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ABSTRACT MEASURES OF NONCOMPACTNESS AND FIXED POINTS FOR NONLINEAR MAPPINGS

DAVID ARIZA-RUIZ* AND JESÚS GARCIA-FALSET**

*Departamento de Análisis Matemático, Universidad de Sevilla,
Apdo. 1160, 41080-Sevilla, Spain
E-mail: dariza@us.es

**Departament d'Anàlisi Matemàtica, Universitat de València,
Dr. Moliner 50, 46100, Burjassot, València, Spain
E-mail: garciaf@uv.es
Corresponding author

Abstract. In this paper, we study the existence of fixed points for a mapping by using abstract measures of noncompactness. Thus, we can obtain some generalizations of Darbo and Sadovskiĭ's theorems and we also give a characterization for the existence of fixed points of a mapping which is not necessarily continuous. Finally, we solve an open problem proposed by I.A. Rus in 2001.
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1. Introduction

From a mathematical point of view, many problems arising from diverse areas of natural science involve the existence of solutions of nonlinear equations with the form

$$Tu = u, \ u \in C, \tag{1.1}$$

where C is a closed convex subset of a Banach space $\mathbb{B}$, and $T : C \to \mathbb{B}$ is a nonlinear mapping. Fixed point theory plays an important role in order to solve (1.1). In this theory one of the most important results was obtained by Brouwer [8] in 1912: *Every continuous mapping from a convex compact subset of $\mathbb{R}^n$ into itself has a fixed point.* A generalization of Brouwer's theorem was given by Schauder [22] in 1930: *Every continuous and compact self-mapping from a closed, convex and bounded subset of a Banach space has a fixed point.* The condition of Schauder's theorem involves continuity and compactness. Since the infinite dimensional Banach spaces are not locally compact, this condition seems quite strong. By this reason it is interesting to know how far is a set to be compact, the degree of noncompactness of a set is measured using functions μ called measures of noncompactness. The first such measure was defined by Kuratowski [18] in 1930. Kuratowski's initial interest in these measures was connected with certain problems in general topology, but the concept received a new impetus due to Darbo [10], who in 1955 using this concept proved a theorem which guarantees the existence of a fixed point of the so-called k-set contraction operators and thus he obtained a generalization of Schauder's theorem. Darbo's result has proved to be a very useful tool in the theory of functional equations, including ordinary differential equations, partial differential equations, integral and integro-differential equations, optimal control theory, etc (for instance see [2, 6]). In 1967 Sadovskiĭ [21] introduced the concept of condensing map and thus a more general fixed point result than Darbo's theorem was obtained. Since in some Banach spaces the complete description of the family of all relatively compact sets is not known (for instance see [6, Chapter 1]), since 1930, when Kuratowski gave his definition of measure of noncompactness a lot of papers using different axioms to define measure of noncompactness have been used in order to obtain fixed point theorems of Darbo's type, see [1, 3, 5, 6, 9, 13, 20].

The aim of this paper is to discuss on the axioms of a measure of noncompactness and thus to obtain fixed point results for continuous and noncontinuous mappings which, among other things, allow us to obtain a characterization for the existence of fixed points of a mapping which is not necessarily continuous. As a consequence of these facts, we give some generalizations of the above mentioned Darbo, Sadovskii fixed point theorems. In particular, we obtain generalizations of both the main result of [1] and [3, Theorem 3.1]. Finally, we solve an open question proposed by Rus [20, Problem 6.4.1].

2. Preliminaries

In this section we shall recall some definitions and results that are needed later on.

Throughout this article $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non-negative real numbers and the set of all positive integer numbers respectively.

Let $(\mathbb{B}, \|\cdot\|)$ be a real Banach space. Let $\mathcal{B} = \mathcal{B}(\mathbb{B})$ denote the collection of all nonempty bounded subsets of $\mathbb{B}$, and $\mathcal{K}(\mathbb{B})$ the family of all relatively compact subsets of $\mathcal{B}(\mathbb{B})$. For a nonempty subset X of $\mathbb{B}$, we denote by $\overline{X}$ and $\mathsf{co}(X)$ the closure and the convex hull of X, respectively. As usual, $B_r(x)$ denotes the closed ball of center x and radius r and given a nonempty set $A \in \mathcal{B}$, $\operatorname{diam}(A) := \sup\{\|x-y\| : \ x, y \in A\}$.

We recall the definition and some facts of the measures of noncompactness (we refer the reader to [4, 5, 6] for a deep study of measures of noncompactness).

Definition 2.1. A mapping $\mu : \mathcal{B} \to \mathbb{R}_+$ is said to be a *measure of noncompactness* on a Banach space $\mathbb{B}$ if it satisfies the following properties:

(P_1) *Consistency*: The family $\ker(\mu) := \{X \in \mathcal{B} : \mu(X) = 0\}$ is nonempty and $\ker(\mu) \subseteq \mathcal{K}(\mathbb{B})$.
(P_2) *Invariant under the closed convex hull*: $\mu(\overline{\mathsf{co}}(X)) = \mu(X)$.
(P_3) *Monotonicity*: if $Y \subseteq X$ then $\mu(Y) \leq \mu(X)$.

Sometimes it is necessary to assume that a measure of noncompactness μ enjoys some additional properties.

(P_4) *Generalized Cantor's intersection theorem*: if $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of closed sets from $\mathcal{B}$ such that $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$ is nonempty.

(P_5) *The maximum property*: For every $x \in \mathbb{B}$ and $X \in \mathcal{B}$, $\mu(X \cup \{x\}) = \mu(X)$.

(P_6) `Algebraic convexity`: $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for every $X, Y \in \mathcal{B}$ and for every $\lambda \in [0,1]$.

The family $\ker(\mu)$ described in (P_1) is called the *kernel* of the measure of noncompactness μ. When $\ker(\mu) = \mathcal{K}(\mathbb{B})$ the measure of noncompactness is called `full`. A measure of noncompactness is said to be `regular` if it enjoys properties $(P_1) - (P_6)$ and it is full. Finally, let us notice that if a measure of noncompactness μ satisfies (P_5), then μ also satisfies (P_4), see [4, p. 19].

Below we list four important examples of measures of noncompactness which quite often arise in the literature. The two first examples are regular measures of noncompactness which were introduced by Kuratowski [18] and Goldenstein et al. [16] respectively. Concerning the last two measures we have to say that they do not fulfill the maximum property neither they are full, see [5] for more details.

Given a subset $A \in \mathcal{B}(\mathbb{B})$, we define

(1) The *Kuratowski measure of noncompactness* of A as:

$$\alpha(A) = \inf\left\{ r > 0 : A \subset \bigcup_{i=1}^{n} D_i, \operatorname{diam}(D_i) \leq r \right\}.$$

(2) The *Hausdorff measure of noncompactness* (or ball measure of noncompactness) of A as:

$$\chi(A) = \inf\left\{ r > 0 : A \subset \bigcup_{i=1}^{n} B_r(x_i),\ x_i \in \mathbb{B} \right\}.$$

(3) The *diameter measure of noncompactness*: $\mu_d(A) = \operatorname{diam}(A)$.

(4) The *supremum measure of noncompactness*: $\mu_s(A) = \sup\big\{ \|x\| : x \in A \big\}$.

At this point, it is worth noting that

$$\ker(\mu_d) = \{A \in \mathcal{B} :\ A \text{ is a singleton}\} \text{ and } \ker(\mu_s) = \{0\}.$$

For our subsequent analysis, we need the following definitions (see [12]):

(i) A Banach space $\mathbb{B}$ is said to be *strictly convex* if for all $x, y \in \mathbb{B}$ with $\|x\| = \|y\| = 1$, one has $\|\lambda x + (1-\lambda)y\| < 1$, for all $\lambda \in (0,1)$.

