contradiction, if (3.4) does not hold, then there exists $\bar{t} \in [t_0, \gamma)$ such that $x(\bar{t}) = \eta_2$ and $0 < x(t) \le \eta_2$ for all $t \in [t_0 - r, \bar{t})$. Calculating the derivative of $x(t)$, we have that

$$\begin{aligned} 0 < \dot{x}(\bar{t}) &= -a(\bar{t})x(\bar{t}) + \sum_{i=1}^{P} \frac{b_i(\bar{t})x^m(t - \tau_i(\bar{t}))}{1 + x^n(\bar{t} - \tau_i(\bar{t}))} \\ &\quad - H(x(\bar{t} - \delta(\bar{t}))) \\ &< -a^-\eta_2 + \sum_{i=1}^{P} b_i^+ - H^- = 0, \end{aligned}$$

which leads to a contradiction. Then, (3.4) is satisfied.
Secondly, we claim that the following inequality holds

$$x(t) \ge \eta_1 > 0, \ t \in [t_0, \tau(\varphi)). \tag{3.5}$$

Otherwise, there exists $\underline{t} \in [t_0, \tau(\varphi))$ such that $x(\underline{t}) = \eta_1$ and $x(t) \ge \eta_1$ for all $t \in [t_0 - r, \underline{t})$. Then from (3.4), we can see that $\eta_1 \le x(t) \le \eta_2$ for all $t \in [t_0 - r, \underline{t})$. Calculating the derivative of $x(t)$, we have that

$$\begin{aligned} 0 < \dot{x}(\underline{t}) &= -a(\underline{t})x(\underline{t}) + \sum_{i=1}^{P} \frac{b_i(\underline{t})x^m(\underline{t} - \tau_i(\underline{t}))}{1 + x^n(\underline{t} - \tau_i(\underline{t}))} \\ &\quad - H(x(\underline{t} - \delta(\underline{t}))) \\ &> -a^+\eta_1 + \sum_{i=1}^{P} \frac{b_i^-}{1 + \eta_2^n} - H^+ = 0, \end{aligned}$$

which also leads to a contradiction. Then, (3.5) is satisfied. Furthermore, By Theorem 2.3.1 in [11], we have that $\tau(\varphi) = +\infty$. Therefore, the proof is completed. □

Theorem 3.5. *Assume that (H1)-(H4) hold and the following condition is satisfied*

(H6) $\dfrac{\sum_{i=1}^{P} b_i^+ \eta_2^m n}{4\eta_1 a} + \dfrac{L^H}{a} + \dfrac{L^I}{1 - e^{-a\sigma}} < 1$, *where* η_1 *and* η_2 *are defined in* (3.3).

Then, (E) *possesses a unique piecewise pseudo almost periodic solution.*

Proof. Let $D \in PC(\mathbb{R},\mathbb{R})$ denote the set of all piecewise pseudo almost periodic functions. Define

$$D^* = \big\{\varphi|\varphi \in D,\ \eta_1 \le \varphi(t) \le \eta_2,\ \forall t \in \mathbb{R}\big\}.$$

where η_1 and η_2 are defined in (3.3).

Define the operator T in D^* by

$$(T\varphi)(t) = \int_{-\infty}^{t} W(t,s)g_\varphi(s)ds + \sum_{t_k<t} W(t,t_k)I_k(\varphi(t_k)), \tag{3.6}$$

where

$$g_\varphi(s) = \sum_{i=1}^{P} \frac{b_i(t)\varphi^m(t-\tau_i(t))}{1+\varphi^n(t-\tau_i(t))} - H(\varphi(t-\delta(t))). \tag{3.7}$$

We shall prove that T is a contraction mapping on D^*. Then by the contraction mapping principle, T has a unique fixed point, say x^*, in D^*. By (3.1), x^* also satisfies (E) and hence x^* is the unique piecewise pseudo almost periodic solution of (E) in D^*.

We divide the proof into the following three steps.

