

ON BISHOP-PHELPS PARTIAL ORDER, VARIATION MAPPINGS AND CARISTI'S FIXED POINT THEOREM IN QUASI-METRIC SPACES

NASEER SHAHZAD* AND OSCAR VALERO**

*Department of Mathematics, King Abdulaziz University
P.O.B. 80203, Jeddah 21589, Saudi Arabia
E-mail: nshahzad@kau.edu.sa

**Departamento de Ciencias Matemáticas e Informática
Universidad de las Islas Baleares
Ctra. de Valldemossa km. 7.5, 07122 Palma de Mallorca, Spain
E-mail: o.valero@uib.es

Abstract. In this paper we continue the study of those conditions that guarantee the existence of fixed points for variation mapping in the spirit of M.R. Tasković. Concretely, we provide a general fixed point result for variation mappings defined in left- K -sequentially complete T_1 quasi-metric spaces in such a way that only lower semicontinuity from above is required instead of lower semicontinuity. We give examples that elucidate that the assumptions in the statement of our main result cannot be weakened. Moreover, it is shown that the CS -convergence condition by Tasković implies left K -sequentially completeness and, thus, we retrieve the fixed point result for variation mappings in T_1 quasi-metric spaces due to Tasković. Furthermore, some fixed point theorems, among other Caristi type fixed point results, for variation mappings are derived as a particular case of our main result when several different quasi-metric notions of completeness are considered. Finally, we provide a characterization of left K -sequentially completeness for T_1 quasi-metric spaces via variation mappings.

Key Words and Phrases: Quasi-metric, left K -sequentially completeness, variation mapping, Caristi mapping, fixed point.

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