(ii) A Banach space $\mathbb{B}$ is said to have *Kadec-Klee property* if whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{B}$ satisfies both $x_n \rightharpoonup x \in \mathbb{B}$ (i.e., $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x) and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$.

We next show the existence of other measures of noncompactness different from μ_d satisfying that its kernel is reduced to the singleton sets and which, in general, does not fulfill property (P_4), see Example 2.3.

Given a Banach space $(\mathbb{B}, \|\cdot\|)$, we define $\mu_{si} : \mathcal{B}(\mathbb{B}) \to \mathbb{R}$ as

(5) $\mu_{si}(A) := \sup\left\{\, \|x\| : x \in A\right\} - \inf\left\{\, \|y\| : y \in \overline{\mathsf{co}}(A)\right\}.$

Proposition 2.2. *If* $(\mathbb{B}, \|\cdot\|)$ *is strictly convex, then* μ_{si} *is a measure of noncompactness.*

Proof. It is clear that μ_{si} satisfies properties (P_2) and (P_3). Let us show that $\mu_{si}(A) = 0$ if and only if A is a singleton set. It is obvious that for every singleton set of $\mathbb{B}$ its μ_{si}-measure is zero. On the other hand, let $A \in \mathcal{B}(\mathbb{B})$ with $\mu_{si}(A) = 0$. Then, $\|x\| = k$ for all $x \in A$, where k is a non-negative constant. Suppose that A is non-singleton, in this case there exist $x, y \in A$ with $x \neq y$. Since $\mathbb{B}$ is strictly convex, we have $\left\|\frac{1}{2}(x+y)\right\| < k$. Bearing in mind that $\frac{1}{2}(x+y) \in \overline{\mathsf{co}}(A)$, we obtain $\mu_{si}(A) > 0$ which is a contradiction. Therefore μ_{si} is a measure of noncompactness on $\mathbb{B}$, whenever $\mathbb{B}$ is a strictly convex Banach space, whose kernel is reduced to the singletons. $\square$

Next, we will see that, in general, (P_4) fails for the measure μ_{si}.

Example 2.3. Let $\mathbb{B} = \mathcal{C}\,([0,1], \mathbb{R})$ be the space of all continuous functions in $[0,1]$ with the norm

$$\|x\| := \|x\|_\infty + \|x\|_2 = \max_{0\le t\le 1} |x(t)| + \left(\int_0^1 x(t)^2\, dt\right)^{\frac{1}{2}}.$$

$(\mathbb{B}, \|\cdot\|)$ is a nonreflexive strictly convex Banach space (see [12, Theorem 4.2.1]). In this case, let us see that μ_{si} does not fulfill (P_4). In order to show that, it will be enough to consider the decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of convex, closed a bounded subsets of $\mathcal{C}([0,1], \mathbb{R})$, defined by

$$C_n := \left\{ x \in \mathcal{C}([0,1], \mathbb{R}) : \|x\|_\infty \le 1, x(0) = 0, x(t) = 1 \text{ if } \frac{1}{n} \le t \le 1 \right\}.$$

In this case, $\mu_{si}(C_n) \to 0$ as $n \to \infty$, because for any $x \in C_n$

$$\|x\| \le 1 + \left(\int_0^{\frac{1}{n}} x(t)^2\, dt\right)^{\frac{1}{2}} + \sqrt{1 - \frac{1}{n}}\,.$$

Since $0 \le x^2(t) \le 1$ for all $t \in [0, \frac{1}{n}]$, we deduce that

$$1 + \sqrt{1 - \frac{1}{n}} \le \|x\| \le 1 + \frac{1}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}}\,.$$

Thus, $0 \le \mu_{si}(C_n) \le 1/\sqrt{n}$ for all $n \in \mathbb{N}$ and, therefore, $\mu_{si}(C_n) \to 0$ as $n \to \infty$. On the other hand, if there exists $x \in \bigcap_{n\in\mathbb{N}} C_n$, then $x(t) = 1$ if $\frac{1}{n} \le t \le 1$, for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have $x(t) = 1$ if $t \in (0,1]$, but $x(0) = 0$. Then, $x \notin \mathcal{C}([0,1], \mathbb{R})$ which is a contradiction. Therefore, $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$.

Nevertheless if $\mathbb{B}$ is a reflexive strictly convex Banach space and it has the Kadec-Klee property, we obtain the following result.

Proposition 2.4. *Let* $(\mathbb{B}, \|\cdot\|)$ *be a reflexive strictly convex Banach space with the Kadec-Klee property. Then* μ_{si} *satisfies* (P_4).

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of closed sets of $\mathcal{B}$ such that $\mu_{si}(X_n) \to 0$ as $n \to \infty$. Let us see that $\bigcap_{n\in\mathbb{N}} X_n$ is nonempty. In order to do this, we define $C_n := \overline{\mathsf{co}}(X_n)$ for $n \in \mathbb{N}$. Note that $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$ and each C_n is weakly compact. Then, for each $n \in \mathbb{N}$, there exists $y_n \in C_n$ such that $\|y_n\| = \min\left\{ \|y\| : y \in C_n \right\}$. Moreover, we can assume that $y_n \rightharpoonup y$ where $y \in \bigcap_{n\in\mathbb{N}} C_n$ and $\|y\| \leq \liminf_{n\in\mathbb{N}} \|y_n\|$ since the norm is weakly lower semi-continuous.

Since $\mu_{si}(X_n) \to 0$, we have

$$\lim_{n\to\infty} \sup_{x\in X_n} \|x\| - \|y\| = 0 \tag{2.1}$$

because, for all $n \in \mathbb{N}$,

$$\mu_{si}(X_n) = \sup_{x\in X_n} \|x\| - \inf_{z\in C_n} \|z\| = \sup_{x\in X_n} \|x\| - \|y_n\| \geq \sup_{x\in X_n} \|x\| - \|y\| \geq 0.$$

For each $n \in \mathbb{N}$, we take $x_n \in X_n$ such that

$$\|x_n\| + \frac{1}{n} \geq \sup_{x\in X_n} \|x\| .$$

We claim that $\|x_n\| \to \|y\|$ as $n \to \infty$. Indeed, for each $n \in \mathbb{N}$

$$0 \leq \|x_n\| + \frac{1}{n} - \|y\| \leq \sup_{x\in K_n} \|x\| + \frac{1}{n} - \|y\| .$$

Taking limits as $n \to \infty$ and bearing in mind (2.1), we obtain that $\|x_n\| \to \|y\|$. On the other hand, $x_n \rightharpoonup z$ where $z \in \bigcap_{n\in\mathbb{N}} C_n$, that is, $z \in C_n$ for all $n \in \mathbb{N}$. Then, $\|z\| \geq \|y_n\|$ for every $n \in \mathbb{N}$, which implies $\|z\| \geq \liminf_{n\to\infty} \|y_n\| \geq \|y\|$. Furthermore, note that $\|z\| \leq \liminf_{n\in\mathbb{N}} \|x_n\| = \|y\|$.

Hence, we have a sequence $\{x_n\}$ with $x_n \in X_n$, for each $n \in \mathbb{N}$, such that $x_n \rightharpoonup z$ and $\|x_n\| \to \|z\|$. By using Kadec-Klee property, we deduce that $x_n \to z$ and, therefore, $z \in \bigcap_{n\in\mathbb{N}} X_n$, that is, $\bigcap_{n\in\mathbb{N}} X_n$ is nonempty. □

Finally we finish this section with an open question related to the measure of noncompactness μ_{si}.