Step 1. Firstly, we show that T is a self-mapping in D^*. Note that for $t_j \le t < t_{j+1}$, $j \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{t_k<t} e^{-a(t-t_k)} &\le \sum_{-\infty<k\le j} e^{-a(j-k)\sigma}\\ &= \sum_{0\le m=j-k<+\infty} e^{-am\sigma} = \frac{1}{1-e^{-a\sigma}}.\end{aligned} \tag{3.8}$$

For $\varphi \in D^*$, by (H5), (H6) and Lemma 3.2, we can get

$$\begin{aligned}\|T\varphi\| &= \sup_{t\in\mathbb{R}}\Bigg\{\int_{-\infty}^{t} W(t,s)\Bigg(\sum_{i=1}^{P}\frac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))} - H(x(t-\delta(t)))\Bigg)ds\\ &\quad + \sum_{t_k<t} W(t,t_k)I_k(\varphi(t_k))\Bigg\}\\ &\le \sup_{t\in\mathbb{R}}\Bigg\{\int_{-\infty}^{t}|W(t,s)|\Bigg(\sum_{i=1}^{P} b_i^+ - H^-\Bigg)ds + \sum_{t_k<t}|W(t,t_k)||I_k(\varphi(t_k))|\Bigg\}\\ &\le \sup_{t\in\mathbb{R}}\Bigg\{\int_{-\infty}^{t} e^{-a(t-s)}\Bigg(\sum_{i=1}^{P} b_i^+ - H^-\Bigg)ds + \sum_{t_k<t} e^{-a(t-t_k)}\mu\Bigg\}\\ &\le \frac{\sum_{i=1}^{P} b_i^+ - H^-}{a} + \frac{\mu}{1-e^{-a\sigma}} < +\infty.\end{aligned}$$

For $\varphi \in D$, by (3.6), it is not difficult to see that $T\varphi \in UPC(\mathbb{R},\mathbb{R})$.

Let $\vartheta_k = I_k(\varphi(t_k))$, then $\vartheta_k \in AP(\mathbb{Z},\mathbb{R})$. Let $\tau \in \Lambda$, $q \in Q$, where Λ, Q are defined in Lemma 3.1. Then, for $t_k < t \le t_{k+1}$, we have

$$
\begin{aligned}
\|(T\varphi)(t+\tau)-(T\varphi)(t)\| &\le \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}|W(t+\tau,s+\tau)-W(t,s)||g_\varphi(s+\tau)|ds\right\}\\
&+\sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}|W(t,s)||g_\varphi(s+\tau)-g_\varphi(s)|ds\right\}\\
&+\sup_{t\in\mathbb{R}}\left\{\sum_{t_k<t}|W(t+\tau,t_{k+q})-W(t,t_k)||\vartheta_{k+q}|\right\}\\
&+\sup_{t\in\mathbb{R}}\left\{\sum_{t_k<t}|W(t,t_k)||\vartheta_{k+q}-\vartheta_k|\right\}\\
&:=\Phi_1+\Phi_2+\Phi_3+\Phi_4.
\end{aligned}
\tag{3.9}
$$

Furthermore, it follows from Lemma 3.3 that

$$
\begin{aligned}
\Phi_1 &= \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}|W(t+\tau,s+\tau)-W(t,s)||g_\varphi(s+\tau)|ds\right\}\\
&\le \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}\varepsilon M e^{-\frac{a}{2}(t-s)}\cdot\left|\sum_{i=1}^{P}\frac{b_i(s+\tau)\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))}\right.\right.\\
&\left.\left.-H(\varphi(s+\tau-\delta(s+\tau)))\right|ds\right\}\\
&\le \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}\varepsilon M e^{-\frac{a}{2}(t-s)}\cdot\left(\sum_{i=1}^{P}b_i^+ + H^+\right)ds\right\}\\
&\le \frac{2M\varepsilon}{a}\left(\sum_{i=1}^{P}b_i^+ + H^+\right).
\end{aligned}
\tag{3.10}
$$

From (3.7) and (H4), we have

$$
\begin{aligned}
|g_\varphi(s+\tau)-g_\varphi(s)| &= \left|\sum_{i=1}^{P}\frac{b_i(s+\tau)\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))}-\sum_{i=1}^{P}\frac{b_i(s)\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))}\right|\\
&+\Big|H(\varphi(s+\tau-\delta(s+\tau)))-H(\varphi(s-\delta(s)))\Big|\\
&\le \left|\sum_{i=1}^{P}\frac{b_i(s+\tau)\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))}-\sum_{i=1}^{P}\frac{b_i(s)\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))}\right|\\
&+\left|\sum_{i=1}^{P}\frac{b_i(s)\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))}-\sum_{i=1}^{P}\frac{b_i(s)\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))}\right|\\
&+\Big|H(\varphi(s+\tau-\delta(s+\tau)))-H(\varphi(s-\delta(s)))\Big|
\end{aligned}
$$

$$\leq \sum_{i=1}^{P} \Big| b_i(s+\tau) - b_i(s) \Big| + \sum_{i=1}^{P} |b_i(s)| \left| \frac{\varphi^m(s+\tau-\tau_i(s+\tau))}{1+\varphi^n(s+\tau-\tau_i(s+\tau))} - \frac{\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))} \right|$$
$$+ L^H \Big| \varphi(s+\tau-\delta(s+\tau)) - \varphi(s-\delta(s)) \Big|,$$

then, it follows from Lemma 3.1 that

$$\begin{aligned} |g_\varphi(s+\tau) - g_\varphi(s)| \leq & P\varepsilon + \sum_{i=1}^{P} b_i^+ \eta_2^m \left| \frac{1}{1+\varphi^n(s+\tau-\tau_i(s+\tau))} - \frac{1}{1+\varphi^n(s-\tau_i(s))} \right| \\ & + L^H \varepsilon \\ \leq & P\varepsilon + \sum_{i=1}^{P} b_i^+ \eta_2^m \cdot \frac{n\rho^{n-1}}{(1+\rho^n)^2} \Big| \varphi(s+\tau-\tau_i(s+\tau)) - \varphi(s-\tau_i(s)) \Big| \\ & + L^H \varepsilon \\ \leq & P\varepsilon + \sum_{i=1}^{P} b_i^+ \eta_2^m \cdot \frac{n\rho^{n-1}}{(2\sqrt{\rho^n})^2} \varepsilon + L^H \varepsilon \\ \leq & \varepsilon \left(P + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right), \end{aligned}$$

where $\varphi(s+\tau-\tau_i(s+\tau))$, $\varphi(s-\tau_i(s)) \in D^*$, ρ lies between $\varphi(s+\tau-\tau_i(s+\tau))$ and $\varphi(s-\tau_i(s))$. Then, by Lemma 3.2, we get

$$\begin{aligned} \Phi_2 = & \sup_{t\in\mathbb{R}} \left\{ \int_{-\infty}^{t} |W(t,s)| |g_\varphi(s+\tau) - g_\varphi(s)| ds \right\} \\ \leq & \sup_{t\in\mathbb{R}} \left\{ \int_{-\infty}^{t} e^{-a(t-s)} \cdot |g_\varphi(s+\tau) - g_\varphi(s)| ds \right\} \\ \leq & \frac{\varepsilon}{a} \left(P + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right). \end{aligned} \tag{3.11}$$

From Lemma 3.1-3.3 and (H4), it follows that

$$\begin{aligned} \Phi_3 = & \sup_{t\in\mathbb{R}} \left\{ \sum_{t_k<t} |W(t+\tau, t_{k+q}) - W(t,t_k)| |\vartheta_{k+q}| \right\} \\ \leq & \sup_{t\in\mathbb{R}} \left\{ \sum_{t_k<t} |W(t+\tau, t_{k+q}) - W(t,t_k)| \cdot \mu \right\} \\ \leq & \frac{M\mu\varepsilon}{1-e^{-a\sigma/2}}, \end{aligned} \tag{3.12}$$

and

$$\Phi_4 = \sup_{t\in\mathbb{R}} \left\{ \sum_{t_k<t} |W(t,t_k)| |\vartheta_{k+q} - \vartheta_k| \right\} \leq \frac{\varepsilon}{1-e^{-a\sigma}}. \tag{3.13}$$

Substituting (3.10), (3.11), (3.12) and (3.13) into (3.9), we can see that

$$\|(T\varphi)(t+\tau)-(T\varphi)(t)\| \le \varepsilon\left[\frac{2M}{a}\left(\sum_{i=1}^{P} b_i^{+}+H^{+}\right)+\frac{1}{a}\left(P+\sum_{i=1}^{P}\frac{b_i^{+}\eta_2^{m}n}{4\eta_1}+L^{H}\right)\right.$$
$$\left.+\frac{M\mu}{1-e^{-a\sigma/2}}+\frac{1}{1-e^{-a\sigma}}\right].$$

This implies that $T\varphi \in D$.

Step 2. Secondly, we show that T is a self-mapping from $PAP_T(\mathbb{R},\mathbb{R})$ to $PAP_T(\mathbb{R},\mathbb{R})$. It is easy to see that $g_\varphi \in PAP_T(\mathbb{R},\mathbb{R})$. Let

$$g_\varphi = \overline{g}_\varphi + \widehat{g}_\varphi,$$

where $\overline{g}_\varphi \in AP_T(\mathbb{R},\mathbb{R})$ and $\widehat{g}_\varphi \in PAP_T^0(\mathbb{R},\mathbb{R})$. Since $\vartheta_k = I_k(\varphi(t_k)) \in PAP(\mathbb{Z},\mathbb{R})$, let $\vartheta_k = \overline{\vartheta}_k + \widehat{\vartheta}_k$, where $\overline{\vartheta}_k$ and $\widehat{\vartheta}_k$. Then, we can have

$$T\varphi = T_1\varphi + T_2\varphi,$$

where

$$T_1\varphi = \int_{-\infty}^{t} W(t,s)\overline{g}_\varphi(s)ds + \sum_{t_k<t} W(t,t_k)\overline{\vartheta}_k,$$

$$T_2\varphi = \int_{-\infty}^{t} W(t,s)\widehat{g}_\varphi(s)ds + \sum_{t_k<t} W(t,t_k)\widehat{\vartheta}_k.$$