> ***Open question.*** *Can we drop the hypothesis of reflexivity or Kadec-Klee property in the previous result? In other words, are reflexivity and Kadec-Klee property necessary conditions for (P_4) to hold for μ_{si}.*

3. A fixed point result for continuous mappings

If one reads carefully the proof of any recent generalization of Darbo theorem (for instance see [1, 3, 11, 13] and the references therein), one notices that the essential ingredient is the following result which was explicitly proved by Nussbaum [19, Proposition 10] in less generality: for the Kuratowski measure.

Proposition 3.1. [19] *Let C be a closed, bounded, convex set in a Banach space $\mathbb{B}$. Let $T : C \to C$ be a continuous mapping. Let $C_1 := \overline{\text{co}}(TC)$ and $C_n := \overline{\text{co}}(TC_{n-1})$ for $n > 1$. Assume that $\alpha(C_n) \to 0$, as $n \to \infty$. Then, T has a fixed point.*

It is worth to note that the previous result holds for any measure noncompactness satisfying (P_4). However, the converse implication generally fails. Indeed, it is enough to consider the identity mapping defined on the unit ball of any infinite dimensional Banach space.

We next prove an extension of Nussbaum's result considering a more general condition which will be used in order to characterize the existence of fixed points for continuous mappings.

Proposition 3.2. *Let C be a closed, bounded, convex set in a Banach space $\mathbb{B}$. Let $T : C \to C$ be a continuous mapping. If there exists a decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of convex closed and T-invariant subsets of C such that $\mu(C_n) \to 0$, as $n \to \infty$, for a measure of nonconpactness μ on $\mathbb{B}$, satisfying (P_4), then T has at least a fixed point in C.*

Proof. By property (P_4), the intersection set $C_\infty := \bigcap_{n\in\mathbb{N}} C_n$ is nonempty and since

$$\mu(C_\infty) = \mu\left(\bigcap_{n\in\mathbb{N}} C_n\right) \le \mu(C_n)$$

for all $n \in \mathbb{N}$, then taking limits as $n \to \infty$ we get $\mu(C_\infty) = 0$, i.e., $C_\infty \in \ker(\mu)$. Furthermore, C_∞ is convex and T-invariant because each C_n is also convex and T-invariant. Therefore, $T : C_\infty \to C_\infty$ satisfies all hypothesis of Schauder theorem, which implies that T has at least a fixed point in $C_\infty \subseteq C$. □

The following example shows that the converse implication of Proposition 3.2 can fail if we do not require any additional hypothesis of measure of noncompactness.

Example 3.3. Consider the measure of noncompactness μ_s. Let $T : [1, 2] \to [1, 2]$ be the identity mapping. It is clear that the set of fixed point of T is $[1, 2]$. But, for any decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of convex closed and T-invariant subsets of $[1, 2]$, we have $\mu_s(C_n) \not\to 0$ as $n \to \infty$, because $\mu_s(X) \ge 1$ for any subset X of $[1, 2]$.

Nevertheless if we assume that the kernel of the measure of noncompactness contains the singleton sets, then we obtain the following result.

Theorem 3.4. *Let C be a closed, bounded, convex set in a Banach space $\mathbb{B}$ and μ be a measure of noncompactness on $\mathbb{B}$ satisfying that any singleton subset of C belongs to $\ker(\mu)$ and (P_4) holds. If $T : C \to C$ is continuous, then the following assertions are equivalent:*

(a) *there exists a decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of convex, closed and T-invariant subsets of C such that $\mu(C_n) \to 0$, as $n \to \infty$.*
(b) *T has at least a fixed point in C.*

Proof. The necessary condition follows from Proposition 3.2. Let us prove the sufficient condition. Assume that T has at least a fixed point p in C. Note that the sequence of sets defined by $C_n := \{p\}$ satisfies (a) because p is a fixed point of T and any singleton set belongs to $\ker(\mu)$. □

Remark 3.5. Notice that both Proposition 3.2 and Theorem 3.4 hold without the second condition (P_2) of the definition of the measures of noncompactness.

3.1. **Generalizations of Darbo's theorem.** In 1955 Darbo [10] used Kuratowski measure of noncompactness to generalize Schauder's theorem to k-set contractive mappings. It is well-known that this theorem can be established for any regular measure of noncompactness.

Theorem 3.6. *Let C be a nonempty, closed, bounded, and convex subset of a Banach space $\mathbb{B}$, μ a regular measure of noncompactness on $\mathbb{B}$. Suppose that $T : C \to C$ is a continuous and k-set-contractive mapping (that is, there exists $k \in [0, 1)$ such that $\mu(TX) \leq k\,\mu(X)$ for any nonempty subset X of C). Then, T has a fixed point in C.*

Next we will give two generalizations of Darbo's theorem which can be obtained by applying Proposition 3.2.

We will denote by $\mathcal{G}$ the set of all functions $g : \mathbb{R}_+ \to [0, 1)$ such that if $\{t_n\}_{n\in\mathbb{N}}$ is a monotone decreasing sequence in $(0,\infty)$ and $g(t_n) \to 1$ then $t_n \to 0$. Note that this class of functions was introduced by Geraghty [14], who did not make any assumption of continuity upon g in order to get an extension of Banach contraction principle.

By $\mathcal{F}$ denote the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}_+$, with $F(t) = 0 \iff t = 0$, and satisfying that either F is continuous on $\mathbb{R}_+$ or $F(t) \geq t$ for all $t \geq 0$.

Theorem 3.7. *Let μ be a measure of noncompactness satisfying (P_4). Let C be a nonempty, closed, bounded and convex subset of a Banach space $\mathbb{B}$ and $T : C \to C$ be a continuous mapping. If there exist two functions $g \in \mathcal{G}$ and $F \in \mathcal{F}$ such that, for every T-invariant subset X of C,*

$$F(\mu(TX)) \leq g(\mu(X))\, F(\mu(X)), \tag{3.1}$$

then T has at least one fixed point in C.

Proof. By Proposition 3.2, it will be enough to find a decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of convex closed and T-invariant subsets of C such that $\mu(C_n) \to 0$, as $n \to \infty$.

Let $C_1 := C$ and define the sequence $\{C_n\}_{n\in\mathbb{N}}$ by $C_{n+1} := \overline{\mathsf{co}}(T(C_n))$ for each $n \in \mathbb{N}$. Note that $T(C_1) = T(C) \subseteq C = C_1$, $C_2 = \overline{\mathsf{co}}(T(C_1)) \subseteq C = C_1$, and by induction we get $\{C_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of T-invariant subsets, since for each $n \in \mathbb{N}$, we have $T(C_n) \subseteq \overline{\mathsf{co}}(T(C_n)) = C_{n+1} \subseteq C_n$.

If there exists a natural number $n_0 \in \mathbb{N}$ such that $F(\mu(C_{n_0})) = 0$, then $\mu(C_{n_0}) = 0$ since $F \in \mathcal{F}$. By (P_3), $\mu(C_n) = 0$ for every $n \geq n_0$ and, thus, $\mu(C_n) \to 0$, as $n \to \infty$.