Similar as the previous proof in Step 1, we can see that

$$T_1\varphi \in AP_T(\mathbb{R},\mathbb{R}). \tag{3.14}$$

Moreover, for $r>0$, by Lemma 3.2, we have

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\left\|\int_{-\infty}^{t} W(t,s)\widehat{g}_\varphi(s)ds\right\|dt &\le \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} e^{-a(t-s)}\|\widehat{g}_\varphi(s)\|dsdt\\ &= \frac{1}{2r}\int_{-r}^{r}\int_{0}^{+\infty} e^{-as}\|\widehat{g}_\varphi(t-s)\|dsdt\\ &= \int_{0}^{+\infty} e^{-as}\Phi(s)ds,\end{aligned}$$

where

$$\Phi(s) = \frac{1}{2r}\int_{-r}^{r}\|\widehat{g}_\varphi(t-s)\|dt.$$

Since $\widehat{g}_\varphi \in PAP_T^0(\mathbb{R},\mathbb{R})$, it follows that $\widehat{g}(\cdot - s) \in PAP_T^0(\mathbb{R},\mathbb{R})$ for each $s\in\mathbb{R}$. Thus, we have

$$\lim_{r\to+\infty}\Phi(s) = 0, \quad \text{for all } s\in\mathbb{R}.$$

By applying the Lebesgue dominated convergence theorem, we can get

$$\int_{-\infty}^{t} W(t,s)\widehat{g}_{\varphi}(s)ds \in PAP_T^0(\mathbb{R},\mathbb{R}). \tag{3.15}$$

For a given $k \in \mathbb{Z}$, define

$$\Psi(t) = W(t,t_k)\widehat{\vartheta}_k,\ t_k < t \le t_{k+1},$$

then, by Lemma 3.2, we have that

$$\lim_{t\to+\infty} |\Psi(t)| = \lim_{t\to+\infty} |W(t,t_k)||\widehat{\vartheta}_k| \le \lim_{t\to+\infty} e^{-a(t-s)}|\widehat{\vartheta}_k| = 0,$$

which implies that $\Psi(t) \in PC_T^0(\mathbb{R},\mathbb{R})$, and it follows from Remark 2.7 that $\Psi(t) \in PAP_T^0(\mathbb{R},\mathbb{R})$. Furthermore, define $\Psi_m : \mathbb{R} \to \mathbb{R}$ by

$$\Psi_m(t) = W(t,t_{k-m})\widehat{\vartheta}_{k-m},\ m \in \mathbb{N}^+,\ t_k < t \le t_{k+1}.$$

Obviously, $\Psi_m \in PAP_T^0(\mathbb{R},\mathbb{R})$. Then, It follows from Lemma 3.2 that

$$|\Psi_m(t)| = \Big|W(t,t_{k-m})\vartheta_{k-m}^2\Big| \le \sup_{k\in\mathbb{Z}}\Big\{|\vartheta_k^2| \cdot e^{-a(t-t_k)} \cdot e^{-a\sigma m}\Big\}.$$

Thus, we can see that the series $\sum_{m=1}^{\infty} \Psi_m$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.8, we obtain

$$\sum_{t_k<t} W(t,t_k)\widehat{\vartheta}_k \in PAP_T^0(\mathbb{R},\mathbb{R}). \tag{3.16}$$

From (3.14), (3.15) and (3.16), we can see that $T\varphi \in PAP_T(\mathbb{R},\mathbb{R})$. Therefore T is a self-mapping from $PAP_T(\mathbb{R},\mathbb{R})$ to $PAP_T(\mathbb{R},\mathbb{R})$.
Step 3. We show that T is a contraction mapping in D. For any φ, $\psi \in D$,

$$\begin{aligned}
|T\varphi - T\psi| &\le \int_{-\infty}^{t} |W(t,s)||g_\varphi(s) - g_\psi(s)|ds + \sum_{t_k<t} |W(t,t_k)||I_k(\varphi(t_k)) - I_k(\psi(t_k))| \\
&\le \int_{-\infty}^{t} |W(t,s)|\Bigg|\sum_{i=1}^{P} \frac{b_i(s)\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))} - H(\varphi(s-\delta(s))) \\
&- \sum_{i=1}^{P} \frac{b_i(s)\psi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))} - H(\psi(s-\delta(s)))\Bigg|ds \\
&+ \sum_{t_k<t} |W(t,t_k)|\big|I_k(\varphi(t_k)) - I_k(\psi(t_k))\big| \\
&\le \int_{-\infty}^{t} |W(t,s)|\Bigg|\sum_{i=1}^{P} \frac{b_i(s)\varphi^m(s-\tau_i(s))}{1+\varphi^n(s-\tau_i(s))} - \sum_{i=1}^{P} \frac{b_i(s)\psi^m(s-\tau_i(s))}{1+\psi^n(s-\tau_i(s))}\Bigg| \\
&+ \int_{-\infty}^{t} |W(t,s)|\Big|H(\varphi(s-\delta(s))) - H(\psi(s-\delta(s)))\Big|ds \\
&+ \sum_{t_k<t} |W(t,t_k)|\big|I_k(\varphi(t_k)) - I_k(\psi(t_k))\big|,
\end{aligned}$$

which together with (H4), (H5) and Lemma 3.2 yields

$$
\begin{aligned}
|T\varphi - T\psi| \leq & \int_{-\infty}^{t} e^{-a(t-s)} \cdot \sum_{i=1}^{P} b_i^+ \eta_2^m \cdot \frac{n\omega^{n-1}}{(1+\omega^n)^2} \Big|\varphi(s-\tau_i(s)) - \psi(s-\tau_i(s))\Big| \\
& + \int_{-\infty}^{t} e^{-a(t-s)} \cdot L^H \cdot \Big|\varphi(s-\delta(s)) - \psi(s-\delta(s))\Big| ds \\
& + \sum_{t_k<t} e^{-a(t-t_k)} \cdot L^I \cdot \Big|\varphi(t_k) - \psi(t_k)\Big| \\
\leq & \int_{-\infty}^{t} e^{-a(t-s)} \cdot \sum_{i=1}^{P} b_i^+ \eta_2^m \cdot \frac{n}{4\eta_1} \Big|\varphi(s-\tau_i(s)) - \psi(s-\tau_i(s))\Big| \\
& + \int_{-\infty}^{t} e^{-a(t-s)} \cdot L^H \cdot \Big|\varphi(s-\delta(s)) - \psi(s-\delta(s))\Big| ds \\
& + \sum_{t_k<t} e^{-a(t-t_k)} \cdot L^I \cdot \Big|\varphi(t_k) - \psi(t_k)\Big| \\
\leq & \left[\frac{\sum_{i=1}^{P} b_i^+ \eta_2^m n}{4\eta_1 a} + \frac{L^H}{a} + \frac{L^I}{1-e^{-a\sigma}}\right] \cdot \|\varphi - \psi\|,
\end{aligned}
$$

where $\varphi(s-\delta(s))$, $\psi(s-\delta(s)) \in D^*$, ω lies between $\varphi(s-\delta(s))$ and $\psi(s-\delta(s))$. Thus, we can obtain

$$
\|T\varphi - T\psi\| \leq \left[\frac{\sum_{i=1}^{P} b_i^+ \eta_2^m n}{4\eta_1 a} + \frac{L^H}{a} + \frac{L^I}{1-e^{-a\sigma}}\right] \cdot \|\varphi - \psi\|.
$$

It follows from (H6) that the mapping T is a contraction.

Hence, the mapping T possesses a unique fixed point $x^* \in D$ with $Tx^* = x^*$. By (3.1), x^* satisfies (E). Therefore, we can conclude that (E) possesses a unique piecewise pseudo almost periodic solution $x^* \in PAP_T(\mathbb{R}, \mathbb{R})$. □

4. Globally exponentially stable of piecewise pseudo almost periodic solution

In order to discuss the exponential stability of the unique positive piecewise pseudo almost periodic solutions, firstly, together with (E), we consider the following equation with initial condition:

$$
\begin{cases}
\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{P} \dfrac{b_i(t)x^m(t-\tau_i(t))}{1+x^n(t-\tau_i(t))} - H(x(t-\delta(t))), \ t \in \mathbb{R}, \ t \neq t_k, \\
\Delta x(t_k) = \gamma_k x(t_k) + I_k(x(t_k)), \ k \in \mathbb{N}, \\
x_{t_0} = \varphi,
\end{cases} \tag{4.1}
$$

where $\varphi \in PC([-r, 0], \mathbb{R}^+)$.