Otherwise, we can assume that $F(\mu(C_n)) > 0$ for all $n \in \mathbb{N}$. Using (3.1) and the properties of the measure of noncompactness we have

$$F(\mu(C_{n+1})) = F(\mu(\overline{\mathsf{co}}(TC_n))) = F(\mu(TC_n)) \leq g(\mu(C_n))\, F(\mu(C_n)),$$

for each $n \in \mathbb{N}$. This implies that the sequence $\{F(\mu(C_n))\}_{n\in\mathbb{N}}$ is nonincreasing and nonnegative, since $g \in \mathcal{G}$. Thus, we infer that there exists $\ell \geq 0$ such that

$$\lim_{n\to\infty} F(\mu(C_n)) = \ell.$$

We now distinguish two cases:

Case 1. Assume that $\ell > 0$. Since $\dfrac{F(\mu(C_{n+1}))}{F(\mu(C_n))} \leq g(\mu(C_n)) < 1$ for all $n \in \mathbb{N}$, by squeeze theorem, $g(\mu(C_n)) \to 1$ as $n \to \infty$. Since $g \in \mathcal{G}$, we conclude that $\lim_{n\to\infty} \mu(C_n) = 0$.

Case 2. Assume that $\ell = 0$. On the one hand, if F is continuous then $\mu(C_n) \to 0$ as $n \to \infty$, because $F(t) = 0 \iff t = 0$. On the other hand, if F satisfies condition (C_2), then $F(\mu(C_n)) \geq \mu(C_n) \geq 0$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} \mu(C_n) = 0$. □

Remark 3.8. If we consider $F = Id$, we get the main result of [1]. As a particular case, we have Darbo's theorem whenever $g(t) = k$ with $k \in [0, 1)$. On the other hand, for $F(t) = t + \varphi(t)$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function fixed with $\varphi(0) = 0$, we generalize [3, Theorem 3.1], since we do not require the continuity of φ.

Recall that a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called *comparison* if ψ is nondecreasing and $\psi^n(t) \to 0$, as $n \to \infty$, for all $t \geq 0$. We will denote by Ψ the set of comparison functions. For different properties and applications of comparison functions, we refer the reader to [20, Chapter 4] and [17, Lemma 3.1].

Theorem 3.9. *Let μ be a measure of noncompactness satisfying (P_4). Let C be a nonempty, closed, bounded and convex subset of a Banach space $\mathbb{B}$ and $T : C \to C$ be a mapping. If there exist two functions $\psi \in \Psi$ and $F \in \mathcal{F}$ such that, for every T-invariant subset X of C,*

$$F(\mu(TX)) \leq \psi\big(F(\mu(X))\big), \tag{3.2}$$

then T has at least one fixed point in C.

Proof. Similarly as in the proof of the Theorem 3.7, we define by induction the sequence $\{C_n\}_{n\in\mathbb{N}}$ where $C_1 := C$ and $C_{n+1} := \overline{\text{co}}(TC_n)$ for $n \in \mathbb{N}$, with $\mu(C_n) > 0$ for all $n \in \mathbb{N}$. From (3.2), for each $n \in \mathbb{N}$ we have

$$\begin{aligned} F(\mu(C_{n+1})) = F(\mu(\overline{\text{co}}(TC_n))) = F(\mu(TC_n)) &\leq \psi\big(F(\mu(C_n))\big) \\ &\leq \psi^2\big(F(\mu(C_{n-1}))\big) \leq \cdots \leq \psi^n\big(F(\mu(C))\big) \end{aligned}$$

Letting $n \to \infty$ in the above inequality and bearing in mind that $\psi \in \Psi$, we get

$$\lim_{n\to\infty} F(\mu(C_n)) = 0.$$

If F is continuous, we get $\mu(C_n) \to 0$ as $n \to \infty$ since $F(t) = 0 \iff t = 0$. On the other hand, if $F(t) \geq t$ for all $t \in \mathbb{R}_+$, it is clear that $\mu(C_n) \to 0$ as $n \to \infty$. □

The following example shows that the above two results are not a consequence of [9, Theorem 2.1] and, therefore, they cannot be obtained via simulation function.

Example 3.10. Let $\mathbb{B}$ be the space of all continuous functions $x : [0, 2/3] \to \mathbb{R}$ with the maximum norm $\|x\|_\infty := \max_{0 \le t \le \frac{2}{3}} |x(t)|$. Define $T : B^+_{2/3} \to B^+_{2/3}$ as $T(x) = \tau \circ x$, where $B^+_{2/3} := \left\{x \in \mathbb{B} : x(t) \ge 0,\ \|x\|_\infty \le \frac{2}{3}\right\}$ and $\tau : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ is the function given by

$$\tau(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le t \le \frac{1}{2}, \\ 2 - 3t & \text{if } \frac{1}{2} \le t \le \frac{2}{3}. \end{cases}$$

Notice that T is well-defined since τ is a continuous function. Furthermore, the constant function equal to $\frac{1}{2}$ is the unique fixed point of T. Let us see that T satisfies conditions (3.1) and (3.2) with $F = Id$, $g(t) = k \in (3/4, 1)$ and $\psi(t) = kt$ for the measure of noncompactness μ_d.

Let $A \subseteq B^+_{2/3}$ be a T-invariant set. Note that if $\sup\{\|x\| : x \in A\} \le \frac{1}{2}$, then $TA = \left\{\frac{1}{2}\right\}$ and obviously (3.1) and (3.2) hold. Then, without loss of generality, we can assume $\sup\left\{ \|x\| : x \in A\right\} > \frac{1}{2}$.

Define $\sigma = \sup\left\{ \|x\| - \frac{1}{2} : x \in A\right\} > 0$. Then, there exist two sequences $\{x_n\}_{n\in\mathbb{N}} \subset A$ and $\{\sigma_n\}_{n\in\mathbb{N}}$, with $\sigma_n > 0$, such that $\|x_n\|_\infty = \frac{1}{2} + \sigma_n$ for each $n \in \mathbb{N}$ and $\sigma_n \to \sigma$ as $n \to \infty$. For each $n \in \mathbb{N}$ there exists $t_n \in [0, \frac{2}{3}]$ such that $x_n(t_n) = \frac{1}{2} + \sigma_n$. By definition of τ, $Tx(t_n) = \frac{1}{2} - 3\sigma_n$.

Since A is T-invariant, $x_n, Tx_n \in A$ for all $n \in \mathbb{N}$. Then,

$$\mu_d(A) \ge \lim_{n\to\infty} |x_n(t_n) - Tx_n(t_n)| = \lim_{n\to\infty} 4\sigma_n = 4\sigma.$$

On the other hand,

$$\mu_d(TA) \le \sup\left\{ \left\|\tfrac{1}{2} - Tx\right\|_\infty : x \in A\right\} \le 3\sigma$$

since for any $x \in A$, $0 \le Tx(t) \le \frac{1}{2}$ for all $t \in [0, \frac{2}{3}]$ and

$$\left\|\frac{1}{2} - Tx\right\|_\infty = \max_{0\le t\le\frac{2}{3}} \left|\frac{1}{2} - Tx(t)\right| \le \max_{0\le t\le\frac{2}{3}} \left|3x(t) - \frac{3}{2}\right| \le 3\left(\frac{1}{2} + \sigma\right) - \frac{3}{2} = 3\sigma.$$

Thus, for any $k \in (\frac{3}{4}, 1)$, we have $\mu_d(TA) \le 3\sigma \le k\,4\alpha \le k\,\mu_d(A)$. Therefore, T satisfies condition (3.1) with $F = Id$ and $g(t) = k \in (3/4, 1)$ and also satisfies condition (3.2) with $F = Id$ and $\psi(t) = kt$.

However, for $X_0 = \{\frac{1}{2}, \frac{2}{3}\}$, it is easy to see that $\mu_d(TX_0) = \frac{1}{2} > \frac{1}{6} = \mu_d(X_0)$. Then, [9, Theorem 2.1] cannot be applied.