Theorem 4.1. *Assume that the assumptions of Theorem 3.4 hold and*

(H7) $\dfrac{\ln(1+L^I)}{\sigma}+\sum_{i=1}^{P}\dfrac{b_i^+\eta_2^m n}{4\eta_1}+L^H < a$, *where* η_1 *and* η_2 *is defined in Theorem 3.4.*

Then the unique piecewise pseudo almost periodic solution of (E) *is exponential stabile.*

Proof. By using integral form of (E), if $t > t_0$, $t_0 \neq t_k$, $k \in \mathbb{Z}$,

$$\begin{aligned} x(t) =& W(t,t_0)x(t_0)+\int_{t_0}^{t} W(t,s)g_x(s)ds \\ &+\sum_{t_0<t_k<t} W(t,t_k)\left(I_k(x(t_k))+\delta_k\right). \end{aligned}$$

Let $x(t)$ be the unique piecewise pseudo almost periodic solution of system (4.1) and $y(t)$ be an arbitrary solution of system (E) with the following initial condition:

$$x_{t_0}=\xi,\ \xi\in PC([-r,0],\mathbb{R}^+).$$

Then for x, $y \in D^*$, we have

$$\begin{aligned} y(t)-x(t) =& W(t,t_0)(\varphi-\xi)+\int_{t_0}^{t} W(t,s)\big(g_y(s)-g_x(s)\big)ds \\ &+\sum_{t_0<t_k<t} W(t,t_k)\big(I_k(y(t_k))-I_k(x(t_k))\big), \end{aligned}$$

which together with Lemma 3.2, (H4) and (H5) gives

$$\begin{aligned} \|y(t)-x(t)\| \leq& |W(t,t_0)||\varphi-\xi|+\int_{t_0}^{t}|W(t,s)||g_y(s)-g_x(s)|ds \\ &+\sum_{t_0<t_k<t}|W(t,t_k)|\big|I_k(y(t_k))-I_k(x(t_k))\big| \\ \leq& e^{-a(t-t_0)}\cdot\|\varphi-\xi\| \\ &+\int_{t_0}^{t} e^{-a(t-s)}\cdot\left|\sum_{i=1}^{P}\frac{b_i(s)y^m(s-\tau_i(s))}{1+y^n(s-\tau_i(s))}-H(y(s-\delta(s)))\right. \\ &\left.-\sum_{i=1}^{P}\frac{b_i(s)x^m(s-\tau_i(s))}{1+x^n(s-\tau_i(s))}-H(x(s-\delta(s)))\right| \\ &+\sum_{t_0<t_k<t}|W(t,t_k)|\big|I_k(y(t_k))-I_k(x(t_k))\big|, \end{aligned}$$

i.e.,

$$
\begin{aligned}
\|y(t)-x(t)\| \le\, & e^{-a(t-t_0)}\cdot\|\varphi-\xi\| \\
& + \int_{t_0}^{t} e^{-a(t-s)}\cdot\sum_{i=1}^{P} b_i^{+}\eta_2^{m}\cdot\frac{n\omega^{n-1}}{(1+\omega^{n})^2}\Big|y(s-\tau_i(s))-x(s-\tau_i(s))\Big| \\
& + \int_{-\infty}^{t} e^{-a(t-s)}\cdot L^{H}\cdot\Big|y(s-\delta(s))-x(s-\delta(s))\Big|ds \\
& + \sum_{t_0<t_k<t} e^{-a(t-t_k)}\cdot L^{I}\cdot\Big|y(t_k)-x(t_k)\Big| \\
\le\, & e^{-a(t-t_0)}\cdot\|\varphi-\xi\| \\
& + \int_{t_0}^{t} e^{-a(t-s)}\cdot\sum_{i=1}^{P}\frac{b_i^{+}\eta_2^{m}n}{4\eta_1}\cdot\Big|y(s-\tau_i(s))-x(s-\tau_i(s))\Big|ds \\
& + \int_{t_0}^{t} e^{-a(t-s)}\cdot L^{H}\cdot\Big|y(s-\delta(s))-x(s-\delta(s))\Big|ds \\
& + \sum_{t_0<t_k<t} e^{-a(t-t_k)}\cdot L^{I}\cdot\Big|y(t_k)-x(t_k)\Big|. \qquad (4.2)
\end{aligned}
$$