3.2. **Generalizations of Sadovskiĭ's theorem.** Given a measure of noncompactness μ on a Banach space $(\mathbb{B}, \|\cdot\|)$ and a nonempty bounded subset C of $\mathbb{B}$. Recall that a mapping $T : C \to \mathbb{B}$ is called *μ-condensing* if $\mu(TX) < \mu(X)$ for all subset X of C with $\mu(X) > 0$. In 1967 Sadovskiĭ [21] obtained a fixed point result for μ-condensing mappings which is an extension of Darbo's theorem.

Theorem 3.11. [21] *Let C be a nonempty, closed, bounded, and convex subset of a Banach space $\mathbb{B}$, μ a regular measure of noncompactness on $\mathbb{B}$, and let $T : C \to C$ be a continuous μ-condensing mapping. Then, T has a fixed point in C.*

The following example, due to Baronti, Casini and Papini [7], shows that Sadovskiĭ's theorem fails if the measure of noncompactness μ is not regular, in particular if μ does not fulfill the maximum property.

Example 3.12. ([7]) Let $\mathbb{B}$ be the non-reflexive Banach space of all continuous real functions on the closed unit interval, with the norm

$$\|x\| := \|x\|_\infty + \|x\|_1 = \max_{0\leq t\leq 1} |x(t)| + \int_0^1 |x(t)|\, dt.$$

Consider the following closed, convex and bounded subset

$$C = \big\{x \in \mathbb{B} : x(0) = 0, x(1) = 1, 0 \leq x(t) \leq t, x \text{ is monotone nondecreasing}\big\}.$$

The mapping $T : C \to C$ defined by

$$Tx(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ (2t-1)\, x(2t-1) & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

is fixed point free continuous and μ_d-condensing.

If the proof of Theorem 3.11 is checked, one can note that it is only required that T is a μ-condensing mapping for the T-invariant, closed, convex subsets, that is, $\mu(TX) < \mu(X)$ for any closed and convex subset X of C with $\mu(X) > 0$ and $T(X) \subseteq X$.

The following example shows that the above condition is weaker than to be condensing, see also Example 3.10.

Example 3.13. Let B^+ be the first quadrant of the unit ball in the Euclidean plane with its usual norm, i.e.,

$$B^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, \|x\|_2 \leq 1\}.$$

Consider the mapping $T : B^+ \to B^+$ defined by

$$T(x_1, x_2) = P_{B^+}\left(x_1, \frac{x_2 + 1}{2}\right),$$

where $P_{B^+} : \mathbb{R}^2 \to B^+$ denotes to the metric projection onto B^+.

The mapping T is not μ_d-condensing since if we take the subset $X = [0, \frac{1}{2}] \times \{0\}$, clearly $TX = [0, \frac{1}{2}] \times \{\frac{1}{2}\}$ and consequently $\mu_d(TX) = \frac{1}{2} = \mu_d(X)$.

However, we shall prove that T verifies $\mu_d(TX) < \mu_d(X)$, for any T-invariant, convex and closed subset X of B^+.

Let X be a T-invariant, convex and closed subset of B^+. Since T is continuous and its unique fixed point is $(0, 1)$, by Brouwer's theorem, we have $(0, 1) \in X$. Without loss of generality, we can suppose that $0 < \operatorname{diam}(X) = \|x^0 - y^0\|_2$, which implies that $\|u-v\|_2 \leq \|x^0 - y^0\|_2$ for all $u, v \in X$. Then, bearing in mind that the metric projection P_{B^+} is nonexpansive, we have

$$\begin{aligned}\|T(u)-T(v)\|_2 &= \left\| P_{B^+}\left(u_1, \frac{u_2+1}{2}\right) - P_{B^+}\left(v_1, \frac{v_2+1}{2}\right)\right\|_2 \\ &\leq \left\| \left(u_1, \frac{u_2+1}{2}\right) - \left(v_1, \frac{v_2+1}{2}\right)\right\|_2 \\ &= \left\| \left(u_1 - v_1, \frac{u_2-v_2}{2}\right)\right\|_2 \\ &\leq \left\| (u_1 - v_1, u_2 - v_2)\right\|_2 = \|u-v\|_2 .\end{aligned}$$

Thus, $\operatorname{diam}(TX) = \operatorname{diam}(X) \iff {x_2}^0 = {y_2}^0 < 1$ (otherwise, we would obtain that $x^0 = (0,1) = y^0$ which is a contradiction) but in this case we would have that $\overline{\mathsf{co}}(\{(0,1), x^0, y^0\}) \subseteq X$ and

$$\operatorname{diam}(\overline{\mathsf{co}}(\{(0,1), x^0, y^0\})) = \left\|x^0 - y^0\right\|_2 = \left|x_1^0 - y_1^0\right| \leq \max\left\{x_1^0, y_1^0\right\},$$

which is a contradiction because by Pythagorean theorem

$$\left\|(0,1) - x^0\right\|_2 = \sqrt{\left(x_1^0\right)^2 + \left(1 - x_2^0\right)^2} > x_1^0$$

and

$$\left\|(0,1) - y^0\right\|_2 = \sqrt{\left(y_1^0\right)^2 + \left(1 - y_2^0\right)^2} > y_1^0.$$

Therefore, $\operatorname{diam}(TX) < \operatorname{diam}(X)$.

The previous example motivates the following concept which is more general than the definition of condensing mapping.

Definition 3.14. Let μ a measure of noncompactness on a Banach space $\mathbb{B}$ and C be a nonempty bounded subset of $\mathbb{B}$. A mapping $T : C \to \mathbb{B}$ is *μ-quasi-condensing* if $\mu(TX) < \mu(X)$ for all T-invariant, closed, convex subset X of C with $\mu(X) > 0$.

Next we will prove a new result which indicates that, in order to guarantee the existence of a fixed point for a mapping, we can replace the concept of condensing mapping by quasi-condensing.

Theorem 3.15. *Let μ be a measure of noncompactness satisfying (P_5). Let C be a nonempty, closed, bounded and convex subset of a Banach space $\mathbb{B}$ and $T : C \to C$ be a continuous mapping. Suppose that*

$$\begin{aligned}&\textit{there exists a function } h : \mathbb{R}_+ \to \mathbb{R}_+ \textit{ such that } h(\mu(TX)) \neq h(\mu(X)),\\ &\textit{for every } T\textit{-invariant closed, convex subset } X \textit{ of } C \textit{ with } \mu(X) > 0.\end{aligned} \tag{3.3}$$

Then, T has at least one fixed point in C.

Proof. Fix a point $x_0 \in C$ and let Δ denote the family of all closed, convex subsets X of C for which $x_0 \in X$ and $T(X) \subseteq X$. Now we define $X_\infty := \bigcap_{X \in \Delta} X$. Note that X_∞ is nonempty because $x_0 \in X_\infty$.

We claim that $X_\infty = \overline{\mathsf{co}}(T(X_\infty) \cup \{x_0\})$. Indeed, since $T(X_\infty) \subseteq X_\infty$, $x_0 \in X_\infty$ and X_∞ is closed and convex, we have that $\overline{\mathsf{co}}(T(X_\infty) \cup \{x_0\}) \subseteq X_\infty$, therefore,

$T(\overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})) \subseteq T(X_\infty) \subseteq \overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})$ and bearing in mind that $x_0 \in \overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})$ and $\overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})$ is closed and convex, from the definition of X_∞, we have that $X_\infty \subseteq \overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})$. Thus, $X_\infty = \overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})$.