Multiplying the both side of (4.2) by $e^{a(t)}$, we have

$$
\begin{aligned}
e^{at}\cdot\|y(t)-x(t)\| \le\, & e^{at}\cdot e^{-a(t-t_0)}\cdot\|\varphi-\xi\| \\
& + \int_{-\infty}^{t} e^{at}\cdot e^{-a(t-s)}\cdot\sum_{i=1}^{P}\frac{b_i^{+}\eta_2^{m}n}{4\eta_1}\cdot\Big|y(s-\tau_i(s))-x(s-\tau_i(s))\Big|ds \\
& + \int_{-\infty}^{t} e^{at}\cdot e^{-a(t-s)}\cdot L^{H}\cdot\Big|y(s-\delta(s))-x(s-\delta(s))\Big|ds \\
& + \sum_{t_0<t_k<t} e^{at}\cdot e^{-a(t-t_k)}\cdot L^{I}\cdot\Big|y(t_k)-x(t_k)\Big| \\
\le\, & e^{at_0}\cdot\|\varphi-\xi\| + \int_{t_0}^{t} e^{as}\cdot\left(\sum_{i=1}^{P}\frac{b_i^{+}\eta_2^{m}n}{4\eta_1}+L^{H}\right)\cdot\Big\|y(s)-x(s)\Big\|ds \\
& + \sum_{t_0<t_k<t} e^{at_k}\cdot L^{I}\cdot\Big\|y(t_k)-x(t_k)\Big\|. \qquad (4.3)
\end{aligned}
$$

Let $u(t)=\|y(t)-x(t)\|\cdot e^{at}$, then (4.3) can be rewritten by the following form:

$$
u(t) \le u(t_0) + \int_{t_0}^{t}\left(\sum_{i=1}^{P}\frac{b_i^{+}\eta_2^{m}n}{4\eta_1}+L^{H}\right)\cdot u(s)ds + \sum_{t_0<t_k<t} L^{I}u(t_k). \qquad (4.4)
$$

Thus, let (4.4) compare with the two inequalities in Lemma 2.10, we can see that

$$\begin{aligned} u(t) \leq & u(t_0) \prod_{t_0<t_k<t} (1+L^I) \cdot \exp\left(\int_{t_0}^{t} \left(\sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) ds \right) \\ = & u(t_0) \prod_{t_0<t_k<t} (1+L^I) \cdot \exp\left(\left(\sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right). \end{aligned}$$

From (H3), we can know that $\sigma = \inf_{k\in\mathbb{Z}} t_k^1 = \inf\limits_{k\in\mathbb{Z}} (t_{k+1} - t_k) > 0$, then we have

$$\begin{aligned} u(t) \leq & u(t_0) \cdot \prod_{t_0<t_k<t} (1+L^I) \cdot \exp\left(\left(\sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right) \\ \leq & u(t_0) \cdot (1+L^I)^{\frac{t-t_0}{\sigma}} \cdot \exp\left(\left(\sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right) \\ = & u(t_0) \cdot \exp\left(\frac{\ln(1+L^I)}{\sigma} (t-t_0) \right) \cdot \exp\left(\left(\sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right) \\ = & u(t_0) \cdot \exp\left(\left(\frac{\ln(1+L^I)}{\sigma} + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \|y(t) - x(t)\| = & u(t) \cdot e^{-at} \\ \leq & u(t_0) \cdot \exp\left(\left(\frac{\ln(1+L^I)}{\sigma} + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H \right) (t-t_0) \right) \cdot e^{-at} \\ \leq & \|\varphi - \xi\| \cdot \exp\left(\left(\frac{\ln(1+L^I)}{\sigma} + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H - a \right) (t-t_0) \right), \end{aligned}$$

which together with $\frac{\ln(1+L^I)}{\sigma} + \sum_{i=1}^{P} \frac{b_i^+ \eta_2^m n}{4\eta_1} + L^H < a$ yields the the unique piecewise pseudo almost periodic solution of (E) is exponential stable. □

5. Example

In this section, we present an example to demonstrate the main established results.

Example 5.1. Consider the following hematopoiesis model with harvesting terms and impulses:

$$\begin{cases} \dot{x}(t) & = -\frac{|\sin\sqrt{2}t|+1}{24} x(t) + \frac{1}{4} \cdot \frac{x^{\frac{1}{20}}(t-e^{\sin t})}{1+x^{\frac{1}{10}}(t-e^{\sin t})} \\ & \quad + \frac{|\sin\sqrt{2}t|+1}{20} \cdot \frac{x^{\frac{1}{20}}(t-e^{\cos t})}{1+x^{\frac{1}{10}}(t-e^{\cos t})} \\ & \quad - \frac{\sin x(t-\sin\sqrt{2}t)+1}{8}, \ t \in \mathbb{R}, \ t \neq t_k, \\ \Delta x(t_k) & = -\frac{|\sin k|+|\sin \pi k|}{4} \cdot x(t_k) + \frac{|\sin k|}{40} \cdot \cos(x(t_k)). \end{cases} \tag{5.1}$$

Problem (5.1) can be regarded as a problem of the form (E), where

$$P=2,\ a(t)=\frac{|\sin\sqrt{2}t|+1}{12},\ b_1(t)=\frac{1}{4},\ b_2(t)=\frac{|\sin\sqrt{2}t|+1}{6},$$
$$H(x)=\frac{\sin x+2}{60},\ \tau_1(t)=e^{\sin t},\ \tau_2(t)=e^{\cos t},$$
$$\gamma_k=-\frac{|\sin k|+|\sin \pi k|}{4}\in AP(\mathbb{Z},\mathbb{R}),\ t_k=k+\frac{|\sin k-\cos k|}{8},$$
$$I_k(x)=\frac{|\sin k|}{40}\cdot\cos x\in PAP(\mathbb{Z},\mathbb{R}).$$

Then, we can see that

$$m=\frac{1}{20},\ n=\frac{1}{10},\ -1\le\gamma_k\le 0,\ a^+=\frac{1}{6},\ a^-=\frac{1}{12},$$
$$b_1^+=b_1^-=\frac{1}{4},\ b_2^+=\frac{1}{3},\ b_2^-=\frac{1}{6},\ H^+=\frac{1}{20},\ H^-=\frac{1}{30},$$

and conditions (H1), (H2), (H4) and (H5) are satisfied with

$$a^-=a=\frac{1}{12},\ \mu=\frac{1}{40},\ L^H=\frac{1}{60},\ L^I=\frac{1}{40}.$$

Moreover, $\{t_k^j\}$, $k\in\mathbb{Z}$, $j\in\mathbb{Z}$ are equipotentially almost periodic and

$$\begin{aligned}
t_k^1 =&k+1+\frac{1}{8}\Big|\sin(k+1)-(k+1)\Big|-\left(k+\frac{1}{8}\Big|\sin k-\cos k\Big|\right)\\
\ge&1-\frac{1}{8}\Big|\sin(k+1)-\sin k\big]-\big[\cos(k+1)-\cos k\big]\Big|\\
\ge&1-\frac{1}{4}\left|\sin\frac{1}{2}\cos\frac{2k+1}{2}\right|-\frac{1}{4}\left|\sin\frac{1}{2}\sin\frac{(2k+1)}{2}\right|\\
\ge&1-\frac{1}{4}\sin\frac{1}{2}-\frac{1}{4}\sin\frac{1}{2}>\frac{4}{5},
\end{aligned}$$

which leads to

$$\sigma=\inf_{k\in\mathbb{Z}}t_k^1=\inf_{k\in\mathbb{Z}}(t_{k+1}-t_k)=\frac{4}{5}>0,$$

then the condition (H3) holds. Furthermore, by a simple calculation, we have

$$\eta_1=\frac{\sum_{i=1}^P\frac{b_i^-}{1+\eta_2^n}-H^+}{a^+}\approx 1.8324069,\ \eta_2=\frac{\sum_{i=1}^P b_i^+-H^-}{a^-}\approx 6.6,$$
$$\frac{\sum_{i=1}^P b_i^+\eta_2^m n}{4\eta_1 a}+\frac{L^H}{a}+\frac{L^I}{1-e^{-a\sigma}}\approx 0.9846429<1,$$
$$\frac{\ln(1+L^I)}{\sigma}+\sum_{i=1}^P\frac{b_i^+\eta_2^m n}{4\eta_1}+L^H\approx 0.06678543<a=\frac{1}{12},$$

hence, conditions (H6) and (H7) can be easily satisfied. Therefore, by Theorem 3.5 and Theorem 4.1, we can see that (5.1) has a unique positive piecewise pseudo almost periodic solution and the solution is globally exponentially stable.

Remark 5.2. Since there are few paper consider positive piecewise pseudo almost periodic solutions of the generalized hematopoiesis model with harvesting terms and impulses. One can see that all the results in [1]-[31] can not directly be applicable to (5.1) to obtain the existence, uniqueness and globally exponentially stable of the positive piecewise pseudo almost periodic solution. These implies that the results in this paper are essentially new.

6. Conclusion

In this paper, we investigate generalized hematopoiesis model with harvesting terms and impulses, which are more generalized and different from the corresponding ones known in the literature. The results on the existence and uniqueness of positive piecewise pseudo almost periodic solution have been completely established by means of the contraction mapping principle, the global exponential stability of pseudo almost periodic solutions are further obtained by applying the generalized Gronwall-Bellmain inequality. Our results can improve and extend previous works in the literature.

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