Then, by using (P_2), (P_3) and (P_5), we get $\mu(X_\infty) = \mu(T(X_\infty))$ because

$$\begin{aligned} \mu(X_\infty) &= \mu\big(\overline{\mathrm{co}}(T(X_\infty) \cup \{x_0\})\big) \\ &= \mu\big(\mathrm{co}(T(X_\infty) \cup \{x_0\})\big) \\ &= \mu\big(T(X_\infty) \cup \{x_0\}\big) \\ &= \mu\big(TX_\infty\big). \end{aligned}$$

Note that if $\mu(X_\infty) > 0$, then $h(\mu(X_\infty)) = h(\mu(TX_\infty)) \neq h(\mu(X_\infty))$ which is a contradiction. Thus, $\mu(X_\infty) = 0$. If we define $C_n = X_\infty$ for every $n \in \mathbb{N}$, then by Proposition 3.2 we obtain the conclusion. □

Concerning Theorem 3.15 one can think that condition (3.3) is more general than to be quasi-condensing, but both conditions are equivalent, as we show in the following result.

Proposition 3.16. *Let C be a bounded subset of a Banach space $\mathbb{B}$, μ a measure of noncompactness on $\mathbb{B}$. A mapping $T : C \to C$ is μ-quasi-condensing if and only if T satisfies condition* (3.3).

Proof. Suppose that T is quasi-condensing. In this case, it is enough to consider $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(x) = x$. On the other hand, if there exists a function h satisfying condition (3.3) it is clear by (P_3) that $\mu(T(X)) < \mu(X)$ whenever $T(X) \subseteq X$ and $\mu(X) > 0$. □

Since Example 3.13 guarantees the existence of quasi-condensing mappings which are not condensing, then, as a consequence of the above two results, we obtain the following generalization of Sadovskiĭ's theorem.

Corollary 3.17. *Let μ be a measure of noncompactness satisfying (P_5). Let C be a closed, bounded and convex subset of a Banach space $\mathbb{B}$. If $T : C \to C$ is continuous and μ-quasi-condensing, then T has at least one fixed point in C.*

At this point, one may be interested in characterizing the existence of fixed points for the μ-quasi-condensing mappings with respect to a general measure of noncompactness μ. In this direction, we need the concept of minimal T-invariant sets.

Definition 3.18. Let C be a nonempty, convex, closed bounded subset of a Banach space $\mathbb{B}$ and $T : C \to C$. A set $K \subseteq C$ is said to be *minimal T-invariant* if K is non-empty closed, convex, $T(K) \subseteq K$, and whenever Y is a non-empty, closed, convex subset of K with $T(Y) \subseteq Y$,it follows that $Y = K$.

Obviously any singleton T-invariant set, necessarily is a fixed point of the mapping T, and it is minimal. In general, the existence of a minimal invariant subset for a mapping is not assured, see Remark 3.20. Furthermore, identifying the minimal T-invariant sets is not an easy task, as is cited by Goebel and Sims in [15]. Nevertheless, we obtain the following result.

Theorem 3.19. *Let C be a nonempty, convex, closed bounded subset of a Banach space $\mathbb{B}$ and μ be a measure of noncompactness on $\mathbb{B}$. Assume that $T : C \to C$ is a μ-quasi-condensing and continuous mapping. Then, the following assertions are equivalent:*

(a) *T has at least a fixed point in C;*
(b) *there exists a minimal T-invariant subset of C;*
(c) *if K is minimal T-invariant, then K is singleton.*

Proof. $(a) \implies (b)$ is obvious because any fixed point of T is minimal T-invariant. On the other hand, $(c) \implies (a)$ is clear. We next prove $(b) \implies (c)$. Let K be a minimal T-invariant. Then, $\overline{\mathsf{co}}(TK) \subseteq K$ since K is convex and closed. Thus, $T(\overline{\mathsf{co}}(TK)) \subseteq TK \subseteq \overline{\mathsf{co}}(TK)$. Since K is minimal T-invariant, $K = \overline{\mathsf{co}}(TK)$. Bearing in mind the three properties of the measure of noncompactness μ and the fact that T is μ-quasi-condensing, we deduce K is compact and, by Schauder's theorem, there exists $p \in K$ such that $Tp = p$. Then, $\{p\}$ is a subset of K which is T-invariant. Therefore, $K = \{p\}$. □

Remark 3.20. We can use the previous result in order to prove that for the mapping $T : C \to C$ given in Example 3.12 there is no minimal T-invariant subset, because otherwise T would have at least one fixed point in C which is a contradiction.

It is interesting to note that when C is weakly compact the existence of a minimal T-invariant set is obtained by using Zorn's Lemma. As a consequence of this fact and Theorem 3.19, we recapture [13, Theorem 3.1].

Corollary 3.21. *Let $\mathbb{B}$ be a Banach space and consider μ a measure of noncompactness on $\mathbb{B}$. If C is a closed convex subset of $\mathbb{B}$ and T is a continuous μ-condensing mapping from C into itself, then T has at least a fixed point whenever $\overline{T(C)}$ is weakly compact.*

Let μ be a measure of noncompactness on a Banach space $\mathbb{B}$. We will say that $\mathbb{B}$ has the *μ-fixed point property* (shortly, *μ-FPP*) if every μ-quasi-condensing and continuous mapping defined in a nonempty closed, convex and bounded subset of $\mathbb{B}$ has at least a fixed point.

Notice that if $\mathbb{B}$ is a Banach spaces and μ is a measure of noncompactness with the maximum property (P_5), then $\mathbb{B}$ has μ-FPP.

On the other hand, it is well-known in reflexive spaces every closed, bounded and convex set is weakly compact. Therefore, as a particular case of the previous result, we deduce that every reflexive space has μ-FPP for every measure of noncompactness μ. Thus, the natural question arises of whether it is possible to get this reverse implication.

> ***Open question.*** *If a Banach space $\mathbb{B}$ has μ-FPP, for any measure μ, is $\mathbb{B}$ reflexive?*

As we have mentioned in the above section, Darbo's theorem has been extended by using some weaker conditions. These weaker conditions imply that the mapping $T : C \to C$ verifies that $\mu(TX) \leq f\big(\mu(X)\big)$ for every $X \subseteq C$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain function such that $f(t) < t$ for all $t > 0$.

Note that the previous condition implies that the mapping T is μ-condensing. Therefore, any recent generalization of Darbo's theorem can be seen as a consequence of Sadovskiĭ's theorem whenever the measure of noncompactness μ satisfies the maximum property (P_5) or the space $\mathbb{B}$ is reflexive. For this reason, any new extension of Darbo theorem for a regular measure of noncompactness μ (as the Kuratowski measure α or the Hausdorff measure χ) has a lack of interest because its proof would be easily deduced from Sadovskiĭ's theorem. In our opinion, any fixed point result of Darbo type must be stated in the most general context for measures of noncompactness.

The following example shows that Theorem 3.4 is more general than Proposition 3.1 and than any result of Sadowskiĭ or Darbo type.

Example 3.22. Let B_r be the closed ball centered at zero with radius $r > 0$ in an infinite dimensional Banach space $(\mathbb{B}, \|\cdot\|)$. Let $f : [0,1] \to [0,1]$ be a function. We define the mapping $T : B_1 \to B_1$ by $Tx = f(\|x\|)\, x$ for every $x \in B_1$.

(1) If we take $f(t) = k$ for all $t \in [0,1]$, where $k \in [0,1)$, it is easy to see that T is k-set-contractive for the Kuratowski measure of noncompactness α and, therefore, we can apply Darbo's theorem in order to ensure the existence of fixed points of T.

(2) If f is a continuous and strictly decreasing function with $f(0) = 1$, then the mapping T is α-condensing and, by Sadowskiĭ's theorem, we deduce the existence of fixed points of T. Note that, in this case, we do not apply Darbo's theorem because this mapping T is not k-set-contractive for any $k \in [0,1)$, see [4, p. 40] for more details.

(3) If f is continuous, then we have that, for the measure of noncompactness μ_d or α or χ, T satisfies each hypothesis of Proposition 3.2 which guarantees the existence of fixed points of T. Indeed, it is enough to consider the decreasing sequence $\{B_{1/n}\}_{n\in\mathbb{N}}$. However, we want to point out that in general we cannot apply Sadowskiĭ's theorem.

(4) For each $n \in \mathbb{N}$, set $I_n := \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ and consider $f : [0,1] \to [0,1]$ defined by

$$f(t) = \begin{cases} 1 & \text{for } t \in I_{2n-1} \text{ or } t = 0, \\ 1 - 2^{-n} & \text{for } t \in I_{2n}. \end{cases}$$

We claim that T is not μ-condensing for any measure of noncompactness μ. Indeed, it is enough to take the set $A_n := \{x \in B_1 : \|x\| \in I_{2n-1}\}$ for some $n \in \mathbb{N}$, because in this case $TA_n = A_n$ which implies $\mu(TA_n) = \mu(A_n)$ for any measure of noncompactness μ.

It is clear that T is not continuous and, therefore, Proposition 3.2 cannot be applied. However, T has fixed points in B_1. To be more precise, the set of fixed points of T is

$$\bigcup_{n\in\mathbb{N}} \left\{x \in B_1 : \frac{1}{2^{2n-1}} < \|x\| \leq \frac{1}{2^{2(n-1)}}\right\} \cup \{0\}.$$

Moreover, T satisfies condition (a) of Theorem 3.4 for the measure of noncompactness $\mu = \mu_d$ or $\mu = \alpha$ or $\mu = \chi$. Just consider $C_n = B_{1/n}$ for each $n \in \mathbb{N}$.

4. A fixed point result for a not necessarily continuous mapping

In view of Example 3.22.(4), we wonder if condition (a) of Theorem 3.4 is sufficient in order to get the existence of fixed points, that is, if the assumption of continuity of the mapping can be dropped in Proposition 3.2. The answer is affirmative whenever the measure of noncompactness satisfies an additional condition related to its kernel. Moreover, in this case the assumption of convexity is dropped. To be more precise, in this section we will consider a different definition of measure of noncompactness in the setting of metric spaces. This point of view is considered in [20, Chapter 6].

Definition 4.1. Let $(\mathbb{M}, d)$ be a complete metric space. A mapping $\mu : \mathcal{B}(\mathbb{M}) \to \mathbb{R}_+$ is called a *measure of noncompactness* on $\mathbb{M}$ if the four properties $(P_1) - (P_4)$ in Definition 2.1 hold, where (P_2) is replaced by the following property:

(P_2') *Invariant under the closure*: $\mu(\overline{X}) = \mu(X)$ for all $X \in \mathcal{B}(\mathbb{M})$.

Theorem 4.2. *Let μ be a measure of noncompactness on a complete metric space $\mathbb{M}$ such that $\ker(\mu) = \{A \in \mathcal{B}(\mathbb{M}) : A$ is a singleton$\}$. Let $T : \mathbb{M} \to \mathbb{M}$ be any mapping. Then, the following assertions are equivalent:*

- (a) *there exists a decreasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed and T-invariant subsets of $\mathbb{M}$ such that $\mu(X_n) \to 0$, as $n \to \infty$.*
- (b) *T has at least a fixed point in $\mathbb{M}$.*

Furthermore, in this case given a sequence $\{x_n\}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ converges to some fixed point of T.

Proof. $(a) \implies (b)$ Similar to the first part of Proposition 3.2, $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$ is nonempty, closed, T-invariant and, moreover, $\mu(X_\infty) = 0$. Then, $X_\infty = \{p\}$ for some $p \in C$. Since X_∞ is T-invariant, we deduce that p is a fixed point of T.

$(b) \implies (a)$ Let p be a fixed point of T in X. It is enough to consider $X_n = \{p\}$ for all $n \in \mathbb{N}$.

Finally, suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence satisfying (a) and p is a fixed point of T with $X_\infty = \{p\}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \in X_n$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}_{n\in\mathbb{N}}$ converges to p. By contradiction, suppose that $x_n \not\to p$ as $n \to \infty$. Then, there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, p) > \varepsilon_0$ for all $k \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the closed set $S_r := \overline{\{x_{n_k} : k \geq r\}}$. It is clear that $\{S_r\}_{r\in\mathbb{N}}$ is decreasing, for each $r \in \mathbb{N}$, S_r is a subset of X_{n_r} and, thus, $\mu(S_r) \to 0$ as $r \to \infty$. Therefore, by (P_4), $\bigcap_{r\in\mathbb{N}} S_r = \{p\}$, i.e., $p \in S_r$ for all $r \in \mathbb{N}$. Then, for each $r \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ with $k_0 \geq r$ such that $d(x_{n_{k_0}}, p) < \frac{\varepsilon_0}{2}$, which is a contradiction. □

It is worth to note that all hypothesis of the previous result are satisfied by the measure of noncompactness μ_d.

Corollary 4.3. *Let* $\mathbb{M}$ *be a complete metric space. A mapping* $T : \mathbb{M} \to \mathbb{M}$ *has at least a fixed point in* $\mathbb{M}$ *if and only if there exists a decreasing sequence* $\{X_n\}_{n\in\mathbb{N}}$ *of closed and* T*-invariant subsets of* $\mathbb{M}$ *such that* $\mu_d(X_n) \to 0$*, as* $n \to \infty$*. Furthermore, in this case given a sequence* $\{x_n\}$ *such that* $x_n \in X_n$ *for each* $n \in \mathbb{N}$*, then* $\{x_n\}_{n\in\mathbb{N}}$ *converges to some fixed point of* T*.*

On the other hand, if the framework is a reflexive strictly convex space with Kadec-Klee property then, as we have proved in Propositions 2.2 and 2.4, μ_{si} also satisfies all hypothesis of Theorem 4.2.

Corollary 4.4. *Let* $\mathbb{B}$ *be a reflexive strictly convex space with Kadec-Klee property. A mapping* $T : \mathbb{B} \to \mathbb{B}$ *has at least a fixed point in* $\mathbb{B}$ *if and only if there exists a decreasing sequence* $\{X_n\}_{n\in\mathbb{N}}$ *of closed and* T*-invariant subsets of* $\mathbb{B}$ *such that* $\mu_{si}(X_n) \to 0$*, as* $n \to \infty$*. Furthermore, in this case given a sequence* $\{x_n\}$ *such that* $x_n \in X_n$ *for all* $n \in \mathbb{N}$*, then* $\{x_n\}$ *converges to some fixed point of* T*.*

Now we drop the hypothesis related to the kernel of the measure of noncompactness.

Theorem 4.5. *Let* μ *be a measure of noncompactness on a complete metric space* $\mathbb{M}$ *and* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping. Suppose that*

(H_1) *there exists a decreasing sequence* $\{X_n\}_{n\in\mathbb{N}}$ *of closed and* T*-invariant sets of* $\mathbb{M}$ *such that* $\mu(X_n) \to 0$*, as* $n \to \infty$*.*

(H_2) $\mu_d(TA) < \mu_d(A)$ *for all compact subset* $A \subseteq \mathbb{M}$ *with* $\mu_d(A) > 0$ *and* $TA \subseteq A$*.*

Then, T *has a unique fixed point.*

Proof. Since μ is a measure of noncompactness, by (H_1) and property (P_4) we deduce that $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$ is a nonempty T-invariant compact set. Define $\mathcal{S} = \big\{K \subseteq X_\infty : K \text{ is nonempty compact}, TK \subseteq K\big\}$. Note that $\mathcal{S} \neq \emptyset$ because $X_\infty \in \mathcal{S}$. By using Zörn's lemma, we can guarantee that there exists $K_0 \in \mathcal{S}$ such that K_0 is minimal with respect to the inclusion. Let us prove $\overline{TK_0} = K_0$. Since $K_0 \in \mathcal{S}$, $TK_0 \subseteq K_0$, then $\overline{TK_0} \subseteq \overline{K_0} = K_0$. Hence, $T\big(\overline{TK_0}\big) \subseteq TK_0 \subseteq \overline{TK_0}$. Thus, $\overline{TK_0} \in \mathcal{S}$. The minimality of K_0 implies that $\overline{TK_0} = K_0$. From (H_2), we have $\mu_d(K_0) = 0$, that is, K_0 is a singleton set. Thus, $K_0 = \{p\}$ with $p \in \mathbb{M}$. Since K_0 is T-invariant, then p is a fixed point of T.

Now, suppose that there exists two different fixed points p_1 and p_2 of T. Then the compact set $\{p_1, p_2\}$ is T-invariant and, by (H_2), we get $\mu_d(\{p_1, p_2\}) = \mu_d(\{Tp_1, Tp_2\}) < \mu_d(\{p_1, p_2\})$, which is a contradiction. □

In [20, Problem 6.4.1] Rus raises the following open question: Let $(\mathbb{M}, d)$ be a complete metric space and let μ be a measure of noncompactness on $\mathbb{M}$. Suppose that $T : \mathbb{M} \to \mathbb{M}$ is a mapping satisfying:

(a) there exists a comparison function φ such that $\mu(T(A)) \leq \varphi(\mu(A))$ for every T-invariant subset $A \in \mathcal{B}(\mathbb{M})$,

(b) $\mu_d(TA) < \mu_d(A)$ for every T-invariant subset $A \in \mathcal{B}(\mathbb{M})$ with $\mu_d(A) > 0$.

Does T have a fixed point?

Regarding the above problem we have to say that if T is a mapping which does not have any bounded T-invariant subset, then it satisfies obviously conditions (a) and (b) and it is fixed point free. Thus, from now on, we will assume that T has at least a bounded T-invariant subset.

Motivated by the above problem, we now consider the following question related to the existence and uniqueness of fixed point for (μ, φ)-contractions that are diametrally contractive: Let μ be a measure of noncompactness in a complete metric space $(\mathbb{M}, d)$ and $T : \mathbb{M} \to \mathbb{M}$ a mapping such that:

- (i) T is a (μ, φ)-contraction: there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(TA) \leq \varphi\big(\mu(A)\big)$ for every $A \in \mathcal{B}(\mathbb{M})$.
- (ii) T is diametrally contractive, that is, $\mu_d(TA) < \mu_d(A)$ for all $A \in \mathcal{B}(\mathbb{M})$ with $\mu_d(A) > 0$.

Does T have fixed point?

The following example shows that in general the answer of the above question is negative even in $\mathbb{R}$.

Example 4.6. Let $\mathbb{M} = [1, \infty)$ with the usual metric. Consider the mapping $T : [1, \infty) \to [1, \infty)$ defined by $Tx = x + \frac{1}{x}$. Note that T is continuous and compact. Thus, T is a (μ, φ)-contraction for any full measure of noncompactness μ and any function φ, since $\mu(TA) = 0$ for all $A \in \mathcal{B}(\mathbb{M})$. Moreover, T is fixed point free.

Let us prove that T is diametrally contractive. It is easy to see that T is strictly contractive, that is, $|Tx - Ty| < |x - y|$ for all $x, y \in [1, \infty)$. Let A be a bounded subset of $[1, \infty)$. Since TA is bounded, there exist $u_1, u_2 \in \overline{TA}$ such that

$$|u_1 - u_2| = \operatorname{diam}(TA).$$

On the other hand, since T is continuous, there exist $x_1, x_2 \in \overline{A}$ such that $Tx_i = u_i$, for $i = 1, 2$. Then,

$$\operatorname{diam}(TA) = |u_1 - u_2| = |Tx_1 - Tx_2| < |x_1 - x_2| \leq \operatorname{diam}(\overline{A}) = \operatorname{diam}(A).$$

Finally, we give an answer to the open problem proposed by Rus [20, Problem 6.4.1].

Theorem 4.7. *Let μ be a measure of noncompactness on a complete metric space $\mathbb{M}$. Suppose that $T : \mathbb{M} \to \mathbb{M}$ is a mapping satisfying:*

- (a) *there exists a comparison function φ such that $\mu(T(A)) \leq \varphi(\mu(A))$ for every T-invariant subset $A \in \mathcal{B}(\mathbb{M})$,*
- (b) *$\mu_d(TA) < \mu_d(A)$ for every T-invariant subset $A \in \mathcal{B}(\mathbb{M})$ with $\mu_d(A) > 0$.*

Then, T has a unique fixed point p in $\mathbb{M}$ if and only if there exists a closed, bounded and T- invariant set.

Proof. Note that if T has a unique fixed point, then the set $\{p\}$ is closed, bounded and T-invariant. Let us see the reciprocal. In order to do this, we will prove the existence and uniqueness of a fixed point of T as a consequence of Theorem 4.5.

Let A_0 be a closed, bounded and T-invariant set. We define $X_1 = \overline{TA_0}$ and $X_{n+1} = \overline{TX_n}$ for each $n \in \mathbb{N}$. Clearly for each $n \in \mathbb{N}$, X_n is closed, bounded and $X_{n+1} \subseteq X_n$. Furthermore, each X_n is T-invariant. Indeed, since A_0 is closed and

T-invariant, $TX_1 = T\big(\overline{TA_0}\big) \subseteq T(\overline{A_0}) = TA_0 \subseteq \overline{TA_0} = X_1$ and, by induction, if $TX_n \subset X_n$ then $TX_{n+1} = T\big(\overline{TX_n}\big) \subseteq T(\overline{X_n}) = TX_n \subseteq \overline{TX_n} = X_{n+1}$.

From (a), $\mu(X_{n+1}) \leq \varphi^n\big(\mu(X_1)\big)$ for all $n \in \mathbb{N}$. Taking limits and bearing in mind that φ is a comparison function, we have $\mu(X_n) \to 0$ as $n \to \infty$. Therefore, (H_1) holds. Clearly T verifies (H_2). Then, by Theorem 4.5, T has a unique fixed point p in $\mathbb{M}$. □

The following example shows that in general the answer to Rus' problem is negative.

Example 4.8. Let $\mathbb{M} = [0, \infty)$ with the usual metric. Consider the fixed point free mapping $T : [0, \infty) \to [0, \infty)$ given by

$$Tx = \begin{cases} 1 & \text{if} \quad x = 0 \\ x/2 & \text{if} \quad 0 < x < 1 \\ x + 1 & \text{if} \quad x \geq 1. \end{cases}$$

Since T maps bounded sets into bounded sets, T is compact and, therefore, T satisfies (a) for any full measure of noncompactness μ, since $\mu(TA) = 0$ for all $A \in \mathcal{B}(\mathbb{M})$. On the other hand, it is easy to see that the bounded T-invariant sets are either $(0, k]$ with $0 < k < 1$ or $(0, k)$ with $0 < k \leq 1$. Thus, $\mu_d(TA) \leq \mu_d(A)/2$ for all bounded T-invariant set A. Hence, T satisfies (b).

Remark 4.9. The above example and Theorem 4.7 solve completely the problem proposed by Rus [20, Problem 6.4.1]. In general, the answer is negative. However, we can guarantee the existence and uniqueness of fixed points for mappings T satisfying (a) and (b) if, and only if, there exists at least a closed, bounded T-invariant set.

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